The 67th William Lowell Putnam Mathematical Competition
Saturday, December 2, 2006

A1 Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

A2 Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

A3 Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

A4 Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ for $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$ for $k = n$.

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

A5 Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \ldots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

A6 Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

B1 Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

B2 Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

B3 Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

B4 Let Z denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of Z that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z$.) [Editorial note: the proposers probably intended to write $Z(V)$ for $V \cap Z$, but this changes nothing.]

B5 For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\, dx$ and $J(x) = \int_0^1 x\, (f(x))^2\, dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

B6 Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$

Solutions to the 67th William Lowell Putnam Mathematical Competition
Saturday, December 2, 2006 (last updated December 3)

Kiran Kedlaya and Lenny Ng

A1 We change to cylindrical coordinates, i.e., we put $r = \sqrt{x^2+y^2}$. Then the given inequality is equivalent to

$$r^2 + z^2 + 8 \leq 6r,$$

or

$$(r-3)^2 + z^2 \leq 1.$$

This defines a solid of revolution (a solid torus); the area being rotated is the disc $(x-3)^2 + z^2 \leq 1$ in the xz-plane. By Pappus's theorem, the volume of this equals the area of this disc, which is π, times the distance through which the center of mass is being rotated, which is $(2\pi)3$. That is, the total volume is $6\pi^2$.

A2 Suppose on the contrary that the set B of values of n for which Bob has a winning strategy is finite. Then for every nonnegative integer m not in S, Alice must have some move on a heap of n stones leading to a position in which the second player wins. That is, every nonnegative integer not in B can be written as $b+p-1$ for some $b \in B$ and some prime p.

However, there are numerous ways to show that this cannot happen. For instance, suppose $B = \{b_1, \dots, b_m\}$. Let $p_1, \dots, p_{2m}$ be any prime numbers; then by the Chinese remainder theorem, there exists a positive integer x such that

$$\begin{aligned} x - b_1 &\equiv -1 \pmod{p_1 p_{m+1}} \\ &\dots \\ x - b_n &\equiv -1 \pmod{p_m p_{2m}}. \end{aligned}$$

For each $b \in B$, the unique integer p such that $x = b+p-1$ is divisible by at least two primes, and so cannot itself be prime.

A3 We first observe that given any sequence of integers $x_1, x_2, \dots$ satisfying a recursion

$$x_k = f(x_{k-1}, \dots, x_{k-n}) \qquad (k > n),$$

where n is fixed and f is a fixed polynomial of n variables with integer coefficients, for any positive integer N, the sequence modulo N is eventually periodic. This is simply because there are only finitely many possible sequences of n consecutive values modulo N, and once such a sequence is repeated, every subsequent value is repeated as well.

We next observe that if one can rewrite the same recursion as

$$x_{k-n} = g(x_{k-n+1}, \dots, x_k) \qquad (k > n),$$

where g is also a polynomial with integer coefficients, then the sequence extends uniquely to a doubly infinite sequence $\dots, x_{-1}, x_0, x_1, \dots$ which is fully periodic modulo any N. That is the case in the situation at hand, because we can rewrite the given recursion as

$$x_{k-2005} = x_{k+1} - x_k.$$

It thus suffices to find 2005 consecutive terms divisible by N in the doubly infinite sequence, for any fixed N (so in particular for $N = 2006$). Running the recursion backwards, we easily find

$$\begin{aligned} x_1 = x_0 = \cdots = x_{-2004} &= 1 \\ x_{-2005} = \cdots = x_{-4009} &= 0, \end{aligned}$$

yielding the desired result.

