# Putnam 2004

**A1**. Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

**Solution 1**. Suppose that $i$ is the first instance that $\frac{S(i)}{i} \geq 4/5$. Then

$$\frac{S(i)-1}{i-1} = \frac{S(i-1)}{i-1} < 4/5 \text{ and } \frac{S(i)}{i} \geq 4/5.$$

Now,

$$\frac{S(i)-1}{i-1} < 4/5 \Rightarrow S(i) - 1 < 4/5\,(i-1) \Rightarrow 5S(i) - 5 < 4i - 4 \Rightarrow 5S(i) - 4i < 1,$$

and

$$\frac{S(i)}{i} \geq 4/5 \Rightarrow 5S(i) - 4i \geq 0.$$

Thus,

$$0 \leq 5S(i) - 4i < 1.$$

Since $5S(i) - 4i$ is an integer, we must have $5S(i) - 4i = 0$.

**Solution 2**. The curve $C$ connecting the points $(n, S(n)), n = 1, 2, \cdots$, must cross the line $y = 4x/5$ at some point. Thus, either there is a point $(n, S(n))$ on the line (i.e., 80% is achieved) or there is a segment from a point $(i, S(i))$ on $C$ below the line to the point $(i+1, S(i+1)) = (i+1, S(i)+1))$ above the line. Note that $S(i) < \frac{4}{5}i$, since $(i, S(i))$ is below $y = 4x/5$. Since $\frac{4}{5}i$ is a multiple of $\frac{1}{5}$ and $S(i)$ is an integer,

$$S(i) < \frac{4}{5}i \Rightarrow S(i) + \frac{1}{5} \leq \frac{4}{5}i,$$

Then

$$S(i) + 1 = \left(S(i) + \frac{1}{5}\right) + \frac{4}{5} \leq \frac{4}{5}i + \frac{4}{5} = \frac{4}{5}\,(i+1)$$

Thus, the point $(i+1, S(i+1))$ does *not* lie above the line $y = 4x/5$, a contradiction.

**A2**. For $i = 1, 2$ let $T_i$ be a triangle with side lengths $a_i, b_i, c_i$, and area $A_i$. Suppose that $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, and that $T_2$ is an acute triangle. Does it follow that $A_1 \leq A_2$?

**Solution**. Let $\theta_{c_i}$ be the angle between the sides with lengths $a_i$ and $b_i$, and similarly define $\theta_{a_i}$ and $\theta_{b_i}$. We cannot have

$$\theta_{a_2} < \theta_{a_1},\ \theta_{b_2} < \theta_{b_1},\ \textit{and}\ \theta_{c_2} < \theta_{c_1},$$

since then

$$\pi = \theta_{a_2} + \theta_{b_2} + \theta_{c_2} < \theta_{a_1} + \theta_{b_1} + \theta_{c_1} = \pi.$$

Thus, suppose that $\theta_{c_1} \leq \theta_{c_2}$. Then $\sin\theta_{c_1} \leq \sin\theta_{c_2}$, since the acuteness of $T_2$ gives $\theta_{c_2} \leq \pi/2$ and $\sin\theta$ is increasing for $\theta \in [0, \pi/2]$. Hence

$$A_1 = \tfrac{1}{2} a_1 b_1 \sin\theta_{c_1} \leq \tfrac{1}{2} a_2 b_2 \sin\theta_{c_2} = A_2.$$

**A3**. Define a sequence $\{u_n\}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{bmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{bmatrix} = n!$$

for all $n \geq 0$. Show that $u_n$ is an integer for all $n$. (By convention, $0! = 1$.)

**Solution**.

$$\begin{aligned} \det \begin{bmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{bmatrix} &= u_n u_{n+3} - u_{n+1} u_{n+2} = n! \\ u_{n+3} &= \frac{u_{n+1} u_{n+2} + n!}{u_n} \end{aligned}$$

