Putnam 2000

A1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that x_0, x_1, x_2,... are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Solution. Since $0 < x_j < A$,

$$\sum_{j=0}^{\infty} x_j^2 < \sum_{j=0}^{\infty} A x_j = A^2.$$

Thus, the possible values of $\sum_{j=0}^{\infty} x_j^2$ belong to $(0, A^2)$. We show that each point in $(0, A^2)$ is a possible value using the following argument due to Ken Rogers. Let $x_i = ay^i$ for $y \in (0,1)$ and a constant $a > 0$. Then

$$\sum_{j=0}^{\infty} x_j = A \Leftrightarrow \frac{a}{1-y} = \sum_{j=0}^{\infty} ay^j = A \Leftrightarrow a = A(1-y).$$

We have

$$\sum_{j=0}^{\infty} x_j^2 = \sum_{j=0}^{\infty} a^2 \left(y^2\right)^j = \frac{a^2}{1-y^2} = \frac{A^2(1-y)^2}{1-y^2} = \frac{1-y}{1+y} A^2.$$

Then, as y varies from 1 to 0, we have that $\frac{1-y}{1+y}$ varies from 0 to 1.

A2. Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of two squares of integers. [Example: $0 = 0^2 + 0^2$, $1^2 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

Solution. Note that for any integer a

$$(a+1)^2 - 1 = a^2 + 2a, \quad (a+1)^2 + 0^2, \quad (a+1)^2 + 1^2$$

will be a triple of such numbers if $2a$ is a square. Thus choose $a = 2m^2$, $m = 0, 1, \cdots$.

A3. The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5 and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

Solution. A square of area 5, has edge length $\sqrt{5}$, and diagonal length $\sqrt{10}$ which is the diameter of the circle. If x and y (say $0 < x < y$) are the dimensions of the rectangle $P_2P_4P_6P_8$, then

$$x^2 + y^2 = 10 \text{ and } xy = 4,$$

and the relevant solution is $x = \sqrt{2}$, $y = 2\sqrt{2}$. We may assume that $P_1 = \left(\frac{1}{2}\sqrt{10}, 0\right)$, and

$$P_2 = \left(\tfrac{1}{2}\sqrt{10}\cos\theta, \tfrac{1}{2}\sqrt{10}\sin\theta\right)$$

for some $\theta \in \left(0, \frac{\pi}{2}\right)$, and $P_2P_4 = \sqrt{2}$ (otherwise rotate all by $90°$). Then

$$P_2 = \left(\tfrac{1}{2}\sqrt{10}\cos(\theta+\alpha), \tfrac{1}{2}\sqrt{10}\sin(\theta+\alpha)\right)$$

where α is the angle subtended by the side P_2P_4. Since $y = 2x$, $\alpha = 2\arctan\left(\frac{1}{2}\right)$. Thus,

$$\begin{aligned}
\cos\alpha &= \cos\left(2\arctan\left(\tfrac{1}{2}\right)\right) = \cos^2\left(\arctan\left(\tfrac{1}{2}\right)\right) - \sin^2\left(\arctan\left(\tfrac{1}{2}\right)\right) \\
&= \left(\tfrac{2}{\sqrt{5}}\right)^2 - \left(\tfrac{1}{\sqrt{5}}\right)^2 = \frac{3}{5} \text{ and} \\
\sin\alpha &= \frac{4}{5}.
\end{aligned}$$

The area of an isosceles triangle with equal sides r at an angle β is $\frac{1}{2}r^2\sin\beta$. Thus, the area of the octagon is