A4 By the linearity of expectation, the average number of local maxima is equal to the sum of the probability of having a local maximum at k over $k = 1, \dots, n$. For $k = 1$, this probability is 1/2: given the pair $\{\pi(1), \pi(2)\}$, it is equally likely that $\pi(1)$ or $\pi(2)$ is bigger. Similarly, for $k = n$, the probability is 1/2. For $1 < k < n$, the probability is 1/3: given the pair $\{\pi(k-1), \pi(k), \pi(k+1)\}$, it is equally likely that any of the three is the largest. Thus the average number of local maxima is

$$2 \cdot \frac{1}{2} + (n-2) \cdot \frac{1}{3} = \frac{n+1}{3}.$$

A5 Since the desired expression involves symmetric functions of $a_1, \dots, a_n$, we start by finding a polynomial with $a_1, \dots, a_n$ as roots. Note that

$$1 \pm i \tan\theta = e^{\pm i\theta} \sec\theta$$

so that

$$1 + i\tan\theta = e^{2i\theta}(1 - i\tan\theta).$$

Consequently, if we put $\omega = e^{2in\theta}$, then the polynomial

$$Q_n(x) = (1+ix)^n - \omega(1-ix)^n$$

has among its roots $a_1, \dots, a_n$. Since these are distinct and Q_n has degree n, these must be exactly the roots.

If we write

$$Q_n(x) = c_n x^n + \cdots + c_1 x + c_0,$$

then $a_1 + \cdots + a_n = -c_{n-1}/c_n$ and $a_1 \cdots a_n = -c_0/c_n$, so the ratio we are seeking is c_{n-1}/c_0. By inspection,

$$\begin{aligned} c_{n-1} &= ni^{n-1} - \omega n(-i)^{n-1} = ni^{n-1}(1-\omega) \\ c_0 &= 1 - \omega \end{aligned}$$

so

$$\frac{a_1 + \cdots + a_n}{a_1 \cdots a_n} = \begin{cases} n & n \equiv 1 \pmod 4 \\ -n & n \equiv 3 \pmod 4). \end{cases}$$

Remark: The same argument shows that the ratio between any two *odd* elementary symmetric functions of $a_1, \dots, a_n$ is independent of θ.

A6 **First solution:** (by Daniel Kane) The probability is $1 - \frac{35}{12\pi^2}$. We start with some notation and simplifications. For simplicity, we assume without loss of generality that the circle has radius 1. Let E denote the expected value of a random variable over all choices of P, Q, R. Write $[XYZ]$ for the area of triangle XYZ.

If P, Q, R, S are the four points, we may ignore the case where three of them are collinear, as this occurs with probability zero. Then the only way they can fail to form the vertices of a convex quadrilateral is if one of them lies inside the triangle formed by the other three. There are four such configurations, depending on which point lies inside the triangle, and they are mutually exclusive. Hence the desired probability is 1 minus four times the probability that S lies inside triangle PQR. That latter probability is simply $E([PQR])$ divided by the area of the disc.

Let O denote the center of the circle, and let P', Q', R' be the projections of P, Q, R onto the circle from O. We can write

$$[PQR] = \pm[OPQ] \pm [OQR] \pm [ORP]$$

for a suitable choice of signs, determined as follows. If the points P', Q', R' lie on no semicircle, then all of the signs are positive. If P', Q', R' lie on a semicircle in that order and Q lies inside the triangle OPR, then the sign on $[OPR]$ is positive and the others are negative. If P', Q', R' lie on a semicircle in that order and Q lies outside the triangle OPR, then the sign on $[OPR]$ is negative and the others are positive.

We first calculate

$$E([OPQ] + [OQR] + [ORP]) = 3E([OPQ]).$$

Write $r_1 = OP, r_2 = OQ, \theta = \angle POQ$, so that

$$[OPQ] = \frac{1}{2} r_1 r_2 (\sin\theta).$$

The distribution of r_1 is given by $2r_1$ on $[0,1]$ (e.g., by the change of variable formula to polar coordinates), and similarly for r_2. The distribution of θ is uniform on $[0, \pi]$. These three distributions are independent; hence

