Putnam 2001

A1. Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

Solution. We have $b = ((b * a) * b) * (b * a) = a * (b * a)$.

A2. You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.

Solution. Let O_n be the event that n tosses result in an odd number of heads, and let E_n be the event that n tosses result in an even number of heads. We have $P(O_1) = 1/3$ and

$$\begin{aligned}
P(O_n) &= P(C_n \text{ lands on } T)\, P(O_{n-1}) + P(C_n \text{ lands on } H)\, P(E_{n-1}) \\
&= \left(1 - \frac{1}{2n+1}\right) P(O_{n-1}) + \frac{1}{2n+1}(1 - P(O_{n-1})) \\
&= \left(1 - \frac{1}{2n+1} - \frac{1}{2n+1}\right) P(O_{n-1}) + \frac{1}{2n+1} \\
&= \frac{2n-1}{2n+1} P(O_{n-1}) + \frac{1}{2n+1} = \frac{(2(n-1)+1)\, P(O_{n-1}) + 1}{2n+1} \\
&= \frac{(2(n-1)+1)\left(\frac{(2(n-2)+1)P(O_{n-2})+1}{2(n-1)+1}\right) + 1}{2n+1} \\
&= \frac{(2(n-2)+1)\, P(O_{n-2}) + 2}{2n+1} \\
&= \cdots = \frac{(2(n-i)+1)\, P(O_{n-i}) + i}{2n+1} \\
&= \frac{(2(n-(n-1))+1)\, P\left(O_{n-(n-1)}\right) + n - 1}{2n+1} \\
&= \frac{3 \cdot \frac{1}{3} + n - 1}{2n+1} = \frac{n}{2n+1}.
\end{aligned}$$

A3. For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

Solution. We first find the roots r_1, r_2, r_3, r_4 of $P_m(x)$. Then

$$P_m(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4).$$

Thus, one of the nonconstant polynomial factors would be of form $x - r_1$ for some root r_1, or $(x - r_1)(x - r_2)$ for some roots r_1 and r_2. The quadratic formula yields

$$\begin{aligned} r^2 &= \frac{1}{2}\left((2m+4) \pm \sqrt{(2m+4)^2 - 4(m-2)^2}\right) \\ &= m + 2 \pm \sqrt{(m+2)^2 - (m-2)^2} = (m+2) \pm 2\sqrt{2}\sqrt{m} \\ &= \left(\sqrt{m} \pm \sqrt{2}\right)^2. \end{aligned}$$

So

$$\{r_1, r_2, r_3, r_4\} = \left\{\sqrt{m} + \sqrt{2}, \sqrt{m} - \sqrt{2}, -\sqrt{m} + \sqrt{2}, -\sqrt{m} - \sqrt{2}\right\}$$

If one of the nonconstant polynomial factors is of form $x - r_1$, then the only possibility is $m = 2$ and $r_1 = 0$. This possibility is realized, since

$$P_2(x) = x^4 - 8x^2 = x^2\left(x^2 - 8\right).$$

If one of the nonconstant polynomial factors is of form $(x - r_1)(x - r_2)$, then note that

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2$$

in which case $r_1 + r_2$ and $r_1 r_2$ must be integers. The possibilities for $r_1 + r_2$ are 0, $\pm 2\sqrt{m}$, and $\pm 2\sqrt{2}$ (not an integer). We have

$$\begin{aligned} r_1 + r_2 &= 0 \Leftrightarrow \{r_1, r_2\} = \begin{cases} \{\sqrt{m} + \sqrt{2}, -\sqrt{m} - \sqrt{2}\} \\ \{\sqrt{m} - \sqrt{2}, -\sqrt{m} + \sqrt{2}\} \end{cases} \\ &\Rightarrow r_1 r_2 = \begin{cases} -\left(m + 2\sqrt{2}\sqrt{m} + 2\right) \\ -\left(m - 2\sqrt{2}\sqrt{m} + 2\right) \end{cases}, \end{aligned}$$

which is an integer only if m is $2n^2$ for some integer $n \geq 0$. We then have

$$\begin{aligned} \{r_1, r_2, r_3, r_4\} &= \left\{\sqrt{m} + \sqrt{2}, \sqrt{m} - \sqrt{2}, -\sqrt{m} + \sqrt{2}, -\sqrt{m} - \sqrt{2}\right\} \\ &= \left\{\sqrt{2}n + \sqrt{2}, \sqrt{2}n - \sqrt{2}, -\sqrt{2}n + \sqrt{2}, -\sqrt{2}n - \sqrt{2}\right\} \end{aligned}$$