$$u_{n+3} = \frac{u_{n+1} u_{n+2} + n!}{u_n}$$

$$\begin{aligned}
u_3 &= u_{0+3} = \frac{u_1 u_2 + 0!}{u_0} = \frac{1 \cdot 1 + 1}{1} = 2 \\
u_4 &= u_{1+3} = \frac{u_2 u_3 + 1!}{u_1} = \frac{1 \cdot 2 + 1}{1} = 3 \\
u_5 &= u_{2+3} = \frac{u_3 u_4 + 2!}{u_2} = \frac{2 \cdot 3 + 2}{1} = 8 \\
u_6 &= u_{3+3} = \frac{u_4 u_5 + 3!}{u_3} = \frac{3 \cdot 8 + 6}{2} = 15 = 5 \cdot 3 \\
u_7 &= u_{4+3} = \frac{u_5 u_6 + 4!}{u_4} = \frac{8 \cdot 15 + 24}{3} = 48 = 6 \cdot 4 \cdot 2 \\
u_8 &= u_{5+3} = \frac{u_6 u_7 + 5!}{u_5} = \frac{15 \cdot 48 + 120}{8} = 105 = 7 \cdot 5 \cdot 3 \\
u_9 &= u_{6+3} = \frac{u_7 u_8 + 6!}{u_6} = \frac{48 \cdot 105 + 720}{15} = 384 = 8 \cdot 6 \cdot 4 \cdot 2
\end{aligned}$$

Thus, we conjecture that $u_k = (k-1)(k-3)(k-5)\cdots 3$ (or 2). We prove this by induction. The first three base cases have been shown. Assume that this holds for $k \leq n+2$. Then

$$\begin{aligned}
u_{n+3} &= \frac{u_{n+1} u_{n+2} + n!}{u_n} = \frac{u_{n+1}(n+1)u_n + n!}{u_n} = u_{n+1}(n+1) + \frac{n!}{u_n} \\
&= u_{n+1}(n+1) + \frac{n!}{(n-1)(n-3)(n-5)\cdots} = u_{n+1}(n+1) + n(n-2)(n-3)\cdots \\
&= u_{n+1}(n+1) + u_{n+1} = (n+2)u_{n+1} = (n+2)n(n-2)\cdots.
\end{aligned}$$

**A4**. Show that for any positive integer $n$ there is an integer $N$ such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i \left(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\right)^n ,$$

where the $c_i$ are rational numbers and each $a_{ij}$ is one of the numbers, $-1, 0, 1$.

**Solution**. We have

$$\begin{aligned} & \sum_{i=1}^{N} c_i \left(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\right)^n \\ = & \sum_{i=1}^{N} \sum_{k_1+\cdots+k_n=n} c_i \frac{n!}{k_1!k_2!\cdots k_n!} \left(a_{i1}x_1\right)^{k_1} \left(a_{i2}x_2\right)^{k_2} \cdots \left(a_{in}x_n\right)^{k_n} \\ = & \sum_{k_1+\cdots+k_n=n} \frac{n!}{k_1!k_2!\cdots k_n!} \left(\sum_{i=1}^{N} c_i a_{i1}^{k_1} a_{i2}^{k_2} \cdots a_{in}^{k_n}\right) x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}. \end{aligned}$$

Thus, we need to choose $c_i$ and $a_{i1}, a_{i2}, \ldots, a_{in}$, so that whenever $(k_1, \ldots, k_n) \neq (1, \ldots, 1)$, or equivalently $k_i = 0$ for some $i$, we have

$$\sum_{i=1}^{N} c_i a_{i1}^{k_1} a_{i2}^{k_2} \cdots a_{in}^{k_n} = 0.$$

Let $i$ range over all of $N = 2^n$ subsets of $\{x_1, \ldots, x_n\}$ and let $c_i = (-1)^{\#i}$

$$a_{ij} := \begin{cases} 1 & \text{if } x_j \in i \\ -1 & \text{if } x_j \notin i. \end{cases}$$

Suppose that one of the $k_j = 0$, say $k_1 = 0$. Each subset $i$ with $x_1 \notin i$ is paired with a subset $i \cup \{x_1\}$ with one more element, namely $x_1$. Thus, the term

$$(-1)^{\#i} a_{i1}^{0} a_{i2}^{k_2} \cdots a_{in}^{k_n} \text{ cancels with } (-1)^{\#i\cup\{x_1\}} a_{(i\cup\{x_1\})1}^{0} a_{(i\cup\{x_1\})2}^{k_2} \cdots a_{(i\cup\{x_1\})n}^{k_n}$$