$$\begin{aligned}
&A(\theta) := 2\cdot\tfrac{1}{2}\cdot\left(\tfrac{1}{2}\sqrt{10}\right)^2\left(\sin\theta + \sin\left(\tfrac{\pi}{2}-\theta\right) + \sin\left(\alpha+\theta-\tfrac{\pi}{2}\right) + \sin\left(\pi-(\alpha+\theta)\right)\right) \\
&= \tfrac{5}{2}\left(\sin\theta + \cos\theta - \cos(\alpha+\theta) + \sin(\alpha+\theta)\right) \\
&= \tfrac{5}{2}\left(\sin\theta + \cos\theta - (\cos\alpha\cos\theta - \sin\alpha\sin\theta) + (\sin\alpha\cos\theta + \cos\alpha\sin\theta)\right) \\
&= \tfrac{5}{2}\left((1+\sin\alpha+\cos\alpha)\sin\theta + (1+\sin\alpha-\cos\alpha)\cos\theta\right) \\
&= \tfrac{5}{2}\left(\left(1+\tfrac{4}{5}+\tfrac{3}{5}\right)\sin\theta + \left(1+\tfrac{4}{5}-\tfrac{3}{5}\right)\cos\theta\right) \\
&= 6\sin\theta + 3\cos\theta.
\end{aligned}$$

There is a further restriction on θ, namely $\theta \geq \frac{\pi}{2} - \alpha$, or else P_4 will come before P_3. For $\theta = \frac{\pi}{2} - \alpha$, $\sin\theta = \cos\alpha = 3/5$ and $\cos\theta = 4/5$, and

$$\begin{aligned}
A\left(\tfrac{\pi}{2}-\alpha\right) &= 6\cdot\tfrac{3}{5} + 3\cdot\tfrac{4}{5} = 6, \text{ while} \\
A\left(\tfrac{\pi}{2}\right) &= 6\cdot 1 + 3\cdot 0 = 6.
\end{aligned}$$

Now

$$\begin{aligned}
A'(\theta) &= \tfrac{d}{d\theta}(6\sin\theta + 3\cos\theta) = 6\cos\theta - 3\sin\theta = 0 \Rightarrow \theta = \arctan 2, \text{ and} \\
A(\arctan 2) &= 6\sin(\arctan 2) + 3\cos(\arctan 2) = 6\tfrac{2}{\sqrt{5}} + 3\tfrac{1}{\sqrt{5}} = 3\sqrt{5} > 6.
\end{aligned}$$

As $\theta = \arctan 2$, the rectangle is "vertical" and the maximum area is $3\sqrt{5}$.

A4. Show that the improper integral

$$\lim_{B\to\infty}\int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

Solution. We use

$$\sin a \sin b = \tfrac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$$

to get

$$\begin{aligned}
\sin(x)\sin(x^2) &= \tfrac{1}{2}\left(\cos\left(x-x^2\right)-\cos\left(x+x^2\right)\right)\\
&= \tfrac{1}{2}\left(\cos\left(x^2-x\right)-\cos\left(x^2+x\right)\right)\\
&= \tfrac{1}{2}\left(\cos\left(\left(x-\tfrac{1}{2}\right)^2-\tfrac{1}{4}\right)-\cos\left(\left(x+\tfrac{1}{2}\right)^2-\tfrac{1}{4}\right)\right)\\
&= \tfrac{1}{2}\left(\cos\left(\left(x-\tfrac{1}{2}\right)^2-\tfrac{1}{4}\right)-\cos\left(\left(x+\tfrac{1}{2}\right)^2-\tfrac{1}{4}\right)\right)\\
&= \tfrac{1}{2}\left(\begin{array}{c}\cos\tfrac{1}{4}\cos\left(\left(x-\tfrac{1}{2}\right)^2\right)+\sin\tfrac{1}{4}\sin\left(\left(x-\tfrac{1}{2}\right)^2\right)\\ -\left(\cos\tfrac{1}{4}\cos\left(\left(x+\tfrac{1}{2}\right)^2\right)+\sin\tfrac{1}{4}\sin\left(\left(x+\tfrac{1}{2}\right)^2\right)\right)\end{array}\right)
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^B \cos\left(\left(x\pm\tfrac{1}{2}\right)^2\right)dx &= \int_{\pm\frac{1}{2}}^{B\pm\frac{1}{2}} \cos\left(u^2\right)du, \text{ and}\\
\int_0^B \sin\left(\left(x\pm\tfrac{1}{2}\right)^2\right)dx &= \int_{\pm\frac{1}{2}}^{B\pm\frac{1}{2}} \sin\left(u^2\right)du.
\end{aligned}$$