and

$$\begin{aligned} &\left(x - \left(\sqrt{2}n + \sqrt{2}\right)\right)\left(x - \left(-\sqrt{2}n - \sqrt{2}\right)\right) \cdot \\ &\left(x - \left(\sqrt{2}n - \sqrt{2}\right)\right)\left(x - \left(-\sqrt{2}n + \sqrt{2}\right)\right) \\ = &\left(x^2 - 2n^2 - 4n - 2\right)\left(x^2 - 2n^2 + 4n - 2\right) = P_{2n^2}(x). \end{aligned}$$

In the remaining case,

$$r_1 + r_2 = \pm 2\sqrt{m} \Leftrightarrow \{r_1, r_2\} = \left\{\pm\sqrt{m} + \sqrt{2}, \pm\sqrt{m} - \sqrt{2}\right\} \Rightarrow r_1 r_2 = m - 2$$

and m must either be 2 (as before) or the square of an integer $n \geq 0$. If $m = n^2$, then

$$\{r_1, r_2, r_3, r_4\} = \left\{n + \sqrt{2}, n - \sqrt{2}, -n + \sqrt{2}, -n - \sqrt{2}\right\},$$

and indeed

$$\begin{aligned} & \left(x - \left(n + \sqrt{2}\right)\right)\left(x - \left(n - \sqrt{2}\right)\right)\left(x - \left(-n + \sqrt{2}\right)\right)\left(x - \left(-n - \sqrt{2}\right)\right) \\ = & \left(x^2 - 2xn + n^2 - 2\right)\left(x^2 + 2xn + n^2 - 2\right) \\ = & x^4 - (2n^2 + 4)x^2 + (n^2 - 2)^2 = P_{n^2}(x). \end{aligned}$$

Thus, m must be of the form $2n^2$ or n^2 for an integer $n \geq 0$.

A4. Triangle ABC has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.

Solution. Choose r, s, t, so that

$$\frac{CE}{CB} = r, \ \frac{AF}{AC} = s, \ \frac{BG}{BA} = t$$

Let $\mu(UVW)$ denote the area of a triangle with vertices U, V, W. Then

$$\mu(AEB) = \mu(AEF),$$

since the triangles have the same base AE and since AE bisects FB the same altitude as well. Also

$$\mu(ABE) = \frac{BE}{BC}\mu(ABC) = \frac{BC - EC}{BC} \cdot 1 = 1 - r.$$

Similarly,

$$\begin{aligned} \mu(ACE) &= \frac{CE}{CB}\mu(ACB) \\ \mu(CEF) &= \frac{CF}{CA}\mu(CEA). \end{aligned}$$

Thus,

$$\begin{aligned} \mu(CEF) &= \frac{CF}{CA}\mu(CEA) = \frac{CF}{CA}\mu(ACE) = \frac{CF}{CA}\frac{CE}{CB}\mu(ACB) \\ &= (1 - s)\, r\mu(ACB) = (1 - s)\, r. \end{aligned}$$

Hence,

$$\begin{aligned} 1 &= \mu(ABC) = \mu(AEB) + \mu(AEF) + \mu(CEF) \\ &= (1 - r) + (1 - r) + (1 - s)\, r \\ &= 2 - r - rs \Rightarrow r(1 + s) = 1 \end{aligned}$$

Similarly, $s\left(1+t\right)=1$ and $t\left(1+r\right)=1$. Thus,

$$r\left(1+s\right)=1,\ s\left(1+t\right)=1,\ t\left(1+r\right)=1$$

Now

$$\begin{aligned} r &= \frac{1}{1+s}=\frac{1}{1+\frac{1}{1+t}}=\frac{1+t}{2+t} \text{ and } t\left(1+r\right)=1 \\ &\Rightarrow t\left(1+\frac{1+t}{2+t}\right)=1\Rightarrow t\left(2+t+1+t\right)=2+t \\ &\Rightarrow t^2+t-1=0 \\ &\Rightarrow t=\tfrac{1}{2}\left(\sqrt{5}-1\right) \text{ or } -\tfrac{1}{2}\left(\sqrt{5}+1\right) \text{ (negative)} \end{aligned}$$