Hence, the only sum $\sum_{i=1}^{N} c_i a_{i1}^{k_1} a_{i2}^{k_2} \cdots a_{in}^{k_n}$ that can be nonzero is that with $(k_1, \ldots, k_n) = (1, \ldots, 1)$; i.e., the sum which is the coefficient of $x_1 x_2 \cdots x_n$. In this case,

$$\begin{aligned} \sum_{i=1}^{N} c_i a_{i1}^{k_1} a_{i2}^{k_2} \cdots a_{in}^{k_n} &= \sum_{i=1}^{N} (-1)^{\#i} a_{i1} a_{i2} \cdots a_{in} \\ &= \sum_{i=1}^{N} (-1)^{\#i} (-1)^{n-\#i} = \sum_{i=1}^{N} (-1)^n = (-1)^n 2^n \neq 0. \end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{i=1}^{N} c_i \left(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\right)^n \\
= &\sum_{k_1+\cdots+k_n=n} \frac{n!}{k_1!k_2!\cdots k_n!} \left(\sum_{i=1}^{N} c_i a_{i1}^{k_1} a_{i2}^{k_2} \cdots a_{in}^{k_n}\right) x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \\
= &\left(n! \left(-1\right)^n 2^n\right) x_1 x_2 \cdots x_n
\end{aligned}$$

Replacing $c_i$ by $c_i / \left(n! \left(-1\right)^n 2^n\right) = \left(-1\right)^{\#i} / \left(n! \left(-1\right)^n 2^n\right)$, we obtain the desired result.

**A5**. An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, $p$ and $q$, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at $p$ and ending at $q$, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

**Solution**. For a given coloring of the rectangle, consider the decomposition of the rectangle into faces, edges and vertices, as follows. The faces are the CMR's (connected monochromatic regions). The edges are just the edges of length 1 making up the borders of the CMRs, and the vertices are the endpoints of the edges. For a rectangle with Euler characteristic 1, Euler's formula gives

$$f - e + v = 1.$$

However, this assumes that the faces are simply-connected (without holes). If the holes in the faces which are not simply-connected are filled in, then the number of faces decreases to say $f'$ without affecting $v - e$. Thus, allowing faces with holes we have

$$f - e + v \geq f' - e + v = 1 \quad \text{or} \quad f \geq e - v + 1.$$

Now take the expectation of each side:

$$E\left(f\right) \geq E(e) - E(v) + 1$$

Note that the $2(m+n)$ edges and $2(m+n)$ vertices on the border of the $m \times n$ rectangle cancel in the sum $e - v + 1$. Thus, we need only count internal edges and vertices in $e$ and $v$. However, one of the $\left(m-1\right) n + \left(n-1\right) m = 2mn - \left(n+m\right)$ potential internal edges is edge of a decomposition only if the squares on either side are of different colors, which happens with probability $1/2$. Thus,

$$E(e) = \tfrac{1}{2}\left(2mn - \left(n+m\right)\right) = mn - \tfrac{1}{2}\left(n+m\right)$$

One of the $\left(m-1\right)\left(n-1\right)$ potential internal vertices is a vertex of a given decomposition unless the four squares surrounding the vertex are of the same color, which happens with probability $1/8$. Thus,

$$E(v) = \tfrac{7}{8}\left(m-1\right)\left(n-1\right)$$

Now,

$$\begin{aligned} E(f) &\geq E(e) - E(v) + 1 = mn - \tfrac{1}{2}(n+m) - \tfrac{7}{8}(m-1)(n-1) + 1 \\ &= \frac{1}{8}mn + \frac{3}{8}n + \frac{3}{8}m + \frac{1}{8} > \frac{mn}{8}. \end{aligned}$$

**A6**. Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)dx\right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy\right)^2 dx \leq \left(\int_0^1 \int_0^1 f(x,y)dxdy\right)^2 + \int_0^1 \int_0^1 f(x,y)^2 dxdy$$

**Solution**. Note that

$$\begin{aligned} \left(\int_0^1 \int_0^1 f(x,y)dxdy\right)^2 &= \int_0^1 \int_0^1 f(x,y)dxdy \int_0^1 \int_0^1 f(z,w)dzdw \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x,y)f(z,w)\, dxdydzdw \\ &= \int_{[0,1]^4} f(x,y)f(z,w)\, dxdydzdw \end{aligned}$$