$$\begin{aligned} &E([OPQ]) \\ &= \frac{1}{2}\left(\int_0^1 2r^2\,dr\right)^2 \left(\frac{1}{\pi}\int_0^\pi \sin(\theta)\,d\theta\right) \\ &= \frac{4}{9\pi}, \end{aligned}$$

and

$$E([OPQ] + [OQR] + [ORP]) = \frac{4}{3\pi}.$$

We now treat the case where P', Q', R' lie on a semicircle in that order. Put $\theta_1 = \angle POQ$ and $\theta_2 = \angle QOR$; then the distribution of θ_1, θ_2 is uniform on the region

$$0 \leq \theta_1, \quad 0 \leq \theta_2, \quad \theta_1 + \theta_2 \leq \pi.$$

In particular, the distribution on $\theta = \theta_1 + \theta_2$ is $\frac{2\theta}{\pi^2}$ on $[0, \pi]$. Put $r_P = OP, r_Q = OQ, r_R = OR$. Again, the distribution on r_P is given by $2r_P$ on $[0,1]$, and similarly for r_Q, r_R; these are independent from each other and from the joint distribution of θ_1, θ_2. Write $E'(X)$ for the expectation of a random variable X restricted to this part of the domain.

Let χ be the random variable with value 1 if Q is inside triangle OPR and 0 otherwise. We now compute

$$\begin{aligned} &E'([OPR]) \\ &= \frac{1}{2}\left(\int_0^1 2r^2\,dr\right)^2 \left(\int_0^\pi \frac{2\theta}{\pi^2}\sin(\theta)\,d\theta\right) \\ &= \frac{4}{9\pi} \\ &E'(\chi[OPR]) \\ &= E'(2[OPR]^2/\theta) \\ &= \frac{1}{2}\left(\int_0^1 2r^3\,dr\right)^2 \left(\int_0^\pi \frac{2\theta}{\pi^2}\theta^{-1}\sin^2(\theta)\,d\theta\right) \\ &= \frac{1}{8\pi}. \end{aligned}$$

Also recall that given any triangle XYZ, if T is chosen uniformly at random inside XYZ, the expectation of $[TXY]$ is the area of triangle bounded by XY and the centroid of XYZ, namely $\frac{1}{3}[XYZ]$.

Let χ be the random variable with value 1 if Q is inside triangle OPR and 0 otherwise. Then

$$\begin{aligned} &E'([OPQ] + [OQR] + [ORP] - [PQR]) \\ &= 2E'(\chi([OPQ] + [OQR]) + 2E'((1-\chi)[OPR]) \\ &= 2E'(\frac{2}{3}\chi[OPR]) + 2E'([OPR]) - 2E'(\chi[OPR]) \\ &= 2E'([OPR]) - \frac{2}{3}E'(\chi[OPR]) = \frac{29}{36\pi}. \end{aligned}$$

Finally, note that the case when P', Q', R' lie on a semicircle in some order occurs with probability $3/4$. (The case where they lie on a semicircle proceeding clockwise from P' to its antipode has probability 1/4; this case and its two analogues are exclusive and exhaustive.) Hence

$$\begin{aligned}
&E([PQR]) \\
&= E([OPQ] + [OQR] + [ORP]) \\
&\quad - \frac{3}{4}E'([OPQ] + [OQR] + [ORP] - [PQR]) \\
&= \frac{4}{3\pi} - \frac{29}{48\pi} = \frac{35}{48\pi},
\end{aligned}$$

so the original probability is

$$1 - \frac{4E([PQR])}{\pi} = 1 - \frac{35}{12\pi^2}.$$

Second solution: (by David Savitt) As in the first solution, it suffices to check that for P, Q, R chosen uniformly at random in the disc, $E([PQR]) = \frac{35}{48\pi}$. Draw the lines PQ, QR, RP, which with probability 1 divide the interior of the circle into seven regions. Put $a = [PQR]$, let b_1, b_2, b_3 denote the areas of the three other regions sharing a side with the triangle, and let c_1, c_2, c_3 denote the areas of the other three regions. Put $A = E(a)$, $B = E(b_1)$, $C = E(c_1)$, so that $A + 3B + 3C = \pi$.