By symmetry,

$$r=s=t=\tfrac{1}{2}\left(\sqrt{5}-1\right).$$

Note that

$$\mu\left(RST\right)=\mu\left(ABC\right)-\left(\mu\left(ABR\right)+\mu\left(BCS\right)+\mu\left(CAT\right)\right).$$

Also,

$$\begin{aligned} \mu\left(ABR\right) &= \frac{BR}{BF}\mu\left(ABF\right)=\tfrac{1}{2}\mu\left(ABF\right)=\tfrac{1}{2}\frac{AF}{AC}\mu\left(ABC\right) \\ &= \tfrac{1}{2}s\mu\left(ABC\right)=\tfrac{1}{2}t, \end{aligned}$$

and similarly

$$\mu\left(BCS\right)=\mu\left(CAT\right)=\tfrac{1}{2}t.$$

Thus,

$$\mu\left(RST\right)=1-\left(\tfrac{1}{2}t+\tfrac{1}{2}t+\tfrac{1}{2}t\right)=1-\tfrac{3}{2}\cdot\tfrac{1}{2}\left(\sqrt{5}-1\right)=\frac{7-3\sqrt{5}}{4}\approx .0729\ldots.$$

A5. Prove that there are unique positive integers a, n such that $a^{n+1}-(a+1)^n=2001$.

Solution. Suppose $a^{n+1}-(a+1)^n=2001$. Notice that

$$2002=a^{n+1}-(a+1)^n+1\equiv 0 \bmod a$$

Thus a is a factor of $2002=2\times 7\times 11\times 13$. Note that $a\neq 2$, since then

$$2^{n+1}-3^n=2001,$$

which is impossible since 2001 is divisible by 3 unlike 2^{n+1}. More generally since

$$a^{n+1} - (a+1)^n = 2001 \equiv 0 \bmod 3$$

$a \equiv 2 \bmod 3$ is impossible, as is $a \equiv 0 \bmod 3$. Thus, $a \equiv 1 \bmod 3$, and

$$(-1)^n \equiv (a+1)^n \equiv a^{n+1} \equiv 1 \bmod 3 \Rightarrow n \text{ even.}$$

Recall that $a^{n+1} + 1$ has a factor of $a + 1$ when n is even. Thus, $a + 1$ is a factor of

$$a^{n+1} + 1 - (a+1)^n = 2002 = 2 \times 7 \times 11 \times 13$$

as is a itself. Only two factors differ by 1, namely 13 and 14. Thus, $a = 13$ and the smallest value for n is 2. We have

$$13^{2+1} - 14^2 = 2001.$$

It remains to show that no even $n > 2$ satisfies

$$13^{n+1} - 14^n = 2001$$

If $n > 2$, $14^n \equiv 0 \bmod 8$, and $2001 \equiv 1 \bmod 8$. Thus,

$$13^{n+1} \equiv 1 \bmod 8$$

However, $13 \equiv 5 \bmod 8$ and $13^2 \equiv 5^2 \equiv 1 \bmod 8$. Hence, $13^{n+1} \equiv 5 \bmod 8$ if n is even Thus, $n > 2$ and n even is not possible.

A6. Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

Solution. The answer is yes. Consider the arc of the parabola $y = \frac{1}{2}cx^2$ inside the circle $x^2 + (y-1)^2 = 1$, where $c > 1$.

$$\begin{aligned} x^2 + (\tfrac{1}{2}cx^2 - 1)^2 &= 1 \Rightarrow x^2 + \tfrac{1}{4}c^2x^4 - cx^2 = 0 \Rightarrow x^2\left(\tfrac{1}{4}c^2x^2 + 1 - c\right) = 0 \\ &\Rightarrow x = 0 \text{ or } x = \pm\frac{2\sqrt{c-1}}{c} \end{aligned}$$

