Putnam 2003

A1. Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \ldots + a_k;$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_1 + 1$? For example, with $n = 4$, there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.

Solution. We use induction to prove that the number $N(n)$ of ways is n. For $n = 1$ this is clear. By gathering the equal terms (either a_1 or $a_1 + 1$), each equation $n = a_1 + a_2 + \ldots + a_k$ for given n and k can be uniquely rewritten in the form

$$E(n, k, a, r) : n = ra + (k - r)(a + 1)$$

for some $r \in \{1, \ldots, k\}$ which depends uniquely on $k, a,$and n (indeed, $r = k(a+1) - n$). Let

$$\mathcal{E}_n := \left\{ E(n, k, a, r) : E(n, k, a, r) \text{ holds for some } (k, a, r) \in \mathbb{Z}_+^3 \right\}.$$

Note that

$$\begin{aligned} n &= ra + (k - r)(a + 1) \Rightarrow \\ n + 1 &= (r - 1)a + (k - (r - 1))(a + 1). \end{aligned}$$

Thus, we have a map

$$F : \mathcal{E}_n \to \mathcal{E}_{n+1},$$

defined by

$$F(E(n, k, a, r)) := \begin{cases} E(n + 1, k, a + 1, k) & r = 1 \\ E(n + 1, k, a, r - 1) & 2 \leq r \leq k \leq n. \end{cases}$$

Note that F is clearly $1-1$. It is not onto, since it misses the value

$$E(n + 1, n + 1, 1, n + 1),$$

but this is the only value it misses, since

$$E(n + 1, k, a, s) = \begin{cases} F(E(n, k, a, s + 1)) & \text{if } s \leq k - 1 \\ F(E(n, k, a - 1, k)) & \text{if } s = k \leq n. \end{cases}$$

Hence, $N(n) = n \Rightarrow N(n + 1) = n + 1$.

A2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{\frac{1}{n}} + (b_1 b_2 \cdots b_n)^{\frac{1}{n}} \leq ((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n))^{\frac{1}{n}}.$$

Solution. Using the Lagrange multiplier method, for $y_i \geq 0$ $(i = 1, \ldots n)$, the maximum of the function $y_1 y_2 \cdots y_n$ with the constraint $y_1 + y_2 + \cdots + y_n = 1$, is achieved when $y_1 = y_2 = \cdots = y_n = \frac{1}{n}$. Thus,

$$(y_1 y_2 \cdots y_n)^{\frac{1}{n}} \leq \frac{1}{n} \text{ if } y_1 + y_2 + \cdots + y_n = 1 \text{ and } y_i \geq 0 \ (i = 1, \ldots n).$$

For $x_i \geq 0$ $(i = 1, \ldots n)$ and $s = x_1 + x_2 + \cdots + x_n$, taking $y_i = x_i / s$, we obtain

$$\begin{aligned} \left(\frac{x_1}{s}\frac{x_2}{s}\cdots\frac{x_n}{s}\right)^{\frac{1}{n}} &\leq \frac{1}{n} \text{ or} \\ (x_1 x_2 \cdots x_n)^{\frac{1}{n}} &\leq \frac{1}{n}(x_1 + x_2 + \cdots + x_n), \end{aligned} \tag{1}$$

which says that the geometric mean is not greater than the arithmetic mean. We may assume that $a_i + b_i > 0$ for all i, since the result is clearly true if $a_i + b_i = 0$ for some i. Apply (1) for $x_i = \frac{a_i}{a_i + b_i}$ to get

$$\left(\frac{a_1}{a_1 + b_1} \cdot \frac{a_2}{a_2 + b_2} \cdots \frac{a_n}{a_n + b_n}\right)^{\frac{1}{n}} \leq \frac{1}{n}\left(\frac{a_1}{a_1 + b_1} + \frac{a_2}{a_2 + b_2} + \cdots + \frac{a_n}{a_n + b_n}\right).$$

Now apply it for $x_i = \frac{b_i}{a_i + b_i}$ to get

$$\left(\frac{b_1}{a_1 + b_1} \cdot \frac{b_2}{a_2 + b_2} \cdots \frac{b_n}{a_n + b_n}\right)^{\frac{1}{n}} \leq \frac{1}{n}\left(\frac{b_1}{a_1 + b_1} + \frac{b_2}{a_2 + b_2} + \cdots + \frac{b_n}{a_n + b_n}\right).$$

Adding the above, we get

$$\left(\frac{a_1}{a_1 + b_1} \cdot \frac{a_2}{a_2 + b_2} \cdots \frac{a_n}{a_n + b_n}\right)^{\frac{1}{n}} + \left(\frac{b_1}{a_1 + b_1} \cdot \frac{b_2}{a_2 + b_2} \cdots \frac{b_n}{a_n + b_n}\right)^{\frac{1}{n}} \leq \frac{1}{n} \cdot n = 1$$

Now multiply by $\left((a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)\right)^{\frac{1}{n}}$.

