Putnam 1999

A1. Find polynomials $f(x)$,$g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if} \quad x < -1 \\ 3x+2 & \text{if} \quad 1 \leq x \leq 0 \\ -2x+2 & \text{if} \quad x > 0 \end{cases}$$

Solution. Let $h(x) = ax + b$. Note that there are corners in the graph of this function at $x = -1$ and $x = 0$. Thus, we expect $f(x) = c\,(x+1)$ and $g(x) = dx$ for some positive constants c and d. For $x < -1$, we have

$$\begin{aligned} -1 &= |f(x)| - |g(x)| + h(x) = -c\,(x+1) + dx + ax + b \\ &= (d - c + a)\,x + b - c \end{aligned}$$

Thus, $b - c = -1$ and $d - c + a = 0$. For $x > 0$,

$$\begin{aligned} -2x+2 &= |f(x)| - |g(x)| + h(x) = c\,(x+1) - dx + ax + b \\ &= (c - d + a)\,x + b + c \end{aligned}$$

Thus, $b + c = 2$ and $c - d + a = -2$. Now

$$\begin{aligned} b - c &= -1 \text{ and } b + c = 2 \Rightarrow b = \tfrac{1}{2}, c = \tfrac{3}{2} \\ d - c + a &= 0 \text{ and } c - d + a = -2 \Rightarrow a = -1, d = c - a = \tfrac{5}{2} \end{aligned}$$

Thus,

$$f(x) = \tfrac{3}{2}\,(x+1)\,,\ g(x) = \tfrac{5}{2}x,\ h(x) = -x + \tfrac{1}{2}$$

is correct for $x < -1$ and $x > 0$. Continuity gives $3x + 2$ for $-1 \leq x \leq 0$.

A2. Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

Solution. Clearly $p(x)$ has real coefficients, and we may assume $p(x)$ is monic. Let $p(x) = r\,(x)\,c\,(x)$ where the roots of $r\,(x)$ are all real and the roots of $c\,(x)$ are not real. We have

$$\begin{aligned} r\,(x) &= \prod_j (x - r_j)^2 = \left(\prod_j (x - r_j) \right)^2 = s\,(x)^2 \text{ and} \\ c\,(x) &= \prod_k \left((x - a_k)^2 + b_k^2 \right) \end{aligned}$$

Note that when $c(x)$ is multipied out it is a sum of squares of polynomials, say $c(x) = \sum_{m=1}^{K} g_m(x)^2$, and

$$p(x) = s(x)^2 c(x) = \sum_{m=1}^{K} (s(x) g_m(x))^2,$$

as required.

A3. Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

Solution. We do a partial fractions decomposition:

$$\begin{aligned}
\frac{1}{1-2x-x^2} &= \frac{1}{2-(x+1)^2} \\
&= \frac{1}{\left(\sqrt{2}-(x+1)\right)\left(\sqrt{2}+(x+1)\right)} \\
&= \frac{A}{\sqrt{2}-1-x} + \frac{B}{\sqrt{2}+1+x} \\
&\Rightarrow 1 = A\left(\sqrt{2}+1+x\right) + B\left(\sqrt{2}-1-x\right) \\
&\Rightarrow A = B = \frac{1}{2\sqrt{2}}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{1-2x-x^2} &= \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}-1-x} + \frac{1}{\sqrt{2}+1+x}\right) \\
&= \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}-1}\left(\frac{1}{1-\frac{x}{\sqrt{2}-1}}\right) + \frac{1}{\sqrt{2}+1}\left(\frac{1}{1+\frac{x}{\sqrt{2}+1}}\right)\right) \\
&= \frac{1}{2\sqrt{2}}\left(\begin{array}{c} \frac{1}{\sqrt{2}-1}\left(\sum_{n=0}^{\infty} \frac{x^n}{\left(\sqrt{2}-1\right)^n}\right) \\ +\frac{1}{\sqrt{2}+1}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\left(\sqrt{2}+1\right)^n}\right) \end{array}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{2\sqrt{2}}\left(\frac{1}{\left(\sqrt{2}-1\right)^{n+1}} + \frac{(-1)^n}{\left(\sqrt{2}+1\right)^{n+1}}\right) x^n
\end{aligned}$$