The length of the parabolic arc from $x = 0$ to $x = \frac{2\sqrt{c-1}}{c}$ is then

$$\begin{aligned} &\int_0^{\frac{2\sqrt{c-1}}{c}} \sqrt{1 + c^2x^2}\, dx = (\text{using } u = cx) \\ = \;& \frac{1}{c}\int_0^{2\sqrt{c-1}} \sqrt{1+u^2}\, du = (\text{using the substitution } z = \sinh u) \\ = \;& \frac{1}{c}\left(\tfrac{1}{2}\operatorname{arcsinh} u + \tfrac{1}{2}u\sqrt{1+u^2}\right)\Big|_0^{2\sqrt{c-1}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2c}\left(\operatorname{arcsinh}\left(2\sqrt{c-1}\right) + 2\sqrt{c-1}\sqrt{1+\left(2\sqrt{c-1}\right)^2}\right) \\
&= \frac{1}{2c}\left(\operatorname{arcsinh}\left(2\sqrt{c-1}\right) + 2\sqrt{c-1}\sqrt{1+4\left(c-1\right)}\right) \\
&= \frac{1}{2c}\operatorname{arcsinh}\left(2\sqrt{c-1}\right) + 2\sqrt{1-\frac{7}{4c}+\frac{3}{4c^2}}
\end{aligned}
$$

We have

$$2\sqrt{1-\frac{7}{4c}+\frac{3}{4c^2}} > 2\sqrt{1-\frac{7}{4c}} \approx 2\left(1-\tfrac{1}{2}\left(\frac{7}{4c}\right)\right) > 2-\frac{2}{c},$$

for c sufficiently large. Thus, it suffices to prove that

$$\frac{1}{2c}\operatorname{arcsinh}\left(2\sqrt{c-1}\right) > \frac{2}{c}$$

for c sufficiently large, but this is so since $\operatorname{arcsinh} x$ increases without limit as $x \to \infty$. Indeed,

$$\lim_{x\to\infty}\frac{\operatorname{arcsinh} x}{\log x} = \lim_{x\to\infty}\frac{\frac{d}{dx}\operatorname{arcsinh}(x)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x}{\sqrt{1+x^2}} = 1.$$

Note: Mathematica yields a maximal length of $4.0025901\ldots$ for $c = 199.7944360\ldots$.

B1. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n+1, (k-1)n+2, \ldots, (k-1)n+n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

Solution. Instead of writing the numbers $1, 2, \ldots, n^2$ in the squares, write $0, 1, 2, \ldots, n^2 - 1$. Since half the squares are red and half are black, the assertion to be proven holds iff it holds with this new numbering. Now rewrite each number in base n in the form d_2d_1, where we take $d_2 = 0$ in the first row. Note that d_2 is the same in each row, while d_1 is the same in each column. Since half of the squares in each row and in each column are red and the other half are black, the sum of the d_2's for the red squares is the same as the sum of the d_2's for the black squares in any row, and hence the sum of all red d_2's and all black d_2's is the same. Considering columns, the same equality holds for the sums of d_1's for red and d_1's for black squares. Thus, adding base n yields the same sum for red and black squares.

B2. Find all pairs of real numbers (x, y) satisfying the system of equations

$$
\begin{aligned}
\frac{1}{x} + \frac{1}{2y} &= (x^2+3y^2)(3x^2+y^2) \\
\frac{1}{x} - \frac{1}{2y} &= 2(y^4 - x^4).
\end{aligned}
$$

Solution. By adding and subtracting the two given equations, we obtain the equivalent set of equations

$$2/x = x^4 + 10x^2y^2 + 5y^4$$
$$1/y = 5x^4 + 10x^2y^2 + y^4.$$

Multiplying the first by x and the second by y, we get

$$\begin{aligned} 2 &= x^5 + 10x^3y^2 + 5xy^4 \\ 1 &= 5x^4y + 10x^2y^3 + y^5 \end{aligned}$$

Adding these yields

$$3 = x^5 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 = (x+y)^5.$$

Subtracting yields

$$1 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5 = (x-y)^5$$

Thus,

$$\begin{aligned} x + y &= 3^{1/5} \\ x - y &= 1, \end{aligned}$$

and so $x = (3^{1/5} + 1)/2$ and $y = (3^{1/5} - 1)/2$.