Thus, it suffices to show the existence of improper integrals of the form

$$\lim_{B\to\infty}\int_0^B \sin(x^2)\,dx \text{ and } \lim_{B\to\infty}\int_0^B \cos(x^2)\,dx.$$

Consider the parametric curve

$$\mathbf{r}(u) = (x(u), y(u)) = \left(\int_0^u \cos(t^2)\,dt, \int_0^u \sin(t^2)\,dt\right).$$

The components are integrals of Fresnel type. Note that

$$\mathbf{r}'(u) = (x'(u), y'(u)) = \left(\cos(u^2), \sin(u^2)\right) \text{ and } \|\mathbf{r}'(u)\| = 1.$$

The curvature is given by

$$\begin{aligned}
\kappa(u) &= x'(u)\,y''(u) - y'(u)\,x''(u) = \cos(u^2)\cos(u^2)2u + \sin(u^2)\sin(u^2)2u\\
&= 2u\left(\sin^2(u^2)+\cos^2(u^2)\right) = 2u.
\end{aligned}$$

Since the curvature increases with arc length, the curve spirals inward to a limit point and the integrals then converge. Indeed, the center of curvature of $\mathbf{r}$ at $\mathbf{r}(u)$ is

$$\begin{aligned}
\mathbf{c}(u) &= \mathbf{r}(u) + \frac{1}{2u}\left(-y'(u), x'(u)\right)\\
&= \left(\int_0^u \cos(t^2)\,dt, \int_0^u \sin(t^2)\,dt\right) + \frac{1}{2u}\left(-\sin(u^2), \cos(u^2)\right), \text{ and}\\
\mathbf{c}'(u) &= \left(\cos(u^2), \sin(u^2)\right) + \left(-\cos(u^2), -\sin(u^2)\right) + \frac{1}{2u^2}\left(-\sin(u^2), \cos(u^2)\right)\\
&= \frac{1}{2u^2}\left(-\sin(u^2), \cos(u^2)\right).
\end{aligned}$$

Thus,

$$\int_1^\infty \|\mathbf{c}'(u)\|\,du \le \int_1^\infty \frac{1}{2u^2}du = \frac{1}{2} < \infty$$

and the length of the curve $\mathbf{c}|\,[1,\infty)$ is finite, implying that $\mathbf{L} := \lim_{u\to\infty}\mathbf{c}(u)$ exists. Now

$$\|\mathbf{r}(u) - \mathbf{L}\| \le \|\mathbf{r}(u) - \mathbf{c}(u)\| + \|\mathbf{L} - \mathbf{c}(u)\| = \frac{1}{2u} + \|\mathbf{L} - \mathbf{c}(u)\| \to 0$$

as $u \to \infty$. Thus, $\lim_{u\to\infty}\mathbf{r}(u) = \mathbf{L}$, as desired.

A5. Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

Solution. Consider the triangle T with sides of length a, b, and c connecting these points. We first show the standard (?) fact that the area A of T is given by

$$A = \frac{abc}{4r},$$

where r is the radius of the circumcircle of T. Let α be the angle of T opposite a. Then (from the cross product) we have $A = \frac{1}{2}bc\sin\alpha$. On the hand, the *central* angle opposite a is known to be 2α and so $a = 2r\sin\alpha$. Thus,

$$A = \tfrac{1}{2}bc\sin\alpha = \tfrac{1}{2}bc\frac{1}{2r}(2r\sin\alpha) = \frac{abc}{4r}.$$

If $d = \max(a,b,c)$, then

$$\frac{d^3}{4r} \ge \frac{abc}{4r} = A.$$

But, using the cross-product again, we know that $2A^2$ is a determinant of a 2×2 integer matrix, and hence a positive integer. Thus, $2A^2 \ge 1$ or $A \ge 1/\sqrt{2}$. Hence,

$$d^3 \ge 4r/\sqrt{2} = 2\sqrt{2}r \Rightarrow d \ge \left(2^{3/2}r\right)^{1/3} = (2r)^{1/3} > r^{1/3}.$$

A6. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n > 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