Hence,

$$a_n = \frac{1}{2\sqrt{2}}\left(\frac{1}{\left(\sqrt{2}-1\right)^{n+1}} + \frac{(-1)^n}{\left(\sqrt{2}+1\right)^{n+1}}\right)$$

and

$$
\begin{aligned}
a_n^2 + a_{n+1}^2 = \frac{1}{8}\left(\frac{1}{\left(\sqrt{2}-1\right)^{n+1}} + \frac{(-1)^n}{\left(\sqrt{2}+1\right)^{n+1}}\right)^2 \\
+\frac{1}{8}\left(\frac{1}{\left(\sqrt{2}-1\right)^{n+2}} + \frac{(-1)^{n+1}}{\left(\sqrt{2}+1\right)^{n+2}}\right)^2 \\
= \frac{1}{8}\left(\frac{1}{\left(\sqrt{2}-1\right)^{2n+2}} + 2\left(-1\right)^n + \frac{1}{\left(\sqrt{2}+1\right)^{2n+2}}\right) \\
+\frac{1}{8}\left(\frac{1}{\left(\sqrt{2}-1\right)^{2n+4}} + 2\left(-1\right)^{n+1} + \frac{1}{\left(\sqrt{2}+1\right)^{2n+4}}\right) \\
= \frac{1}{8}\left(\frac{\left(\sqrt{2}-1\right)^2+1}{\left(\sqrt{2}-1\right)^{2n+4}} + \frac{\left(\sqrt{2}+1\right)^2+1}{\left(\sqrt{2}+1\right)^{2n+2}}\right) \\
= \frac{1}{2\sqrt{2}}\left(\frac{\frac{1}{8}2\sqrt{2}\left(4-2\sqrt{2}\right)}{\left(\sqrt{2}-1\right)^{2n+4}} + \frac{\frac{1}{8}2\sqrt{2}\left(4+2\sqrt{2}\right)}{\left(\sqrt{2}+1\right)^{2n+4}}\right) \\
= \frac{1}{2\sqrt{2}}\left(\frac{1}{\left(\sqrt{2}-1\right)^{2n+3}} + \frac{1}{\left(\sqrt{2}+1\right)^{2n+3}}\right) \\
= a_{2n+2}
\end{aligned}
$$

A4. Sum the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{3^m\left(n3^m + m3^n\right)}.$$

Solution. Using the fact that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}$, we have

$$
\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{3^m\left(n3^m + m3^n\right)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\left(3^m/m\right)\left(3^m/m + 3^n/n\right)} \\
= \tfrac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\left(3^m/m\right)\left(3^m/m + 3^n/n\right)} + \frac{1}{\left(3^n/n\right)\left(3^n/n + 3^m/m\right)}\right) \\
= \tfrac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\left(3^m/m\right)\left(3^n/n\right)} \\
= \tfrac{1}{2}\left(\sum_{m=1}^{\infty}\frac{m}{3^m}\right)^2,
\end{aligned}
$$

where we have also used

$$\frac{1}{a\left(a+b\right)} + \frac{1}{b\left(a+b\right)} = \frac{b}{ab\left(a+b\right)} + \frac{a}{ba\left(a+b\right)} = \frac{1}{ab}.$$

Now

$$\sum_{m=1}^{\infty} \frac{m}{3^m} = \sum_{m=1}^{\infty} \frac{m}{3^m} x^{m-1} \Big|_{x=1} = \tfrac{d}{dx} \left(\sum_{m=1}^{\infty} \left(\frac{x}{3} \right)^m \right) \Big|_{x=1}$$
$$= \tfrac{d}{dx} \left(\frac{x}{3-x} \right) \Big|_{x=1} = \frac{3}{(-3+1)^2} = \frac{3}{4}$$