A3. Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

Solution.

$$\begin{aligned} \tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} = \frac{1}{\cos x \sin x} \\ \sec x + \csc x &= \frac{1}{\cos x} + \frac{1}{\sin x} = \frac{\sin x + \cos x}{\cos x \sin x}. \end{aligned}$$

Let $u = \sin x + \cos x$ and note that $-\sqrt{2} \leq u \leq \sqrt{2}$. Then

$$\begin{aligned} u^2 &= \sin^2 x + 2\cos x \sin x + \cos^2 x = 1 + 2\cos x \sin x \\ &\Rightarrow \cos x \sin x = \frac{u^2 - 1}{2}. \end{aligned}$$

Thus,

$$\sin x + \cos x + \tan x + \cot x + \sec x + \csc x = u + \frac{2}{u^2 - 1} + \frac{2u}{u^2 - 1}$$

$$= u + \frac{2(u+1)}{u^2 - 1} = u + \frac{2(u+1)}{(u+1)(u-1)} = u + \frac{2}{u-1} := f(u).$$

We have

$$f'(u) = 1 - \frac{2}{(u-1)^2} = 0 \text{ for } u = 1 \pm \sqrt{2}$$

Now,

$$f\left(1 \pm \sqrt{2}\right) = 1 \pm \sqrt{2} + \frac{2}{1 \pm \sqrt{2} - 1} = 1 \pm 2\sqrt{2}.$$

We check the endpoints:

$$\begin{aligned} f\left(\sqrt{2}\right) &= \sqrt{2} + \frac{2}{\sqrt{2} - 1} = \sqrt{2} + \frac{2\left(\sqrt{2} + 1\right)}{\left(\sqrt{2} - 1\right)\left(\sqrt{2} + 1\right)} \\ &= \sqrt{2} + 2\left(\sqrt{2} + 1\right) = 2 + 3\sqrt{2} \text{ and} \\ f\left(-\sqrt{2}\right) &= -\sqrt{2} + \frac{2}{-\sqrt{2} - 1} = -\sqrt{2} - \frac{2\left(\sqrt{2} - 1\right)}{\left(\sqrt{2} + 1\right)\left(\sqrt{2} - 1\right)} \\ &= -\sqrt{2} - 2\left(\sqrt{2} - 1\right) = 2 - 3\sqrt{2}. \end{aligned}$$

We have

$$\min\left\{\left|1 - 2\sqrt{2}\right|, \left|1 + 2\sqrt{2}\right|, \left|2 - 3\sqrt{2}\right|, \left|2 + 3\sqrt{2}\right|\right\}$$

$$= \min\left\{2\sqrt{2} - 1, 3\sqrt{2} - 2 = \left(2\sqrt{2} - 1\right) + \sqrt{2} - 1\right\} = 2\sqrt{2} - 1 \approx 1.828.$$

Thus, the answer is $2\sqrt{2} - 1$.

A4. Suppose that a, b, c, A, B, C are real numbers, $a \neq 0$ and $A \neq 0$, such that

$$\left|ax^2 + bx + c\right| \leq \left|Ax^2 + Bx + C\right|$$

for all real numbers x. Show that

$$\left|b^2 - 4ac\right| \leq \left|B^2 - 4AC\right|.$$

Solution. By replacing a, b, c by $-a, -b, -c$, we may assume that $a > 0$, and similarly we may assume that $A > 0$. Note that none of the above absolute values are affected by such replacements. For large x

$$\begin{aligned} \left|ax^2 + bx + c\right| &\leq \left|Ax^2 + Bx + C\right| \Rightarrow \left|a + bx^{-1} + cx^{-2}\right| \leq \left|A + Bx^{-1} + Cx^{-2}\right| \\ &\Rightarrow a \leq A, \text{ taking the limit as } x \to \infty. \end{aligned}$$

Let $D := B^2 - 4AC$ and $d := b^2 - 4ac$.