B3. For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

Solution. Since $(m - 1/2)^2 = m^2 - m + 1/4$ and $(m + 1/2)^2 = m^2 + m + 1/4$, we have that $\langle n \rangle = m$ if and only if

$$m^2 - m + 1/4 \le n \le m^2 + m + 1/4$$

However, since n is an integer, this is equivalent to

$$m^2 - m + 1 \le n \le m^2 + m$$

Hence

$$\sum_{n=1}^{\infty}\frac{2^{\langle n\rangle}+2^{-\langle n\rangle}}{2^n}=\sum_{m=1}^{\infty}\sum_{\langle n\rangle=m}\frac{2^{\langle n\rangle}+2^{-\langle n\rangle}}{2^n}=\sum_{m=1}^{\infty}\sum_{n=m^2-m+1}^{m^2+m}\frac{2^m+2^{-m}}{2^n}$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty}(2^m+2^{-m})\sum_{n=m^2-m+1}^{m^2+m}2^{-n}=\sum_{m=1}^{\infty}(2^m+2^{-m})2^{-\left(m^2-m+1\right)}\sum_{n=0}^{2m-1}2^{-n}\\
&= \sum_{m=1}^{\infty}(2^m+2^{-m})2^{-\left(m^2-m+1\right)}\frac{\left(1-2^{-2m}\right)}{\frac{1}{2}}=\sum_{m=1}^{\infty}(2^m+2^{-m})2^{-\left(m^2-m\right)}\left(1-2^{-2m}\right)\\
&= \sum_{m=1}^{\infty}(2^m+2^{-m})\left(2^{-m^2+m}-2^{-m^2-m}\right)=\sum_{m=1}^{\infty}\left(2^{-m^2+2m}-2^{-m^2-2m}\right)\\
&= \sum_{m=1}^{\infty}\left(2^{-m(m-2)}-2^{-(m+2)m}\right)=\sum_{m=1}^{\infty}2^{-m(m-2)}-\sum_{m=3}^{\infty}2^{-m(m-2)}\\
&= \sum_{m=1}^{2}2^{-m(m-2)}=2+1=3.
\end{aligned}$$

B4. Let S denote the set of rational numbers different from $\{-1,0,1\}$. Define $f:S\to S$ by $f(x)=x-1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty}f^{(n)}(S)=\emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

Solution. Suppose that $\frac{n}{m}$ is reduced to lowest terms; i.e., n and m have no common prime factors. Then

$$f(\frac{n}{m})=\frac{n}{m}-\frac{m}{n}=\frac{n^2-m^2}{mn}=\frac{(n+m)\,(n-m)}{mn}$$

We claim that $\frac{n^2-m^2}{mn}$ is reduced to lowest terms: If a prime p divides mn then it divides m or it divides n. If p divides m, then it cannot divide n, and hence it cannot divide $n\pm m$. If p divides n, then it cannot divide m, and hence it cannot divide $n\pm m$. In either case, p cannot divide $(n+m)\,(n-m)$, and the claim holds.

Define the degree of $\frac{n}{m}$ (reduced to lowest terms) by $\deg\left(\frac{n}{m}\right)=|m|+|n|$. Thus,

$$\deg\left(f(\frac{n}{m})\right)=\left|n^2-m^2\right|+|mn|\geq 3+|mn|,$$

since $|n^2-m^2|$ is smallest (i.e., 3) for $\{n,m\}=\{1,2\}$. Hence,

$$\begin{aligned}
&\deg\left(f(\frac{n}{m})\right)-\deg\left(\frac{n}{m})\right)=3+|mn|-(|m|+|n|)\\
=&\ 2+1+|mn|-(|m|+|n|)=2+(|m|-1)\,(|n|-1)\geq 2
\end{aligned}$$

Hence the degree of any fraction in $f^{(n)}(S)$ is at least $2n$. Since any rational number has finite degree, $\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset$.

B5. Let a and b be real numbers in the interval $(0, 1/2)$, and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

Solution. Pick x_0 arbitrary, and define x_n recursively by $x_{n+1} = g(x_n)$ for $n \geq 0$.

$$x_{n+2} = g(x_{n+1}) = g(g(x_n)) = ag(x_n) + bx_n = ax_{n+1} + bx_n \tag{1}$$

We can find solutions of this relation of the form $x_n = r^n$. To determine r, note that

$$\begin{aligned} r^{n+2} &= ar^{n+1} + br^n \Leftrightarrow 0 = r^{n+2} - ar^{n+1} - br^n = r^n \left(r^2 - ar - b\right) \\ &\Leftrightarrow \ r = 0 \text{ or } r = r_{\pm} := \tfrac{1}{2}\left(a \pm \sqrt{a^2 + 4b}\right). \end{aligned}$$