Thus, the original sum is then

$$\tfrac{1}{2} \left(\frac{3}{4} \right)^2 = \frac{9}{32}.$$

A5. Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^{1} |p(x)| \, dx.$$

Solution. We may assume that $p(0) \neq 0$ and that $p(x)$ is monic. Then

$$p(x) = \prod_{k=1}^{1999} (x - r_k)$$

where the $r_k \neq 0$ are the possibly complex roots, ordered by modulus. We have

$$\frac{|p(0)|}{\int_{-1}^{1} |p(x)| \, dx} = \frac{\prod_{k=1}^{1999} |r_k|}{\int_{-1}^{1} \prod_{k=1}^{1999} |x - r_k| \, dx} = \frac{1}{\int_{-1}^{1} \prod_{k=1}^{1999} |1 - \frac{x}{r_k}| \, dx}$$

We need to show that as a function of the r_k, $\int_{-1}^{1} \prod_{k=1}^{1999} |\frac{x}{r_k} - 1| \, dx$ is bounded below. For $|r_k| > 2$, we have

$$|1 - \frac{x}{r_k}| > 1 - \left| \frac{x}{r_k} \right| > 1 - \frac{1}{|r_k|} > \tfrac{1}{2}$$

Thus,

$$\int_{-1}^{1} \prod_{k=1}^{1999} |\frac{x}{r_k} - 1| \, dx > \tfrac{1}{2^{1999-N}} \int_{-1}^{1} \prod_{k=1}^{N} |\frac{x}{r_k} - 1| \, dx$$

where the product is over those $|r_k|$ with $|r_k| \leq 2$. For each N,

$$f_N(r_1, r_2, \cdots, r_N) := \int_{-1}^{1} \prod_{k=1}^{N} |\frac{x}{r_k} - 1| \, dx$$

is a continuous positive function of $(r_1, r_2, \cdots, r_N) \in \{|z| \leq 2\}^N$, a compact set. Thus, f_N has a positive minimum, say M_N. Finally,

$$\int_{-1}^{1} \prod_{k=1}^{1999} |\frac{x}{r_k} - 1| \, dx > \min_{1 \leq N \leq 1999} \left\{ \tfrac{1}{2^{1999-N}} M_N \right\} > 0,$$

We can take $C = \min_{1\le N\le 1999}\left\{\frac{1}{2^{1999-N}}M_N\right\}^{-1}$.

A6. The sequence $(a_n)_{n\ge 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \ge 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

Solution.

$$\begin{aligned} a_n &= \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}} \Rightarrow \\ \frac{a_n}{a_{n-1}} &= \frac{6a_{n-1}a_{n-3} - 8a_{n-2}^2}{a_{n-2}a_{n-3}} = 6\frac{a_{n-1}}{a_{n-2}} - 8\frac{a_{n-2}}{a_{n-3}}. \end{aligned}$$

Thus, with $b_n := \frac{a_n}{a_{n-1}}$, we have

$$b_n = 6b_{n-1} - 8b_{n-2},$$

Trying $b_n = r^n$, we get

$$\begin{aligned} 0 &= r^n - 6r^{n-1} + 8r^{n-2} \\ 0 &= \left(r^2 - 6r + 8\right) r^{n-2} = (r-2)(r-4)\, r^{n-2} \end{aligned}$$

Hence,

$$b_n = c_1 2^n + c_2 4^n$$

and

$$\begin{aligned} b_2 &= \frac{a_2}{a_1} = 2 \Rightarrow 2 = c_1 2^2 + c_2 4^2 = 4c_1 + 16c_2 \\ b_3 &= \frac{a_3}{a_2} = 12 \Rightarrow 12 = c_1 2^3 + c_2 4^3 = 8c_1 + 64c_2. \end{aligned}$$

Thus, $c_1 = -\frac{1}{2}, c_2 = \frac{1}{4}$ and

$$\begin{aligned} b_n &= -2^{n-1} + 2^{2n-2} = 2^{n-1}\left(2^{n-1} - 1\right) \\ a_n &= b_n a_{n-1} = b_n b_{n-1} a_{n-2} = \cdots \\ &= b_n b_{n-1} \cdots b_2 a_1 = b_n b_{n-1} \cdots b_2 \\ &= 2^{(n-1)+(n-2)+\cdots+1} \prod_{i=1}^{n-1} (2^i - 1) \\ &= 2^{n(n-1)/2} \prod_{i=1}^{n-1} (2^i - 1). \end{aligned}$$