There are three cases

$$(i)\ D \geq 0, \quad (ii)\ D < 0 \text{ and } d > 0, \quad (iii)\ D < 0 \text{ and } d < 0$$

Case (i) $D \geq 0$: If $D \geq 0$, then $Ax^2 + Bx + C$ has real zeros $r_2 \geq r_1$, and by the quadratic formula, $r_2 - r_1 = \frac{\sqrt{D}}{A}$. Note that r_1 and r_2 are also zeros of $ax^2 + bx + c$, since $|ax^2 + bx + c| \leq |Ax^2 + Bx + C|$. Thus,

$$\frac{\sqrt{d}}{a} = r_2 - r_1 = \frac{\sqrt{D}}{A} \Rightarrow \sqrt{d} = \frac{a\sqrt{D}}{A} \leq \sqrt{D} \Rightarrow d \leq D.$$

Case (ii) $D < 0$ and $d > 0$. If $D < 0$, then $0 < Ax^2 + Bx + C$ and

$$\left|ax^2 + bx + c\right| \leq \left|Ax^2 + Bx + C\right| = Ax^2 + Bx + C$$

Thus,$\pm\left(ax^2 + bx + c\right) \leq Ax^2 + Bx + C$ and so

$$(A \pm a)\, x^2 + (B \pm b)\, x + (C \pm c) \geq 0$$

Hence

$$\begin{aligned} (B \pm b)^2 - 4\,(A \pm a)\,(C \pm c) &\leq 0, \\ \left(B^2 - 4AC\right) + \left(b^2 - 4ac\right) &\leq \pm\left(-2Bb + 4Ac + 4aC\right). \end{aligned}$$

Since one of $\pm\left(-2Bb + 4Ac + 4aC\right)$ is ≤ 0, we have

$$D + d \leq 0 \text{ and so } |d| = d \leq |D|.$$

Case (iii) $D < 0$ and $d < 0$: In this case, $0 \leq ax^2 + bx + c \leq Ax^2 + Bx + C$ and consequently

$$\min\left\{ax^2 + bx + c\right\} \leq \min\left\{Ax^2 + Bx + C\right\}$$

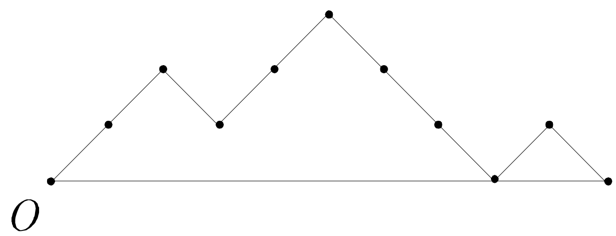
Now,

$$\min\left\{ax^2 + bx + c\right\} = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c = \frac{4ac - b^2}{4a} = \frac{-d}{4a}.$$

Thus,

$$\frac{|d|}{4a} = \frac{-d}{4a} \le \frac{-D}{4A} = \frac{|D|}{4A} \text{ and } |d| \le \frac{a}{A}|D| \le |D|.$$

A5. A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.

Solution. Let us denote a Dyck n-path by its sequence of steps $v_1, v_2, \ldots, v_{2n}$, where $v_i = (1,1)$ or $(1,-1)$. Denote the set of Dyck n-paths by D_n. We define a function

$$F : D_{n-1} \to D_n(\sim e) := D_n \setminus \{\text{paths in } D_n \text{ with a return of even length}\}$$

If $v_1, v_2, \ldots, v_{2(n-1)} \in D_{n-1}(\sim e)$, then we set

$$F\left(v_1, v_2, \ldots, v_{2(n-1)}\right) = (1,1), (1,-1), v_1, v_2, \ldots, v_{2(n-1)} \in D_n(\sim e).$$

If $v_1, v_2, \ldots, v_{2(n-1)} \in D_{n-1} \setminus D_{n-1}(\sim e)$ (i.e., a Dyck $(n-1)$-path with a return of even length), let the final return of even length end with the step v_f. Note that in the Dyck n-path $(1,1), v_1, v_2, \ldots, v_f, (1,-1), v_{f+1}, \ldots, v_{2(n-1)}$ the original subpath $v_1, v_2, \ldots, v_f$ has been translated up one unit and to the right one unit so that $(1,1), v_1, v_2, \ldots, v_f, (1,-1)$ has only one return, namely a final return of odd length, while (by choice of v_f) $v_{f+1}, \ldots, v_{2(n-1)}$ has no return of even length. Thus, for $v_1, v_2, \ldots, v_{2(n-1)} \in D_{n-1} \setminus D_{n-1}(\sim e)$, we set

$$F\left(v_1, v_2, \ldots, v_{2(n-1)}\right) := (1,1), v_1, v_2, \ldots, v_f, (1,-1), v_{f+1}, \ldots, v_{2(n-1)} \in D_n(\sim e).$$

To show that F is a bijection, we need to find an inverse, say G, for F. Let $v_1, v_2, \ldots, v_{2n} \in D_n(\sim e)$ and let $v_1, v_2, \ldots, v_{2i}$ be the first Dyck subpath of $v_1, v_2, \ldots, v_{2n}$. Then set

$$G(v_1, v_2, \ldots, v_{2n}) = v_2, \ldots, v_{2i-1}, v_{2i+1}, \ldots, v_{2n}.$$