The general solution of (1) is then

$$x_n = c_+ r_+^n + c_- r_-^n.$$

If we can show that $c_- = 0$ (resp. $c_+ = 0$) for all choices of x_0, then $c_+ = x_0$ (resp. $c_- = 0$) and

$$g\left(x_0\right) = x_0 r_+ \text{ (resp. } g\left(x_0\right) = x_0 r_-)$$

and we are done. Note that

$$-\left|r_+\right| < r_- < 0 < r_+ < \tfrac{1}{2}\left(\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \tfrac{1}{2}}\right) = \frac{1 + \sqrt{3}}{4},$$

so that $c_+ r_+^n$ dominates for large positive n, while $c_- r_-^n$ dominates for large negative n. However, x_n is not defined for negative n unless g^{-1} exists. We prove that $g : \mathbb{R} \to \mathbb{R}$ is 1-1 and onto as follows. Note that

$$\begin{aligned} g\left(u\right) &= g\left(v\right) \\ &\Rightarrow \ ag(u) + bu = g(g(u)) = g(g(v)) = ag(v) + bv = ag(u) + bv \\ &\Rightarrow \ bu = bv \Rightarrow u = v. \end{aligned}$$

Thus, g is 1-1, and hence must be strictly increasing or strictly decreasing.

If g is strictly increasing and $\lim_{x \to +\infty} g\left(x\right) = L_+ < +\infty$, then we have the contradiction

$$g(L_+) = \lim_{x \to +\infty} g(g(x)) = \lim_{x \to +\infty} (ag(x) + bx) = \infty.$$

Thus, g is strictly increasing $\Rightarrow \lim_{x \to +\infty} g\left(x\right) = +\infty$. If $\lim_{x \to -\infty} g\left(x\right) = L_- > -\infty$, then we have the contradiction

$$g(L_-) = \lim_{x \to -\infty} g(g(x)) = \lim_{x \to -\infty} (ag(x) + bx) = -\infty.$$

Thus, under the assumptions, g strictly increasing implies that $g:\mathbb{R}\to\mathbb{R}$ is 1-1 and onto. If g is strictly decreasing and $\lim_{x\to+\infty} g(x) = L_+ > -\infty$, then we have the contradiction

$$g(L_+) = \lim_{x\to+\infty} g(g(x)) = \lim_{x\to+\infty} (ag(x)+bx) = +\infty.$$

If $\lim_{x\to-\infty} g(x) = L_- < +\infty$, then we have the contradiction

$$g(L_-) = \lim_{x\to-\infty} g(g(x)) = \lim_{x\to-\infty} (ag(x)+bx) = -\infty.$$

Hence in either case, $g:\mathbb{R}\to\mathbb{R}$ is 1-1 and onto (i.e., as x goes from $-\infty$ to $+\infty$, $g(x)$ strictly increases from $-\infty$ to $+\infty$, or $g(x)$ strictly decreases from $+\infty$ to $-\infty$). At any rate, we may define $x_{n-1} = g^{-1}(x_n)$ for $n \le 0$.

Suppose that $c_+ \neq 0$ and $c_- \neq 0$. We will arrive at a contradiction. Note that

$$x_n = c_+ r_+^n + c_- r_-^n = c_- r_-^n \left(1 + \frac{c_+}{c_-}\left(\frac{r_+}{r_-}\right)^n\right).$$

Since $\left|\frac{r_+}{r_-}\right| > 1$, $\left|\frac{c_+}{c_-}\left(\frac{r_+}{r_-}\right)^n\right| < \frac{1}{2}$ for n sufficiently large and negative, say $n < N < 0$. Then for $n < N < 0$, $\mathrm{sign}(x_n) = c_-(-1)^n$ alternates and

$$|x_n| > \left|\tfrac{1}{2}c_- r_-^n\right| \to \infty \text{ as } n \to -\infty.$$

Since $x_n = g(x_{n-1})$, there are arbitrarily large $x > 0$ for which $g(x) < 0$, and so g must be decreasing. We also have

$$x_n = c_+ r_+^n + c_- r_-^n = c_+ r_+^n \left(1 + \frac{c_-}{c_+}\left(\frac{r_-}{r_+}\right)^n\right).$$