We need to show that n divides $a_n = 2^{n(n-1)/2}\prod_{i=1}^{n-1}(2^i - 1)$. Write $n = 2^j k$ where k is odd. Certainly $j \le n \le n(n-1)/2$ for $n \ge 2$ so that 2^j divides $2^{n(n-1)/2}$. It suffices to show that

k divides $2^i - 1$ for some $i < n$. We recall that since $\gcd(k, 2) = 1$, k divides $2^{\phi(k)} - 1$ and $\phi(k)$ is less than n, as required.

B1. Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$.

Solution. Note that $|AB| \cos\theta = |AC| = 1$, so that

$$b := |AB| = 1/\cos\theta = \sec\theta.$$

The line CB has slope

$$\frac{-\sin\theta}{b - \cos\theta} = \frac{-\sin\theta}{\frac{1}{\cos\theta} - \cos\theta} = \frac{-\sin\theta\cos\theta}{1 - \cos^2\theta} = \frac{-\sin\theta\cos\theta}{\sin^2\theta} = -\cot\theta$$

and hence its equation is

$$y = -\cot\theta\,(x - b) = \frac{-x\cos\theta + 1}{\sin\theta}.$$

Also $\angle ADC = (\pi - \theta)/2$ and

$$\angle EDB = \pi - \theta - \angle ADC = \pi - \theta - (\pi - \theta)/2 = (\pi - \theta)/2.$$

Thus, the line DE has equation

$$y = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cot\left(\frac{\theta}{2}\right)(x - 1) = \frac{\sin\theta}{1 - \cos\theta}(x - 1)$$

The lines intersect when

$$\begin{aligned}
\frac{\sin\theta}{1 - \cos\theta}(x - 1) &= \frac{-x\cos\theta + 1}{\sin\theta} \\
(x - 1)\sin^2\theta &= (-x\cos\theta + 1)(1 - \cos\theta) \\
(x - 1)(1 + \cos\theta) &= -x\cos\theta + 1 \\
x(1 + \cos\theta) - (1 + \cos\theta) &= -x\cos\theta + 1 \\
x(1 + 2\cos\theta) &= 2 + \cos\theta \\
x &= \frac{2 + \cos\theta}{1 + 2\cos\theta}
\end{aligned}$$

Then

$$y = -(\cot\theta)\frac{2 + \cos\theta}{1 + 2\cos\theta} + \frac{1}{\sin\theta}.$$

Finally,

$$\begin{aligned}
|EF| &= \frac{y}{\sin\theta} = -\frac{\cot\theta}{\sin\theta}\frac{2+\cos\theta}{1+2\cos\theta} + \frac{1}{\sin^2\theta} \\
&= \frac{-\cos\theta\frac{2+\cos\theta}{1+2\cos\theta} + 1}{\sin^2\theta} \\
&= \frac{-(\cos\theta)(2+\cos\theta) + 1 + 2\cos\theta}{\left(\sin^2\theta\right)(2+\cos\theta)} \\
&= \frac{1}{2+\cos\theta} \to \frac{1}{3} \text{ as } \theta \to 0.
\end{aligned}$$

B2. Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

Solution. Suppose that $P(x)$ does not have n distinct roots. Then $P(x)$ is of the form

$$P(x) = (x-a)^m R(x)$$

for some $m \geq 2$ where $R(a) \neq 0$ and $a \in \mathbb{C}$. Then

$$\begin{aligned}
P''(x) &= \left(m(x-a)^{m-1} R(x) + (x-a)^m R'(x)\right)' \\
&= m(m-1)(x-a)^{m-2} R(x) + 2m(x-a)^{m-1} R'(x) + (x-a)^m R''(x) \\
&= (x-a)^{m-2}\left(m(m-1)R(x) + 2m(x-a)R'(x) + (x-a)^2 R''(x)\right) \\
&= (x-a)^{m-2} S(x)
\end{aligned}$$