Note that $c_1 + c_2 + c_3 + a$ is the area of the region in which we can choose a fourth point S so that the quadrilateral $PQRS$ fails to be convex. By comparing expectations, we have $3C + A = 4A$, so $A = C$ and $4A + 3B = \pi$.

We will compute $B + 2A = B + 2C$, which is the expected area of the part of the circle cut off by a chord through two random points D, E, on the side of the chord not containing a third random point F. Let h be the distance from the center O of the circle to the line DE. We now determine the distribution of h.

Put $r = OD$; the distribution of r is $2r$ on $[0, 1]$. Without loss of generality, suppose O is the origin and D lies on the positive x-axis. For fixed r, the distribution of h runs over $[0, r]$, and can be computed as the area of the infinitesimal region in which E can be chosen so the chord through DE has distance to O between h and $h + dh$, divided by π. This region splits into two symmetric pieces, one of which lies between chords making angles of $\arcsin(h/r)$ and $\arcsin((h + dh)/r)$ with the x-axis. The angle between these is $d\theta = dh/(r^2 - h^2)$. Draw the chord through D at distance h to O, and let L_1, L_2 be the lengths of the parts on opposite sides of D; then the area we are looking for is $\frac{1}{2}(L_1^2 L_2^2) d\theta$. Since

$$\{L_1, L_2\} = \sqrt{1 - h^2} \pm \sqrt{r^2 - h^2},$$

the area we are seeking (after doubling) is

$$2\frac{1 + r^2 - 2h^2}{\sqrt{r^2 - h^2}}.$$

Dividing by π, then integrating over r, we compute the distribution of h to be

$$\begin{aligned}
&\frac{1}{pi}\int_h^1 2\frac{1 + r^2 - 2h^2}{\sqrt{r^2 - h^2}} 2r\,dr \\
&= \frac{16}{3\pi}(1 - h^2)^{3/2}.
\end{aligned}$$

We now return to computing $E(B + 2A)$ Let $A(h)$ denote the smaller of the two areas of the disc cut off by a chord at distance h. The chance that the third point is in the smaller (resp. larger) portion is $A(h)/\pi$ (resp. $1 - A(h)/\pi$), and then the area we are trying to compute is $\pi - A(h)$ (resp. $A(h)$). Using the distribution on h, and the fact that

$$\begin{aligned}
A(h) &= 2\int_h^1 \sqrt{1 - h^2}\,dh \\
&= \frac{\pi}{2} - \arcsin(h) - h\sqrt{1 - h^2},
\end{aligned}$$

we find

$$\begin{aligned}
B + 2A &= \frac{2}{\pi}\int_0^1 A(h)(\pi - A(h))\,\frac{16}{3\pi}(1 - h^2)^{3/2}\,dh \\
&= \frac{35 + 24\pi^2}{72\pi}.
\end{aligned}$$

Since $4A + 3B = \pi$, we solve to obtain $A = \frac{35}{48\pi}$ as in the first solution.

Remark: This is one of the oldest problems in geometric probability; it is an instance of Sylvester's four-point problem, which nowadays is usually solved using a device known as Crofton's formula. We defer to `http://mathworld.wolfram.com/` for further discussion.

B1 The "curve" $x^3 + 3xy + y^3 - 1 = 0$ is actually reducible, because the left side factors as

$$(x + y - 1)(x^2 - xy + y^2 + x + y + 1).$$

Moreover, the second factor is

$$\frac{1}{2}((x + 1)^2 + (y + 1)^2 + (x + y)^2),$$

so it only vanishes at $(-1, -1)$. Thus the curve in question consists of the single point $(-1, -1)$ together with the line $x + y = 1$. To form a triangle with three points on this curve, one of its vertices must be $(-1, -1)$. The other two vertices lie on the line $x + y = 1$, so the length of the altitude from $(-1, -1)$ is the distance from $(-1, -1)$ to $(1/2, 1/2)$, or $3\sqrt{2}/2$. The area of an equilateral triangle of height h is $h^2\sqrt{3}/6$, so the desired area is $3\sqrt{3}/4$.