Since $\left|\frac{r_-}{r_+}\right| < 1$, $\left|\frac{c_-}{c_+}\left(\frac{r_-}{r_+}\right)^n\right| < \frac{1}{2}$ for n sufficiently large, say $n > N > 0$. Thus, if $c_+ > 0$, then for $n > N > 0$,

$$g(x_{n-1}) = x_n \ge \tfrac{1}{2}c_+ r_+^n$$

and so there are arbitrarily large values of x for which $g(x) > 0$. This cannot happen if g is decreasing, as we have just shown is the case. If $c_+ < 0$, then

$$g(x_{n-1}) = x_n \le \tfrac{1}{2}c_+ r_+^n$$

and there are arbitrarily large negative values of x for which $g(x)$ is negative. This is also impossible for g decreasing. Hence, for each x_0, we have $g(x_0) = r_+ x_0$ or $g(x_0) = r_- x_0$. Since g is continuous, both of the sets $\{x_0 \in (0,\infty) : g(x_0) - r_+ x_0 = 0\}$ and $\{x_0 \in (0,\infty) : g(x_0) - r_- x_0 = 0\}$ are closed in $(0,\infty)$. Since $(0,\infty)$ is connected either $g(x_0) = r_+ x_0$ for all $x_0 > 0$, or $g(x_0) = r_- x_0$ for all $x_0 > 0$. Similarly, this holds for $x_0 < 0$. We must eliminate the possibility $g(x) = r_\pm x$ for $x > 0$ and $g(x_0) = r_\mp x$ for $x < 0$, but we know this since g is increasing or decreasing.

B6. Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i}+a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

Solution. Here, "increasing" must mean strictly increasing, since otherwise a constant sequence provides an immediate counter-example. Actually, non-decreasing and "not eventually constant" is enough. The condition $a_{n-i} + a_{n+i} < 2a_n$ means that the line connecting $(n-i, a_{n-i})$ and $(n+i, a_{n+i})$ in $\mathbb{R}^2$ has midpoint $\left(n, \frac{1}{2}\left(a_{n-i} + a_{n+i}\right)\right)$ which lies below (n, a_n). Equivalently, this means that there is a line L_n through (n, a_n) such that $(n-i, a_{n-i})$ and $(n+i, a_{n+i})$ are below L_n. Thus, it suffices to show that there are infinitely many n such that there is a line L_n through (n, a_n) such that all of the points $(n-i, a_{n-i})$ and $(n+i, a_{n+i})$ for $i = 1, 2, \ldots, n-1$ lie below L_n.

The sequence $\left\{\frac{a_i - a_1}{i-1}\right\}$ is (eventually) positive for $i \geq 2$ and has limit 0. Thus, it has a positive maximum, say m_1 which is attained for some *largest* $i_1 > 1$. Thus, the line L_{i_1} through (i_1, a_{i_1}) with slope $m_1 := \frac{a_{i_1} - a_1}{i_1 - 1} > 0$ has the property that all points (i, a_i) with $i < i_1$ are on or below this line, while all points (i, a_i) for $i > i_1$ are strictly below (since i_1 is *largest*). The ratios $\frac{a_i - a_{i_1}}{i - i_1}$ for $i > i_1$ are then all less than m_1 and $\lim_{i\to\infty} \frac{a_i - a_{i_1}}{i - i_1} = 0$. Hence $m_1 - \frac{a_i - a_{i_1}}{i - i_1}$ achieves a positive maximum, say 2ε. The line, say L_{i_1}, with slope $m_1 - \varepsilon$ through (i_1, a_{i_1}) then lies above (i, a_i) for all $i > i_1$. The points (i, a_i) for $i < i_1$ also lie below this line since they are on or below the line with slope m_1 through (i_1, a_{i_1}). We now repeat the same argument, starting with i_1 instead of 1 to produce a largest i_2, such that the maximum of $\frac{a_i - a_{i_1}}{i - i_1}$ for $i > i_1$ is achieved for $i = i_2$. This maximum, say m_{i_1}, is less than $m_1 - \varepsilon$ and so the line with slope m_{i_1} through (i_2, a_{i_2}) will have the property that all points (i, a_i) will lie on or below this line. Again, there is some $\varepsilon > 0$, such that the line with slope $m_{i_1} - \varepsilon$ through (i_2, a_{i_2}) will lie above all other points (i, a_i), $i \neq i_2$. In this way, we produce infinitely many points (i_k, a_{i_k}) such there is a line through (i_k, a_{i_k}) which lies above all points (i, a_i), $i \neq i_k$.