Since $S(a) = m(m-1)R(a) \neq 0$, $S(x)$ has no factors of $(x-a)$. Then

$$(x-a)^m R(x) = P(x) = Q(x)P''(x) = Q(x)(x-a)^{m-2} S(x)$$

implies $Q(x) = c(x-a)^2$ and

$$R(x) = cS(x)$$

Then

$$R(a) = cS(a) = cm(m-1)R(a) \Rightarrow c = \frac{1}{m(m-1)}.$$

Thus,

$$P(x) = Q(x)P''(x) = \frac{1}{m(m-1)}(x-a)^2 P''(x)$$

or

$$P''(x) = m(m-1)\frac{P(x)}{(x-a)^2}$$

However, by expanding $P(x)$ in powers of $(x-a)$ we can see that this is the case only if $P(x) = b(x-a)^m$ for some constant b.

B3. Let $A = \{(x, y) : 0 \le x, y < 1\}$. For $(x, y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2} \le \frac{m}{n} \le 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y)\to(1,1),(x,y)\in A} (1 - xy^2)(1 - x^2 y) S(x, y).$$

Solution. Note that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x^m y^n = \sum_{m=1}^{\infty} x^m \sum_{n=1}^{\infty} y^n = \frac{x}{1-x}\frac{y}{1-y}$$

Moreover,

$$\begin{aligned}
& \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x^m y^n - S(x,y) \\
= & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x^m y^n - \sum_{\frac{1}{2} \le \frac{m}{n} \le 2} x^m y^n \\
= & \sum_{m \ge 2n+1} x^m y^n + \sum_{n \ge 2m+1} x^m y^n \\
= & \sum_{n=1}^{\infty} \frac{x^{2n+1}}{1-x} y^n + \sum_{m=1}^{\infty} \frac{y^{2m+1}}{1-y} x^m \\
= & \frac{x}{1-x} \sum_{n=1}^{\infty} \left(x^2 y\right)^n + \frac{y}{1-y} \sum_{m=1}^{\infty} \left(y^2 x\right)^m \\
= & \frac{x^3 y}{(1-x)(1-x^2 y)} + \frac{y^3 x}{(1-y)(1-y^2 x)}
\end{aligned}$$

Thus,

$$\begin{aligned}
S(x,y) &= \frac{x}{1-x}\frac{y}{1-y} - \frac{x^3 y}{(1-x)(1-x^2 y)} - \frac{y^3 x}{(1-y)(1-y^2 x)} \\
&= \frac{xy\left((1-x^2y)(1-y^2x) - x^2(1-y^2x)(1-y) - y^2(1-x^2y)(1-x)\right)}{(1-x)(1-y)(1-x^2y)(1-y^2x)} \\
&= \frac{xy\left(1 - x^3y^3 - x^2 + x^3y^2 - y^2 + x^2y^3\right)}{(1-x)(1-y)(1-x^2y)(1-y^2x)} \\
&= \frac{xy(1-x)(1-y)(xy + y + x + 1 - x^2y^2)}{(1-x)(1-y)(1-x^2y)(1-y^2x)} \\
&= xy\frac{xy + y + x + 1 - x^2y^2}{(1-x^2y)(1-y^2x)},
\end{aligned}$$

and

$$\lim_{(x,y)\to(1,1),(x,y)\in A}(1-xy^2)(1-x^2y)S(x,y)=3.$$

B4. Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \le f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Solution. Let $c = \lim_{x\to-\infty} f(x) \ge 0$. By replacing $f(x)$ by $f(x) - c$ we may assume $c = 0$. Let $c' = \lim_{x\to-\infty} f'(x) \ge 0$. Then $f'(x) > d$ and for $x < 0$

$$f(0) = f(x) + \int_x^0 f'(x)dx > f(x) + c'x.$$

Thus,

$$f(x) < f(0) - c'x$$

and it follows that $c' = 0$. Similarly, $c'' = \lim_{x\to-\infty} f''(x) = 0$. Since $f'''(x) \le f(x)$,

$$\begin{aligned} f''(x)f'''(x) &\le f''(x)f(x) < f''(x)f(x) + f'(x)^2 \\ \text{or } 0 &< \tfrac{d}{dx}\left(f'(x)f(x) - \tfrac{1}{2}f''(x)^2\right). \end{aligned}$$