Also, the other integrals can be expressed as integrals over $[0,1]^4$ :

$$\begin{aligned} \int_0^1 \left(\int_0^1 f(x,y)dx\right)^2 dy &= \int_0^1 \left(\int_0^1 f(x,y)dx\right)\left(\int_0^1 f(z,y)dz\right) dy \\ &= \int_0^1 \int_0^1 \int_0^1 f(x,y)f(z,y)\, dxdzdy \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 f(x,y)f(z,y)\, dxdzdydw \\ &= \int_{[0,1]^4} f(x,y)f(z,y)\, dxdydzdw, \end{aligned}$$

$$\begin{aligned} \int_0^1 \left(\int_0^1 f(x,y)dy\right)^2 dx &= \int_0^1 \left(\int_0^1 f(x,y)dy\right)\left(\int_0^1 f(x,z)dz\right) dx \\ &= \int_0^1 \int_0^1 \int_0^1 f(x,y)f(x,z)\, dxdzdy \\ &= \int_{[0,1]^4} f(x,y)f(x,z)\, dxdydzdw, \end{aligned}$$

and

$$\int_0^1 \int_0^1 f(x,y)^2 dxdy = \int_{[0,1]^4} f(x,y)^2\, dxdydzdw$$

Thus, we are to prove that

$$\int_{[0,1]^4} \big(f(x,y)^2 + f(x,y)f(z,w) - f(x,y)f(z,y) - f(x,y)f(x,z)\big)\, dxdydzdw \geq 0$$

Note that

$$\begin{aligned}
&\left(f(x,y)+f(z,w)-f(x,w)-f(z,y)\right)^2 = f(x,y)^2 + f(z,w)^2 + f(x,w)^2 + f(z,y)^2 \\
&+2f(x,y)f(z,w) + 2f(x,w)f(z,y) \\
&-2f(x,y)f(x,w) - 2f(x,y)f(z,y) - 2f(z,w)f(x,w) - 2f(z,w)f(z,y)
\end{aligned}$$

The integral this over $[0,1]^4$ is nonnegative and 4 times the preceding integral.

**B1**. Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that $r$ is a rational number such that $P(r) = 0$. Show that the $n$ numbers $c_n r$, $c_n r^2 + c_{n-1}$, $c_n r^3 + c_{n-1}r^2 + c_{n-2}r$, ..., $c_n r^n + c_{n-1}r^{n-1} + \cdots + c_1 r$ are integers.

**Solution**. Let $r = p/q$ in lowest terms. Then we have

$$c_n p^n + c_{n-1}p^{n-1}q + ... + c_0 q^n = 0.$$

For each $j$ between 0 and $n$, $q^j$ divides

$$\begin{aligned}
-\left(c_{n-j}p^{n-j}q^j + ... + c_0 q^n\right) &= c_n p^n + c_{n-1}p^{n-1}q + ... + c_{n-j+1}p^{n-j+1}q^{j-1} \\
&= p^{n-j}\left(c_n p^j + c_{n-1}p^{j-1}q + ... + c_{n-j+1}p^1 q^{j-1}\right)
\end{aligned}$$

Since $p$ and $q$ are coprime, $q^j$ divides

$$c_n p^j + c_{n-1}p^{j-1}q + ... + c_{n-j+1}p^1 q^{j-1}.$$

Dividing by $q^j$, we then have an integer, namely

$$c_n r^j + c_{n-1}r^{j-1} + ... + c_{n-j+1}r.$$

**B2**. Let $m$ and $n$ be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} \leq \frac{m!}{m^m}\frac{n!}{n^n}.$$

**Solution**. Let $A = \{1, 2, ..., m\}$, $B = \{m+1, ..., m+n\}$. The number of functions $f : A \cup B \to A \cup B$ is $(m+n)^{m+n}$. Among those are the functions that send precisely $m$ elements of the domain to $A$, and the other $n$ elements to $B$. There are $\frac{(m+n)!}{m!n!}$ ways to choose which elements go to $A$, and once those are chosen there are $m^m n^n$ such functions. So $(m+n)^{m+n} \geq \frac{(m+n)!}{m!n!}m^m n^n$.