Here, if $i = 1$, $v_2, \ldots, v_{2i-1}$ is empty, and $G(v_1, v_2, \ldots, v_{2n}) = v_3, \ldots, v_{2n}$. Note that since $v_1, v_2, \ldots, v_{2n} \in D_n(\sim e)$, the first return (if any) of $G(v_1, v_2, \ldots, v_{2n}) = v_2, \ldots, v_{2i-1}, v_{2i+1}, \ldots, v_{2n}$, ends in v_{2i-1}, has even length, and is the last return of even length, in which case $F(G(v_1, v_2, \ldots, v_{2n})) =$

$v_1, v_2, \ldots, v_{2n}$. If $v_2, \ldots, v_{2i-1}, v_{2i+1}, \ldots, v_{2n}$ has no return of even length, then it must have been that $i = 2$, in which case we also have $F(G(v_1, v_2, \ldots, v_{2n})) = (1,1), (1,-1), v_3, \ldots, v_{2n} = v_1, \ldots, v_{2n}$. For $G\left(F\left(v_1, v_2, \ldots, v_{2(n-1)}\right)\right)$, note that if $v_1, v_2, \ldots, v_{2(n-1)} \in D_{n-1}(\sim e)$, then

$$G\left(F\left(v_1, v_2, \ldots, v_{2(n-1)}\right)\right) = G\left((1,1), (1,-1), v_1, v_2, \ldots, v_{2(n-1)}\right) = v_1, v_2, \ldots, v_{2(n-1)}.$$

If $v_1, v_2, \ldots, v_{2(n-1)} \in D_{n-1} \setminus D_{n-1}(\sim e)$, then

$$\begin{aligned} G\left(F\left(v_1, v_2, \ldots, v_{2(n-1)}\right)\right) &= G\left((1,1), v_1, v_2, \ldots, v_f, (1,-1), v_{f+1}, \ldots, v_{2(n-1)}\right) \\ &= v_1, v_2, \ldots, v_{2(n-1)}, \end{aligned}$$

as required.

A6. For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

Solution. Let $0 \in A = \{0, \ldots\}$. Then

$$\begin{aligned} 1 &= 0 + 1 \Rightarrow 1 \in B = \{1, \ldots\}, \\ 2 &= 0 + 2 \Rightarrow 2 \in B = \{1, 2, \ldots\}, \\ 3 &= 0 + 3 = 1 + 2 \Rightarrow 3 \in A = \{0, 3, \ldots\}, \\ 4 &= 0 + 4 = 1 + 3 \Rightarrow 4 \in B = \{1, 2, 4, \ldots\}, \\ 5 &= 0 + 5 = 1 + 4 = 2 + 3 \Rightarrow 5 \in A = \{0, 3, 5, \ldots\}, \\ 6 &= 0 + 6 = 1 + 5 = 2 + 4 \Rightarrow 6 \in A = \{0, 3, 5, 6, \ldots\}, \\ 7 &= 0 + 7 = 1 + 6 = 2 + 5 = 3 + 4 \Rightarrow 7 \in B = \{1, 2, 4, 7, \ldots\}. \end{aligned}$$

Note that thus far A consists of the whole numbers ($\mathbb{Z}_+$) with an *even* number of ones in their base 2 representation, whereas B consists of the whole numbers with an *odd* number of ones in their base 2 representation. Let's prove the conjecture. Let $n \geq 0$ and suppose

$$n = a_1 + a_2 \text{ where } a_1, a_2 \in A,\ a_1 \neq a_2.$$

Now the binary reps of a_1 and a_2 differ in some first digit, say from the right. Change that digit in each of a_1 and a_2, to obtain b_1 and $b_2 \in B$. Note that $b_1 + b_2 = a_1 + a_2 = n$. Thus, we have a bijection

$$\begin{aligned} & \{(a_1, a_2) : a_1 + a_2 = n \text{ with } a_1, a_2 \in A,\ a_1 \neq a_2\} \\ \longleftrightarrow\ & \{(b_1, b_2) : b_1 + b_2 = n \text{ with } b_1, b_2 \in B,\ b_1 \neq b_2\}, \end{aligned}$$

showing that $r_A(n) = r_B(n)$ for all $n \in \mathbb{Z}_+$.

B1. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Solution. No. Choosing $y = 0, y = \pm 1$, we get

$$\begin{aligned} 1 &= c(0)A(x) + d(0)B(x) \\ 1 + x + x^2 &= c(1)A(x) + d(1)B(x) \\ 1 - x + x^2 &= c(-1)A(x) + d(-1)A(x), \end{aligned}$$

where $A(x)$ and $B(x)$ are the truncations of $a(x)$ and $b\,(x)$ to polynomials of degree less than 3; note that any higher degree terms in $a(x)$ and $b\,(x)$ must cancel on the right sides. Since $\{1, 1 + x + x^2, 1 - x + x^2\}$ is a basis of the vector space P_2 of polynomials in x of degree less than 3 and each of $1, 1 + x + x^2, 1 - x + x^2$ is a linear combination of $A(x)$ and $B(x)$, $\{A(x), B(x)\}$ spans P_2, but $\dim P_2 = 3$, and so a spanning set of P_2 must have at least 3 elements.