Also, as

$$\lim_{x\to-\infty}\left(f'(x)f(x) - \tfrac{1}{2}f''(x)^2\right) = c'c - \tfrac{1}{2}\left(c''\right)^2 = 0,$$

we then have

$$f'(x)f(x) > \tfrac{1}{2}f''(x)^2.$$

Thus,

$$\begin{aligned} \tfrac{d}{dx}\left(\tfrac{3}{4}f(x)^2\right) &= \tfrac{3}{2}f(x)f'(x) > f'(x)f(x) + \tfrac{1}{2}f'(x)f'''(x) \\ &> \tfrac{1}{2}\left(f''(x)^2 + f'(x)f'''(x)\right) \\ &= \tfrac{d}{dx}\left(\tfrac{1}{2}f'(x)f''(x)\right) \end{aligned}$$

or

$$\tfrac{d}{dx}\left(\tfrac{3}{4}f(x)^2 - \tfrac{1}{2}f'(x)f''(x)\right) > 0.$$

Since $\lim_{x\to\infty}\left(\frac{3}{4}f(x)^2 - \frac{1}{2}f'(x)f''(x)\right) = \frac{3}{4}c^2 - \frac{1}{2}c'c'' = 0$, we then have

$$\tfrac{3}{4}f(x)^2 - \tfrac{1}{2}f'(x)f''(x) > 0.$$

Hence,

$$\tfrac{3}{4}f(x)^2f'(x) > \tfrac{1}{2}f'(x)^2f''(x)$$

or

$$\tfrac{1}{4}\tfrac{d}{dx}\left(f(x)^3\right) > \tfrac{1}{6}\tfrac{d}{dx}\left(f'(x)^3\right).$$

Thus,

$$f(x)^3 > \tfrac{2}{3}f'(x)^3$$

and

$$f'(x) < \left(\tfrac{3}{2}\right)^{1/3} f(x) < 2f(x).$$

B5. For an integer $n \geq 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

Solution. Let $z = e^{i\theta}$. Then

$$\cos(m\theta) = \tfrac{1}{2}\left(e^{im\theta} + e^{-im\theta}\right) = \tfrac{1}{2}\left(z^m + z^{-m}\right) = \tfrac{1}{2}\left(z^m + \overline{z}^{-m}\right).$$

Let $v = (z, z^2, z^3, \cdots, z^n)$ and $\overline{v} = (\overline{z}, \overline{z}^2, \overline{z}^3, \cdots, \overline{z}^n)$. The matrix A is half the sum of the outer products $v \otimes v$ and $\overline{v} \otimes \overline{v}$. Thus,

$$I + A = I + \tfrac{1}{2}\left(v \otimes v + \overline{v} \otimes \overline{v}\right)$$

For any $w \in \mathbb{C}^n$,

$$(I + A)\, w = w + \tfrac{1}{2}\left((v \cdot w)\, v + (\overline{v} \cdot w)\, \overline{v}\right)$$

where $v \cdot w$ is the usual bilinear dot product. For $u \in \mathbb{C}^n$ orthogonal to both v and $\overline{v}$, we have

$$(I + A)\, u = u.$$

This shows that $I + A$ is the identity on $span\,(v, \overline{v})^{\perp}$. Thus, we need only evaluate $\det(I + A)$ on $span\,(v, \overline{v})$. Note that

$$\begin{aligned}
(I + A)\,(v) &= v + \tfrac{1}{2}\left((v \cdot v)\, v + (\overline{v} \cdot v)\, \overline{v}\right) \\
&= \left(1 + \tfrac{1}{2}(v \cdot v)\right) v + \tfrac{1}{2}(\overline{v} \cdot v)\, \overline{v} \\
(I + A)\,(\overline{v}) &= \overline{v} + \tfrac{1}{2}\left((v \cdot \overline{v})\, v + (\overline{v} \cdot \overline{v})\, \overline{v}\right) = \tfrac{1}{2}(v \cdot \overline{v})\, v + \left(1 + \tfrac{1}{2}(\overline{v} \cdot \overline{v})\right) \overline{v}
\end{aligned}$$