**B3**. Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y) : 0 \le x \le a,\ 0 \le y \le f(x)\}$$

has perimeter $k$ units and area $k$ square units for some real number $k$.

**Solution**. If $a > 2$ we may use a constant function $f(x) = c$, where

$$ac = 2a + 2c \Rightarrow c = \frac{2a}{a-2}.$$

If $a < 2$ we show that there is no such function. If $M := \max f$, the area satisfies $A \le aM$, but the perimeter $P$ is clearly larger than $2M$. Thus, since $a < 2$

$$A \le aM \le 2M < P.$$

**B4**. Let $n$ be a positive integer, $n \ge 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the $xy$-plane, for $k = 1, 2, ..., n$. Let $R_k$ be the map that rotates the plane counterclockwise by the angle $\theta$ about the point $P_k$. Let $R$ denote the map obtained by applying, in order, $R_1$, then $R_2$, ..., then $R_n$. For an arbitrary point $(x, y)$, find, and then simplify, the coordinates of $R(x, y)$.

**Solution 1**. We identify points in the plane with complex numbers. Rotation by $\theta$ about the point $z_0$ takes the point $z$ to $z'$, where

$$z' - z_0 = e^{i\theta}(z - z_0).$$

Denoting this rotation by $R_{z_0}$, we then have

$$R_{z_0}(z) = z' = z_0 + e^{i\theta}(z - z_0) = e^{i\theta} z + z_0\left(1 - e^{i\theta}\right).$$

We compute

$$\begin{aligned}
(R_2 \circ R_1)(z) &= R_2\left(e^{i\theta} z + \left(1 - e^{i\theta}\right)1\right) = e^{i\theta}\left(e^{i\theta} z + \left(1 - e^{i\theta}\right)1\right) + 2\left(1 - e^{i\theta}\right) \\
&= e^{i2\theta} z + e^{i\theta}\left(1 - e^{i\theta}\right)1 + \left(1 - e^{i\theta}\right)2 \\
&= e^{i2\theta} z + \left(e^{i\theta} - e^{i2\theta}\right) + \left(2 - 2e^{i\theta}\right) = e^{i2\theta} z + 2 - e^{i\theta} - e^{i2\theta}.
\end{aligned}$$

$$\begin{aligned}
(R_3 \circ R_2 \circ R_1)(z) &= e^{i\theta}\left(e^{i2\theta} z + 2 - e^{i\theta} - e^{i2\theta}\right) + 3\left(1 - e^{i\theta}\right) \\
&= e^{i3\theta} z + 3 - e^{i\theta} - e^{i2\theta} - e^{i3\theta}
\end{aligned}$$

By induction, we can show

$$\begin{aligned}
(R_n \circ R_{n-1} \circ \cdots \circ R_1)(z) &= e^{in\theta} z + n - \sum_{k=1}^{n} e^{ik\theta} = e^{in\theta} z + n - e^{i\theta}\frac{e^{in\theta} - 1}{e^{i\theta} - 1} \\
&= e^{i2\pi} z + n - e^{i\theta}\frac{e^{i2\pi} - 1}{e^{i2\pi/n} - 1} = z + n.
\end{aligned}$$

Thus, $R_n \circ R_{n-1} \circ \cdots \circ R_1$ sends $(x, y)$ to $(x+n, y)$.

**Solution 2**. Imagine a regular $n$-gon with unit edge length which rolls to the right on the $x$-axis, the rotation it executes as it rotates about the $i$-th vertices when it contacts the $x$-axis is exactly the $R_i$. After the $n$-gon has completed a whole revolution, it becomes to a translation by $n$ units of its original self. If it carries the whole plane with itself, each point of the whole plane is translated by $n$ units.