B2. Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}$, form a new sequence of $n - 1$ entries, $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$, by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n - 2$ entries and continue until the final sequence produced consists of a single number x_n. Show that $x_n < \frac{2}{n}$.

Solution. For an infinite sequence $a_0, a_1, \ldots$, the sequence of first averages is

$$\tfrac{1}{2}\left(a_0 + a_1\right), \tfrac{1}{2}\left(a_1 + a_2\right), \tfrac{1}{2}\left(a_2 + a_3\right), \ldots,$$

the sequence of second averages is

$$\begin{aligned} & \tfrac{1}{2}\left(\tfrac{1}{2}\left(a_0 + a_1\right) + \tfrac{1}{2}\left(a_1 + a_2\right)\right), \tfrac{1}{2}\left(\tfrac{1}{2}\left(a_1 + a_2\right), \tfrac{1}{2}\left(a_2 + a_3\right)\right), \ldots \\ = & \tfrac{1}{2^2}\left(a_0 + 2a_1 + a_2\right), \tfrac{1}{2^2}\left(a_1 + 2a_2 + a_3\right), \ldots, \end{aligned}$$

the sequence of third averages is

$$\begin{aligned} & \tfrac{1}{2^3}\left(\left(a_0 + 2a_1 + a_2\right) + \left(a_1 + 2a_2 + a_3\right)\right), \ldots \\ = & \tfrac{1}{2^3}\left(a_0 + 3a_1 + 3a_2 + a_3\right), \ldots \end{aligned}$$

By induction, the first term of the sequence of k-th averages is

$$\frac{1}{2^k}\left(\sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} a_i\right).$$

The value that we seek to estimate, namely the first term of the sequence of $(n-1)$-th averages of $1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, 0, 0, \ldots$, is then

$$\begin{aligned} \frac{1}{2^{n-1}}\left(\sum_{i=0}^{n-1} \frac{(n-1)!}{i!\,(n-1-i)!} \frac{1}{1+i}\right) &= \frac{1}{n2^{n-1}}\left(\sum_{i=0}^{n-1} \frac{n!}{(i+1)!\,(n-(i+1))!}\right) \\ &= \frac{1}{n2^{n-1}}\left(2^n - 1\right) \le \frac{2}{n}. \end{aligned}$$

B3. Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\left(1, 2, \ldots, \left\lfloor \tfrac{n}{i} \right\rfloor\right)$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

Solution. We use induction, first noting that the case $n = 1$ holds. We need to show that

$$n = \frac{\prod_{i=1}^{n} \operatorname{lcm}\left(1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\right)}{\prod_{i=1}^{n-1} \operatorname{lcm}\left(1, 2, \ldots, \left\lfloor \frac{n-1}{i} \right\rfloor\right)}.$$

Since $\operatorname{lcm}\left(1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\right) = 1$ when $i = n$, we may replace $\prod_{i=1}^{n}$ by $\prod_{i=1}^{n-1}$ in the numerator, whence we are to show that

$$n = \prod_{i=1}^{n-1} \left(\frac{\operatorname{lcm}\left(1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor\right)}{\operatorname{lcm}\left(1, 2, \ldots, \left\lfloor \frac{n-1}{i} \right\rfloor\right)} \right) \tag{2}$$

Note that either $\left\lfloor \frac{n}{i} \right\rfloor = \left\lfloor \frac{n-1}{i} \right\rfloor$ or $\left\lfloor \frac{n}{i} \right\rfloor = \left\lfloor \frac{n-1}{i} \right\rfloor + 1$. If $\left\lfloor \frac{n}{i} \right\rfloor = \left\lfloor \frac{n-1}{i} \right\rfloor$, the i-th factor of 2 is 1. If $\left\lfloor \frac{n}{i} \right\rfloor = \left\lfloor \frac{n-1}{i} \right\rfloor + 1$, then it must be that $\frac{n}{i}$ is an integer. We will use the fact that the lcm of a set of positive integers is the product of their maximal prime power factors. If $\frac{n}{i}$ is not the power of a prime, then all of the exponents of the prime power factors of the numbers $1, 2, \ldots, \left\lfloor \frac{n}{i} \right\rfloor$ are no greater than those of $1, 2, \ldots, \left\lfloor \frac{n-1}{i} \right\rfloor$ in which case the i-th factor of 2 is 1. If $\frac{n}{i}$ is p^m for some prime p, then the i-th factor of 2 is p, since p^{m-1} is among the numbers $1, 2, \ldots, \left\lfloor \frac{n-1}{i} \right\rfloor$, but p^m (and its multiples) is not. Thus, the non-unit factors of the right side of 2 are just the prime factors of n each repeated according to multiplicity. Hence the right side of 2 is the prime factorization of n which of course equals n.