Thus, the matrix of $I + A$ on $span\,(v, \overline{v})$ is

$$\begin{bmatrix} 1 + \frac{1}{2}(v \cdot v) & \frac{1}{2}(\overline{v} \cdot v) \\ \frac{1}{2}(v \cdot \overline{v}) & 1 + \frac{1}{2}(\overline{v} \cdot \overline{v}) \end{bmatrix}$$

and the determinant is (since $\overline{z}^{2n} = 1$ and $z^{2n} = 1$)

$$\begin{aligned}
&\begin{vmatrix} 1 + \frac{1}{2}(v \cdot v) & \frac{1}{2}n \\ \frac{1}{2}n & 1 + \frac{1}{2}(\overline{v} \cdot \overline{v}) \end{vmatrix} \\
=& \begin{vmatrix} 1 + \frac{1}{2}(z^2 + \cdots + z^{2n}) & \frac{1}{2}n \\ \frac{1}{2}n & 1 + \frac{1}{2}(\overline{z}^2 + \cdots + \overline{z}^{2n}) \end{vmatrix} \\
=& \begin{vmatrix} 1 + \frac{1}{2}\left(z^2 \frac{1 - z^{2n}}{1 - z^2}\right) & \frac{1}{2}n \\ \frac{1}{2}n & 1 + \frac{1}{2}\left(\overline{z}^2 \frac{1 - \overline{z}^{2n}}{1 - \overline{z}^2}\right) \end{vmatrix} \\
=& \begin{vmatrix} 1 & \frac{1}{2}n \\ \frac{1}{2}n & 1 \end{vmatrix} = 1 - \frac{n^2}{4}.
\end{aligned}$$

B6. Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s, n) = 1$ or $\gcd(s, n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s, t)$ is prime.

Solution. Let $p_1 < p_2 < \ldots < p_k$ be the set of all of the primes that occur in at least one of the prime factorizations of the elements of S. $S_1 \subseteq S$ be the subset consisting of all those members of S involving p_1 in their prime factorization. If there are $s, t \in S_1$ with $\gcd(s, t) = p_1$, then we are done. If not, let $p_{i_1} = p_1$ and let $p_{i_2} > p_1$ be the smallest prime that occurs in the factorization of members of $S - S_1$, and let $S_2 \subseteq S - S_1$ be the subset consisting of all those members of S involving p_{i_2} in their prime factorization. If there are $s, t \in S_2$ with $\gcd(s, t) = p_{i_2}$, then we are done. We continue in this way obtaining $p_{i_1} < p_{i_2} < \ldots < p_{i_j}$ where the process stops at some $j \leq k$. Note that we either have $s, t \in S_j$ with $\gcd(s, t) = p_{i_j}$ (in which case we are done) or $S = \cup_{m=1}^{j} S_m$. Suppose that $S = \cup_{m=1}^{j} S_m$. Let $n = p_{i_1} p_{i_2} \cdots p_{i_j}$. Then by assumption, there is some $s \in S$, such that

$$\gcd(s, n) = s \text{ or } \gcd(s, n) = 1.$$

Now $\gcd(s, n) = 1$ is impossible, since $S = \cup_{m=1}^{j} S_m$. Thus, $\gcd(s, n) = s$, and so s is a divisor of $n = p_{i_1} p_{i_2} \cdots p_{i_j}$. Let j' be the smallest integer so that there is a divisor $s \in S$ of $p_{i_1} p_{i_2} \cdots p_{i_{j'}}$. Now $p_{i_{j'}}$ must occur (exactly once) as a factor of s or else j' can be reduced. Select $s_{j'} \in S_{j'}$. We claim

$$\gcd(s, s_{j'}) = p_{i_{j'}}.$$

Indeed, by definition $s_{j'}$ has factor $p_{i_{j'}}$, but no factors of $p_{i_1}, p_{i_2}, \cdots p_{i_{j'-1}}$.