Solution. Assume that $a_1 \ne 0$. We must then show that $a_2 = 0$. Note that $a_1 = f(a_0) = f(0)$ and so a_1 is the nonzero constant term in $f(x)$. We have $f(a_{m-1}) = a_m = 0$. Thus, a_{m-1} is an integer zero of $f(x)$. Since $f(x) = (x - a_{m-1})\,g(x)$ for some $g(x)$ with integer coefficients, we have that a_{m-1} divides the constant term of $f(x)$, namely a_1. Since a_1 is the constant term of f, we know that a_1 divides all the iterates $(f\circ\cdots\circ f)(a_1)$. In particular, a_1 divides a_{m-1}, and we have already shown that a_{m-1} divides a_1. Thus, $a_{m-1} = \pm a_1$. If $a_{m-1} = a_1$, then

$$a_2 = f(a_1) = f(a_{m-1}) = a_m = 0.$$

Thus, we are done in the case where $a_n \geq 0$ for all n.

Let $a_{n_0} = \min_{0 \leq n \leq m} \{a_n\}$ and let

$$g(x) = f(x + a_{n_0}) - a_{n_0}.$$

Defining $b_0 = 0$ and $b_{n+1} = g(b_n)$, we have

$$\begin{aligned} b_1 &= g(b_0) = g(0) = f(0 + a_{n_0}) - a_{n_0} = a_{n_0+1} - a_{n_0} \geq 0, \\ b_2 &= g(b_1) = g(a_{n_0+1} - a_{n_0}) = f(a_{n_0+1}) - a_{n_0} = a_{n_0+2} - a_{n_0} \geq 0, \\ &\vdots \\ b_n &= a_{n_0+n} - a_{n_0} \geq 0 \text{ for all } n. \end{aligned}$$

Since $a_{n_0+m} - a_{n_0} = 0$, we may then apply the case we have shown to deduce that if $b_m = 0$ for some $m > 0$, then $0 = b_1 = a_{n_0+1} - a_{n_0}$ or $0 = b_2 = a_{n_0+2} - a_{n_0} = 0$. If $b_1 = 0$, then $g(0) = 0$ and $b_n = a_{n_0+n} - a_{n_0} = 0$ for all n, in which case $a_{n_0+n} = a_{n_0}$ and $\{a_n\}$ is a constant sequence (necessarily zero). If $0 = b_2 = a_{n_0+2} - a_{n_0} = 0$, then $\{b_n\}$ is periodic of period 2 in n and so is a_n, in which case $a_2 = a_0 = 0$.

B1. Let a_j, b_j, c_j, be integers for $1 \leq j \leq N$. Assume, for each j, that at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

Solution. Note that for fixed j, the evenness or oddness of $ra_j + sb_j + tc_j$ depends on the evenness or oddness of r, s, t and a_j, b_j, c_j. Thus, it suffices to consider $(r, s, t) \in S := \{0,1\} \times \{0,1\} \times \{0,1\}$, and we have a map

$$F : \{(j; a_j, b_j, c_j) : j \in \{1, \ldots, N\}\} \to S$$

For $(r, s, t) \in S$, let

$$(r, s, t)_1 := \{(\rho, \sigma, \tau) \in S : (r, s, t) \cdot (\rho, \sigma, \tau) = 1\}.$$

Clearly, $(0,0,0)_1$ is empty, and it is easy to check that each of the remaining 7 sets

$$(1,0,0)_1, (0,1,0)_1, (0,0,1)_1, (1,1,0)_1, (0,1,1)_1, (1,0,1)_1, (1,1,1)_1$$

has exactly 4 elements and their union is $S - \{(0,0,0)\}$. Call these $S_1, \ldots, S_7$. We need to show that for some $i \in \{1, \ldots, 7\}$, $\#F^{-1}(S_i) \geq 4N/7$. Suppose that for all i,

$$\#F^{-1}(S_i) < 4N/7.$$

Then, adding we have

$$\#F^{-1}(S_1) + \cdots + \#F^{-1}(S_7) < 4N. \qquad (*)$$

Now $\cup_{i=1}^{7} F^{-1}(S_i) = F^{-1}(S)$, since $F^{-1}((0,0,0)) = \phi$ by assumption. Each $F((a_j, b_j, c_j))$ belongs to exactly 4 of the S_i, namely those $(r, s, t)_1$ such that $(r, s, t) \in F((a_j, b_j, c_j))_1$. Thus, the left side of $(*)$ is $4N$, and we have a contradiction.