B4. Let $f(z) = az^4 + bz^3 + cz^2 + dz + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number, and if $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

Solution. Note that

$$\begin{aligned} & (z - r_1)(z - r_2)(z - r_3)(z - r_4) \\ = \; & z^4 - (r_1 + r_2 + r_4 + r_3) z^3 + (r_1 r_2 + r_3 r_4 + r_2 r_4 + r_2 r_3 + r_1 r_4 + r_1 r_3) z^2 \\ & - (r_2 r_3 r_4 + r_1 r_3 r_4 + r_1 r_2 r_4 + r_1 r_2 r_3) z + r_1 r_2 r_3 r_4 \end{aligned}$$

Thus,

$$\begin{aligned} (r_1 + r_2) + (r_3 + r_4) &= r_1 + r_2 + r_4 + r_3 = -b/a \\ (r_1 + r_2)(r_3 + r_4) + r_1 r_2 + r_3 r_4 &= r_1 r_2 + r_3 r_4 + r_2 r_4 + r_2 r_3 + r_1 r_4 + r_1 r_3 = c/a \\ r_3 r_4 (r_2 + r_1) + r_1 r_2 (r_4 + r_3) &= r_2 r_3 r_4 + r_1 r_3 r_4 + r_1 r_2 r_4 + r_1 r_2 r_3 = -d/a \\ (r_1 r_2)(r_3 r_4) &= r_1 r_2 r_3 r_4 = e/a \end{aligned}$$

and the left sides are then rational. With $s = r_1 + r_2$, $t = r_3 + r_4$, $u = r_1 r_2$ and $v = r_3 r_4$, these become

$$\begin{aligned} s + t &= -b/a \\ st + u + v &= c/a \\ vs + ut &= -d/a \\ uv &= e/a. \end{aligned}$$

We conclude that t is rational, and $u + v = c/a - st$ is rational. Then $s(u+v)$ is rational and

$$u(t-s) = vs + ut - s(u+v) = -d/a - s(u+v)$$

is then rational. If $t \neq s$, then

$$r_1 r_2 = u = \frac{-d/a - s(u+v)}{t-s}$$

is rational as required.

B5. Let A, B and C be equidistant points on the circumference of a circle of unit radius centered at O, and let P be any point in the circle's interior. Let a, b, c be the distances from P to A, B, C respectively. Show that there is a triangle with side lengths a, b, c, and that the area of this triangle depends only on the distance from P to O.

Solution. Choose coordinates in the complex plane so that $A = 1$, $B = e^{2\pi i/3}$, $C = e^{4\pi i/3}$. If $\beta = e^{2\pi i/3}$, then $C = \beta^2$ and $A = \beta^3 = 1$. Let $P = z$. Then

$$a = |z-1|, \; b = |z-\beta|, \text{ and } c = \left|z - \beta^2\right|.$$

To show that a, b, c are side lengths of a triangle we need unit complex numbers $\alpha_1, \alpha_2, \alpha_3$ so that

$$\begin{aligned} \alpha_1(z-1) + \alpha_2(z-\beta) + \alpha_3\left(z-\beta^2\right) &= 0, \text{ or equivalently,} \\ (\alpha_1 + \alpha_2 + \alpha_3)z - \left(\alpha_1 + \alpha_2\beta + \alpha_3\beta^2\right) &= 0 \end{aligned}$$

We know that $1 + \beta + \beta^2 = 0$. For $(\alpha_1, \alpha_2, \alpha_3)$, it is then natural to try permutations of $\left(1, \beta, \beta^2\right)$. Indeed, $(\alpha_1, \alpha_2, \alpha_3) = \left(1, \beta, \beta^2\right)$ (or any other permutation) works fine, since

$$\alpha_1 + \alpha_2\beta + \alpha_3\beta^2 = 1 + \beta^2 + \beta^4 = 1 + \beta^2 + \beta = 0.$$

For a triangle with vector sides (a, b) and (c, d), the area Δ is given by

$$\begin{aligned} 2 \cdot \Delta &= \left|\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = |bc - ad| = |\mathrm{Im}\,((a+ib)(c-id))| \\ &= \left|\mathrm{Im}\left((a+ib)\overline{(c+id)}\right)\right| = \tfrac{1}{2}\left|\left((a+ib)\overline{(c+id)} - \overline{(a+ib)}(c+id)\right)\right| \end{aligned}$$