**B5**. Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left( \frac{1+x^{n+1}}{1+x^n} \right)^{x^n}.$$

**Solution**. Note that

$$\frac{1+x^{n+1}}{1+x^n} = \frac{1+x^n+x^{n+1}-x^n}{1+x^n} = 1 + \frac{x^{n+1}-x^n}{1+x^n} = 1 + \frac{x^n (x-1)}{1+x^n}$$

Thus,

$$\begin{aligned} \ln \left( \prod_{n=0}^{N} \left( 1 + \frac{x^n (x-1)}{1+x^n} \right)^{x^n} \right) &= \sum_{n=0}^{N} x^n \ln \left( 1 + \frac{x^n (x-1)}{1+x^n} \right) \\ &= \sum_{n=0}^{N} x^n \left( \frac{x^n (x-1)}{1+x^n} + \mathrm{O}\left( (x-1)^2 \right) \right) \end{aligned}$$

Note that for $x \in [0, 1)$,

$$\mathrm{O}\left( (x-1)^2 \right) \sum_{n=0}^{\infty} x^n = \mathrm{O}\left( (x-1)^2 \right) \frac{1}{1-x} = \mathrm{O}\left( x-1 \right)$$

Thus, we may neglect the $\mathrm{O}\big((x-1)^2\big)$ expression as $x \to 1^-$. Now, for $x \in (0, 1)$,

$$\sum_{n=0}^{N} x^n \left( \frac{x^n (x-1)}{1+x^n} \right) = (x-1) \sum_{n=0}^{N} \frac{x^{2n}}{1+x^n} = (x-1) \sum_{n=0}^{N} \left( x^{2n} - x^{3n} + x^{4n} - x^{5n} + \cdots \right).$$

Then by virtue of the alternating series, for each $k = 1, 2, \ldots,$ we have

$$\begin{aligned} & (x-1) \left( \frac{1}{1-x^2} - \frac{1}{1-x^3} + \cdots + \frac{1}{1-x^{2k}} \right) \\ \leq\ & \sum_{n=0}^{\infty} x^n \left( \frac{x^n (x-1)}{1+x^n} \right) \\ \leq\ & (x-1) \left( \frac{1}{1-x^2} - \frac{1}{1-x^3} + \cdots - \frac{1}{1-x^{2k+1}} \right) \end{aligned}$$

Taking the limit as $x \to 1^-$, we get

$$-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{2k} \leq \lim_{x\to 1^-}\sum_{n=0}^{\infty} x^n\left(\frac{x^n\,(x-1)}{1+x^n}\right) \leq -\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2k+1}$$

Thus,

$$\lim_{x\to 1^-}\sum_{n=0}^{\infty} x^n\left(\frac{x^n\,(x-1)}{1+x^n}\right) = -\frac{1}{2}+\frac{1}{3}-\cdots$$

Recall that

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \sum_{n=0}^{\infty}(-t)^n\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n\,x^{n+1}}{n+1} \text{ for } x \in (0,1)$$

leads to

$$\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\cdots \text{ or } \ln(2/e) = \ln 2 - 1 = -\frac{1}{2}+\frac{1}{3}-\cdots$$

Thus, $\lim_{x\to 1^-}\prod_{n=0}^{\infty}\left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n} = 2/e$.

**B6**. Let $\mathcal{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathcal{A}$ not exceeding $x$. Let $\mathcal{B}$ denote the set of positive integers $b$ that can be written in the form $b = a - a'$ with $a \in \mathcal{A}$ and $a' \in \mathcal{A}$. Let $b_1 < b_2 < \ldots$ be the members of $\mathcal{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then $\lim_{x\to\infty} N(x)/x = 0$.

**Solution**. Let $\mathbb{N} = \{1, 2, 3, \cdots\}$. Note that

$$\mathcal{B}^c := \mathbb{N} \setminus \mathcal{B} = \{k \in \mathbb{N} : \mathcal{A} \cap (\mathcal{A} + k) = \phi\}.$$

The condition that $\{b_{i+1} - b_i\}$ is unbounded means that there are arbitrarily large strings of consecutive integers in $\mathcal{B}^c$. We construct a sequence $\{K_n\}_{n=1}^{\infty}$ of these strings, say

$$K_n = (k_n - L_n + 1, \ldots, k_n - 2, k_n - 1, k_n) \text{ (of length } L_n)$$

in such a way that $L_{n+1} > k_n$ (i.e., the length $L_{n+1}$ of string $K_{n+1}$ is greater than the largest element $k_n$ of the string $K_n$). Consider the translates of $\mathcal{A}$ given by

$$\mathcal{A}, \mathcal{A} + k_1, \mathcal{A} + k_2, \ldots.$$

We claim that these translates are pairwise disjoint. Indeed, first note that $\mathcal{A} \cap (\mathcal{A} + k_1) = \phi$ since $k_1 \in \mathcal{B}^c$. Now

$$(\mathcal{A} + k_1) \cap (\mathcal{A} + k_2) = \phi \Leftrightarrow \mathcal{A} \cap (\mathcal{A} + k_2 - k_1) = \phi.$$