B2 Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. For $i = 0, \dots, n$, put $s_i = x_1 + \cdots + x_i$ (so that $s_0 = 0$). Sort the numbers $\{s_0\}, \dots, \{s_n\}$ into ascending order, and call the result $t_0, \ldots, t_n$. Since $0 = t_0 \leq \cdots \leq t_n < 1$, the differences

$$t_0, t_1 - t_0, \ldots, t_n - t_{n-1}$$

add up to no more than 1. By the pigeonhole principle, one of these differences is no more than $1/(n+1)$; it equals $\pm(\{s_i\} - \{s_j\})$ for some $0 \leq i < j \leq n$. Put $S = \{s_{i+1}, \ldots, s_j\}$ and $m = \lfloor s_i \rfloor - \lfloor s_j \rfloor$; then

$$\begin{aligned} \left| m + \sum_{s \in S} s \right| &= |m + s_j - s_i| \\ &= |\{s_j\} - \{s_i\}| \\ &\leq \frac{1}{n+1}, \end{aligned}$$

as desired.

B3 The maximum is $\binom{n}{2} + 1$. If the maximum value of L_S is achieved by some S, it is also achieved by any nearby configuration. Hence by varying the points slightly, we can always achieve a maximal configuration in which no two of the lines joining points of S are parallel. It suffices to prove that in this case L_S is always equal to $\binom{n}{2} + 1$, which will then be the desired maximum in any configuration. For convenience, we assume $n \geq 3$, as the cases $n = 1, 2$ are easy.

Let P be the line at infinity in the real projective plane; i.e., P is the set of possible directions of lines in the plane, viewed as a circle. Remove the directions corresponding to lines through two points of S; this leaves behind $\binom{n}{2}$ intervals.

Given a direction in one of the intervals, consider the set of linear partitions achieved by lines parallel to that direction. Note that the resulting collection of partitions depends only on the interval. Then note that the collections associated to adjacent intervals differ in only one element.

The trivial partition that puts all of S on one side is in every such collection. We now observe that for any other linear partition $\{A, B\}$, the set of intervals to which $\{A, B\}$ is:

(a) a consecutive block of intervals, but
(b) not all of them.

For (a), note that if ℓ_1, ℓ_2 are nonparallel lines achieving the same partition, then we can rotate around their point of intersection to achieve all of the intermediate directions on one side or the other. For (b), the case $n = 3$ is evident; to reduce the general case to this case, take points P, Q, R such that P lies on the opposite side of the partition from Q and R.

It follows now that that each linear partition, except for the trivial one, occurs in exactly one place as the partition associated to some interval but not to its immediate counterclockwise neighbor. In other words, the number of linear partitions is one more than the number of intervals, or $\binom{n}{2} + 1$ as desired.

B4 The maximum is 2^k, achieved for instance by the subspace

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = \cdots = x_{n-k} = 0\}.$$

First solution: More generally, we show that any affine k-dimensional plane in $\mathbb{R}^n$ can contain at most 2^k points in Z. The proof is by induction on $k + n$; the case $k = n = 0$ is clearly true.

Suppose that V is a k-plane in $\mathbb{R}^n$. Denote the hyperplanes $\{x_n = 0\}$ and $\{x_n = 1\}$ by V_0 and V_1, respectively. If $V \cap V_0$ and $V \cap V_1$ are each at most $(k-1)$-dimensional, then $V \cap V_0 \cap Z$ and $V \cap V_1 \cap Z$ each have cardinality at most 2^{k-1} by the induction assumption, and hence $V \cap Z$ has at most 2^k elements. Otherwise, if $V \cap V_0$ or $V \cap V_1$ is k-dimensional, then $V \subset V_0$ or $V \subset V_1$; now apply the induction hypothesis on V, viewed as a subset of $\mathbb{R}^{n-1}$ by dropping the last coordinate.