The area of the triangle above with sides $z-1, \beta(z-\beta)$ (and $\beta^2(z-\beta)$) is then (noting that $\bar{\beta}=\beta^2$)

$$\begin{aligned}
& \tfrac{1}{4}\left|\left((z-1)\overline{\beta(z-\beta)}-\overline{(z-1)}\beta(z-\beta)\right)\right| \\
= & \tfrac{1}{4}\left|\left((z-1)\beta^2\left(\bar{z}-\beta^2\right)-(\bar{z}-1)\beta(z-\beta)\right)\right| \\
= & \tfrac{1}{4}|((z-1)(\beta\bar{z}-1)-(\bar{z}-1)(z-\beta))||\beta| \\
= & \tfrac{1}{4}|((\beta z\bar{z}-\beta\bar{z}-z+1)-(z\bar{z}-z-\beta\bar{z}+\beta))| \\
= & \tfrac{1}{4}\left|\left(\left(\beta|z|^2+1\right)-\left(|z|^2+\beta\right)\right)\right| \\
= & \tfrac{1}{4}\left|\left((\beta-1)\left(|z|^2-1\right)\right)\right|=\tfrac{1}{4}|(\beta-1)|\left(1-|z|^2\right)=\tfrac{\sqrt{3}}{4}\left(1-|z|^2\right),
\end{aligned}$$

since

$$|(\beta-1)|^2=(\cos(2\pi/3)-1)^2+\sin^2(2\pi/3)=2-2\cos(2\pi/3)=3.$$

B6. Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that

$$\int_0^1\int_0^1|f(x)+f(y)|\ dxdy \geq \int_0^1|f(x)|\ dx.$$

Solution. Let $A^+=\{x\in[0,1]: f(x)>0\}$ and let $A^-=\{x\in[0,1]: f(x)\leq 0\}$. For $I^+:=\int_{A^+} f(x)\ dx$ and $I^-:=-\int_{A^-} f(x)\ dx$, we have

$$\int_0^1|f(x)|\ dx=\int_{A^+} f(x)\ dx-\int_{A^-} f(x)\ dx=I^++I^-.$$

Also,

$$\begin{aligned}
& \int\int_{A^+\times A^+}|f(x)+f(y)|\ dxdy \\
= & \int\int_{A^+\times A^+} f(x)+f(y)\ dxdy=m\left(A^+\right)\int_{A^+} f(x)\,dx+m\left(A^+\right)\int_{A^+} f(y)\,dy \\
= & 2m\left(A^+\right)I^+
\end{aligned}$$

and similarly

$$\int\int_{A^-\times A^-}|f(x)+f(y)|\ dxdy=2m\left(A^-\right)I^-.$$

We have

$$\begin{aligned}
& \int\int_{A^+\times A^-}|f(x)+f(y)|\ dxdy \\
\geq & \pm\left(\int\int_{A^+\times A^-}|f(x)|-|f(y)|\ dxdy\right) \\
= & \pm\left(\int\int_{A^+\times A^-} f(x)\ dxdy-\int\int_{A^+\times A^-}|f(y)|\ dxdy\right) \\
= & \pm\left(m\left(A^-\right)I^+-m\left(A^+\right)I^-\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}&\int\int_{A^{-}\times A^{+}}\left|f\left(x\right)+f\left(y\right)\right|\;dxdy\\ \geq\;&\pm\left(m\left(A^{+}\right)I^{-}-m\left(A^{-}\right)I^{+}\right)\end{aligned}$$

Thus,

$$\begin{aligned}&\int_0^1\int_0^1\left|f\left(x\right)+f\left(y\right)\right|\;dxdy\\ =\;&\int\int_{(A^{+}\times A^{+})\cup(A^{-}\times A^{-})\cup(A^{+}\times A^{-})\cup(A^{-}\times A^{+})}\left|f\left(x\right)+f\left(y\right)\right|\;dxdy\\ \geq\;&2m\left(A^{+}\right)I^{+}+2m\left(A^{-}\right)I^{-}\pm\left(m\left(A^{-}\right)I^{+}-m\left(A^{+}\right)I^{-}\right)\pm'\left(m\left(A^{+}\right)I^{-}-m\left(A^{-}\right)I^{+}\right)\\ =\;&2m\left(A^{+}\right)I^{+}+2m\left(A^{-}\right)I^{-}\pm m\left(A^{-}\right)I^{+}\mp m\left(A^{+}\right)I^{-}\pm' m\left(A^{+}\right)I^{-}\mp' m\left(A^{-}\right)I^{+}\\ =\;&\left(2m\left(A^{+}\right)\pm m\left(A^{-}\right)\mp' m\left(A^{-}\right)\right)I^{+}+\left(2m\left(A^{-}\right)\mp m\left(A^{+}\right)\pm' m\left(A^{+}\right)\right)I^{-}.\end{aligned}$$