B2. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$. [Here $\binom{n}{m} = \frac{n!}{m!(n-m)!}$, and $gcd(m,n)$ is the greatest common divisor of m and n.]

Solution. Note $\gcd(m,n)\,\mathrm{lcm}(m,n) = mn$. Thus,

$$\begin{aligned} & n \text{ divides } \gcd(m,n)\binom{n}{m} \\ \Leftrightarrow\quad & n \text{ divides } \frac{mn}{\mathrm{lcm}(m,n)}\binom{n}{m} \\ \Leftrightarrow\quad & \mathrm{lcm}(m,n) \text{ divides } m\binom{n}{m} \\ \Leftrightarrow\quad & m \text{ divides } m\binom{n}{m}, \text{ and } n \text{ divides } m\binom{n}{m}, \end{aligned}$$

but certainly m divides $m\binom{n}{m}$ while

$$m\binom{n}{m} = m\frac{n!}{m!\,(n-m)!} = n\frac{(n-1)!}{(m-1)!\,(n-m)!} = n\binom{n-1}{m-1}$$

and so n divides $m\binom{n}{m}$.

B3. Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and $a_N \neq 0$. Let N_k denote the number of zeros (including multiplicities) of $\frac{d^k}{dt^k}f(t)$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \dots \quad \text{and} \quad \lim_{k\to\infty} N_k = 2N.$$

Solution. Here, they must have meant to restrict the domain to $[0,1)$, or equivalently, to the circle $\mathbb{R}/\mathbb{Z}$. By Rolle's Theorem, between any two zeros of $f^{(k)}(t)$ there is at least one zero of $f^{(k+1)}(t)$. Since $f^{(k)}$ is defined on a circle, we then have $N_k \leq N_{k+1}$. Note that

$$f^{(4k)}(t) = (2\pi)^{4k}\sum_{j=1}^{N} j^{4k} a_j \sin(2\pi jt)$$

and eventually the maximum and minimum values of the term $N^{4k}a_N\sin(2\pi Nt)$ will dominate the value of the sum of the lower terms, since

$$\sum_{j=1}^{N-1} j^{4k}\,|a_j| \leq (N-1)^{4k}\sum_{j=1}^{N-1}|a_j| \leq N^{4k}\,|a_N|\,,$$

for k sufficiently large. Thus, $f^{(4k)}$ has at least $2N$ zeros for k sufficiently large. Also, for $z = e^{(2\pi t)i}$, we have

$$\begin{aligned} f^{(4k)}(t) &= (2\pi)^{4k} \sum_{j=1}^{N} j^{4k} a_j \frac{\left(z^j - z^{-j}\right)}{2i} \\ &= (2\pi)^{4k} \sum_{j=1}^{N} j^{4k} a_j \frac{\left(z^{N+j} - z^{N-j}\right)}{2iz^N} \\ &= \frac{p(z)}{z^N} \end{aligned}$$

for a polynomial $p(z)$ of degree $2N$. Thus, $f^{(4k)}$ also has at most $2N$ zeros.

B4. Let $f(x)$ be a continuous function such that $f(2x^2-1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \le x \le 1$.

Solution. Note that $\cos(2u) = \cos^2(u) - \sin^2(u) = 2\cos^2 u - 1$. Thus,

$$f(\cos(2u)) = 2\cos(u) f(\cos(u))$$

and

$$\begin{aligned} f(\cos v) &= 2\cos(v/2) f(\cos(v/2)) = \frac{\sin(v)}{\sin(v/2)} f(\cos(v/2)) \\ &= \frac{\sin(v)}{\sin(v/2)} \frac{\sin(v/2)}{\sin(v/4)} f(\cos(v/4)) \\ &= \cdots = \frac{\sin(v)}{\sin(v/2^k)} f\left(\cos\left(v/2^k\right)\right) \end{aligned}$$