However $k_2 - k_1$ is in the sequence $K_2$ since the length of $K_2$ is larger than $k_1$ by construction. In general, each $\mathcal{A} + k_n$ is disjoint from each of $\mathcal{A} + k_1, \mathcal{A} + k_2, \ldots, \mathcal{A} + k_{n-1}$, since (owing to the clear fact that $k_{i+1} > k_i$)

$$k_n - L_n \overset{\text{by construction}}{<} k_n - k_{n-1} < k_n - k_{n-2} < \cdots < k_n - k_1 < k_n,$$

and this implies that each of $k_n - k_{n-1}, k_n - k_{n-2}, \dots, k_n - k_1$ is in the sequence $K_n$. Thus, $\mathcal{A}, \mathcal{A}+k_1, \mathcal{A}+k_2, \dots$ is an infinite sequence of pairwise disjoint translates of $\mathcal{A}$. For large real $M > 0$, consider an interval $[1, Mk_{n-1}]$. The $n$ subsets

$$\mathcal{A} \cap [1, Mk_{n-1}], (\mathcal{A}+k_1) \cap [1, Mk_{n-1}], \dots, (\mathcal{A}+k_{n-1}) \cap [1, Mk_{n-1}],$$

of $[1, Mk_{n-1}]$ are also pairwise disjoint. Moreover, for $M$ large, these have approximately the same cardinality. Indeed, for $i \in \{0, 1, \dots, n-1\}$, we have

$$\begin{aligned} \#\left(\mathcal{A} \cap [1, Mk_{n-1}]\right) - k_{n-1} &\leq \#\left(\mathcal{A} \cap [1, Mk_{n-1}]\right) - k_i \\ &\leq \#\left((\mathcal{A}+k_i) \cap [1, Mk_{n-1}]\right) \leq \#\left(\mathcal{A} \cap [1, Mk_{n-1}]\right) \end{aligned}$$

since at most $k_i$ elements of $\mathcal{A} \cap [1, Mk_{n-1}]$ are eliminated via a translation by $k_i$. Thus,

$$\begin{aligned} \frac{\#\left(\mathcal{A} \cap [1, Mk_{n-1}]\right)}{\#[1, Mk_{n-1}]} - \frac{1}{M} &\leq \frac{\#\left(\mathcal{A} \cap [1, Mk_{n-1}]\right) - k_{n-1}}{\#[1, Mk_{n-1}]} \\ &\leq \frac{\#\left((\mathcal{A}+k_i) \cap [1, Mk_{n-1}]\right)}{\#[1, Mk_{n-1}]} \leq \frac{\#\left(\mathcal{A} \cap [1, Mk_{n-1}]\right)}{\#[1, Mk_{n-1}]} \end{aligned}$$

Since the $n$ subsets $(\mathcal{A}+k_i) \cap [1, Mk_{n-1}]$ are disjoint in $\mathcal{A} \cap [1, Mk_{n-1}]$, we then have

$$n\left(\frac{\#\left(\mathcal{A} \cap [1, Mk_{n-1}]\right)}{\#[1, Mk_{n-1}]} - \frac{1}{M}\right) \leq 1$$

or

$$\frac{\#\left(\mathcal{A} \cap [1, Mk_{n-1}]\right)}{\#[1, Mk_{n-1}]} \leq \frac{1}{n} + \frac{1}{M}.$$

Thus, for $M > n$,

$$\frac{N(Mk_{n-1})}{Mk_{n-1}} = \frac{\#\left(\mathcal{A} \cap [1, Mk_{n-1}]\right)}{\#[1, Mk_{n-1}]} \leq \frac{2}{n}.$$

Hence, as required

$$\lim_{x \to \infty} N(x)/x = 0.$$