There are four possible choices of the signs $\pm$ and $\pm'$, yielding (where we have used $m\left(A^{-}\right)+m\left(A^{+}\right)=1$)

$$\begin{aligned}(+,+')\;&:\;2m\left(A^{+}\right)I^{+}+2m\left(A^{-}\right)I^{-}\\ (+,-')\;&:\;2I^{+}+2\left(m\left(A^{-}\right)-m\left(A^{+}\right)\right)I^{-}\\ (-,+')\;&:\;2\left(m\left(A^{+}\right)-m\left(A^{-}\right)\right)I^{+}+2I^{-}\\ (-,-')\;&:\;2m\left(A^{+}\right)I^{+}+2m\left(A^{-}\right)I^{-}\end{aligned}$$

Note that $(+,+')$ and $(-,-')$ yield the same result. Thus,

$$\int_0^1\int_0^1\left|f\left(x\right)+f\left(y\right)\right|dxdy\geq\max\left\{\begin{array}{c}2m\left(A^{+}\right)I^{+}+2m\left(A^{-}\right)I^{-},\\ 2I^{+}+2\left(m\left(A^{-}\right)-m\left(A^{+}\right)\right)I^{-},\\ 2\left(m\left(A^{+}\right)-m\left(A^{-}\right)\right)I^{+}+2I^{-}\end{array}\right\}$$

To show that

$$\begin{aligned}\int_0^1\int_0^1\left|f\left(x\right)+f\left(y\right)\right|dxdy\;&\geq\;\int_0^1\left|f\left(x\right)\right|\;dx,\;\text{ or equivalently}\\ \int_0^1\int_0^1\left|f\left(x\right)+f\left(y\right)\right|dxdy-\left(I^{+}+I^{-}\right)\;&\geq\;0,\end{aligned}$$

there are four possible cases of inequalities to consider for the pairs $\left(I^{+},I^{-}\right)$ and $\left(m\left(A^{+}\right),m\left(A^{-}\right)\right)$:

Case 1: If $I^{+}\geq I^{-}$ and $m\left(A^{+}\right)\geq m\left(A^{-}\right)$, then

$$\begin{aligned}&2m\left(A^{+}\right)I^{+}+2m\left(A^{-}\right)I^{-}-\left(I^{+}+I^{-}\right)\\ =\;&\left(2m\left(A^{+}\right)-1\right)I^{+}+\left(2m\left(A^{-}\right)-1\right)I^{-}\\ \geq\;&\left(2m\left(A^{+}\right)-1\right)I^{+}+\left(2m\left(A^{-}\right)-1\right)I^{+}\\ \geq\;&\left(\left(2\left(m\left(A^{+}\right)+m\left(A^{-}\right)\right)-2\right)\right)I^{+}=0.\end{aligned}$$

Case 2: If $I^+ \leq I^-$ and $m(A^+) \leq m(A^-)$, then

$$\begin{aligned} & 2m\left(A^+\right)I^+ + 2m\left(A^-\right)I^- - \left(I^+ + I^-\right) \\ = \ & \left(2m\left(A^+\right) - 1\right)I^+ + \left(2m\left(A^-\right) - 1\right)I^- \\ \geq \ & \left(2m\left(A^+\right) - 1\right)I^- + \left(2m\left(A^-\right) - 1\right)I^+ \\ \geq \ & \left(\left(2\left(m\left(A^+\right) + m\left(A^-\right)\right) - 2\right)\right)I^- = 0. \end{aligned}$$

Case 3: If $I^+ \geq I^-$ and $m(A^+) \leq m(A^-)$, then $m(A^+) \leq \frac{1}{2}$ and $m(A^-) \geq \frac{1}{2}$, and so

$$\begin{aligned} & 2I^+ + 2\left(m\left(A^-\right) - m\left(A^+\right)\right)I^- - \left(I^+ + I^-\right) \\ = \ & I^+ + 2\left(m\left(A^-\right) - m\left(A^+\right) - \tfrac{1}{2}\right)I^- \\ \geq \ & I^+ - 2m\left(A^+\right)I^- \geq I^+ - I^- \geq 0. \end{aligned}$$

Case 4: If $I^+ \leq I^-$ and $m(A^+) \geq m(A^-)$, then $m(A^+) \geq \frac{1}{2}$ and $m(A^-) \leq \frac{1}{2}$, and so

$$\begin{aligned} & 2\left(m\left(A^+\right) - m\left(A^-\right)\right)I^+ + 2I^- - \left(I^+ + I^-\right) \\ = \ & 2\left(m\left(A^+\right) - m\left(A^-\right) - \tfrac{1}{2}\right)I^+ + I^- \\ \geq \ & -2m\left(A^-\right)I^+ + I^- \geq -I^+ + I^- \geq 0. \end{aligned}$$