Also $-\cos(2u) = \sin^2(u) - \cos^2(u) = 2\sin^2 u - 1$, so that

$$f(-\cos(2u)) = 2\sin(u) f(\sin(u))$$

Note that f is odd since $2xf(x) = f(2x^2-1)$ is even. Thus,

$$\begin{aligned} 2\sin(u) f(\sin(u)) &= f(-\cos(2u)) = -f(\cos(2u)) = -2\cos(u) f(\cos(u)) \text{ or} \\ f(\cos(u)) &= -\frac{\sin(u)}{\cos(u)} f(\sin(u)) \text{ if } \cos(u) \neq 0. \end{aligned}$$

Hence, as $k \to \infty$ and $f(0) = 0$ due to the oddness of f, we have

$$f(\cos v) = \frac{\sin(v)}{\sin(v/2^k)} f\left(\cos\left(v/2^k\right)\right) = -\frac{\sin(v)}{\cos(v/2^k)} f\left(\sin\left(v/2^k\right)\right) \to 0.$$

B5. Let S_0 be a finite set of positive integers. We define finite sets S_1, $S_2, \ldots$ of positive integers as follows:

Integer a is in S_{n+1}, if and only if exactly one of $a-1$ or a is in S_n.

Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

Solution. Let $p_n(x)$ be a polynomial with coefficients in $\mathbb{Z}_2$, such that the coefficient $a_k(n)$ of x^k in $p_n(x)$ is 1 when $k \in S_k$ and 0 otherwise. Note that the coefficient $a_k(n+1)$ of $p_{n+1}(x)$ is given by

$$a_k(n+1) = \begin{cases} 0 & \text{if } a_k(n) + a_{k-1}(n) = 1 \\ 1 & \text{if } a_k(n) + a_{k-1}(n) = 0. \end{cases}$$

This is also the coefficient of x^k for the polynomial $p_n(x) + xp_n(x) = (1 + x)p_n(x)$. Thus,

$$p_{n+1}(x) = (x+1)p_n(x) \quad \text{and} \quad p_n(x) = (1+x)^n p_0(x).$$

We must show that there are infinitely many N, such that

$$p_N(x) = p_0(x) + x^N p_0(x) = \left(1 + x^N\right) p_0(x).$$

In other words, that there are infinitely many n, such that

$$(1+x)^N = 1 + x^N$$

This, is true for $N = 2$ and note that if it is true for N, then it is true for $2N$, since

$$(1+x)^{2N} = \left((1+x)^N\right)^2 = \left(1 + x^N\right)^2 = 1 + 2x^N + x^{2N} = 1 + x^{2N}.$$

Thus, $S_N = S_0 \cup \{N + a : a \in S_0\}$ for N equal to a power of 2.

B6. Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm1, \pm1, ..., \pm1)$ in n-dimensional space, with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Solution. Let

$$C = \{(x_1, x_2, ..., x_n) : x_i = \pm1\}$$

For a fixed point $p \in C$, let S_p be the set of points in C of minimal distance (namely 2) from p. A point $q \in S_p$ if q differs from p in one coordinate. Note that two points $q_1, q_2 \in S_p$ agree in all but two coordinates, namely the distinct coordinates in which they differ from p. Thus, $\|q_1 - q_2\| = \sqrt{8}$, and hence all points in S_p are equidistant from each other. It suffices to show that there is some $p \in C$, with $\#(B \cap S_p) \geq 3$. Consider the set

$$A = \{(q,p) : q \in B \cap S_p\}$$

Note that for each $q \in B$, there are n points $p_1(q), \ldots, p_n(q)$ with $q \in S_{p_i(q)}$. Thus,

$$\#A = n \cdot \#(B).$$

Also, for each p, the number of $q \in B$ with $(q,p) \in A$ is $\#(S_p \cap B)$. Thus,

$$\sum_{p \in C} \#(B \cap S_p) = \#A = n \cdot \#(B) > n \cdot 2^{n+1}/n = 2^{n+1} = 2 \cdot 2^n$$

Since there are 2^n terms on the left side, one of them must be bigger than 2.