Second solution: Let S be a subset of Z contained in a k-dimensional subspace of V. This is equivalent to asking that any $t_1, \ldots, t_{k+1} \in S$ satisfy a nontrivial linear dependence $c_1 t_1 + \cdots + c_{k+1} t_{k+1} = 0$ with $c_1, \ldots, c_{k+1} \in \mathbb{R}$. Since $t_1, \ldots, t_{k+1} \in \mathbb{Q}^n$, given such a dependence we can always find another one with $c_1, \ldots, c_{k+1} \in \mathbb{Q}$; then by clearing denominators, we can find one with $c_1, \ldots, c_{k+1} \in \mathbb{Z}$ and not all having a common factor.

Let $\mathbb{F}_2$ denote the field of two elements, and let $\overline{S} \subseteq \mathbb{F}_2^n$ be the reductions modulo 2 of the points of S. Then any $t_1, \ldots, t_{k+1} \in \overline{S}$ satisfy a nontrivial linear dependence, because we can take the dependence from the end of the previous paragraph and reduce modulo 2. Hence $\overline{S}$ is contained in a k-dimensional subspace of $\mathbb{F}_{2^n}$, and the latter has cardinality exactly 2^k. Thus $\overline{S}$ has at most 2^k elements, as does S.

B5 The answer is $1/16$. We have

$$\begin{aligned} &\int_0^1 x^2 f(x)\,dx - \int_0^1 x f(x)^2\,dx \\ &= \int_0^1 (x^3/4 - x(f(x) - x/2)^2)\,dx \\ &\leq \int_0^1 x^3/4\,dx = 1/16, \end{aligned}$$

with equality when $f(x) = x/2$.

B6 We start with some easy upper and lower bounds on a_n. We write $O(f(n))$ and $\Omega(f(n))$ for functions $g(n)$ such that $f(n)/g(n)$ and $g(n)/f(n)$, respectively, are bounded above. Since a_n is a nondecreasing sequence, $a_{n+1} - a_n$ is bounded above, so $a_n = O(n)$. That means $a_n^{-1/k} = \Omega(n^{-1/k})$, so

$$a_n = \Omega\left(\sum_{i=1}^n i^{-1/k}\right) = \Omega(n^{(k-1)/k}).$$

Write $b_n = a_n - Ln^{k/(k+1)}$, for a value of L to be determined later. We have

$$\begin{aligned} &b_{n+1} \\ &= b_n + a_n^{-1/k} - L((n+1)^{k/(k+1)} - n^{k/(k+1)}) \\ &= e_1 + e_2, \end{aligned}$$

where

$$\begin{aligned} e_1 &= b_n + a_n^{-1/k} - L^{-1/k} n^{-1/(k+1)} \\ e_2 &= L((n+1)^{k/(k+1)} - n^{k/(k+1)}) \\ &\quad - L^{-1/k} n^{-1/(k+1)}. \end{aligned}$$

We first estimate e_1. For $-1 < m < 0$, by the convexity of $(1+x)^m$ and $(1+x)^{1-m}$, we have

$$\begin{aligned} 1 + mx &\leq (1+x)^m \\ &\leq 1 + mx(1+x)^{m-1}. \end{aligned}$$

Hence

$$\begin{aligned} -\frac{1}{k} L^{-(k+1)/k} n^{-1} b_n &\leq e_1 - b_n \\ &\leq -\frac{1}{k} b_n a_n^{-(k+1)/k}. \end{aligned}$$

Note that both bounds have sign opposite to b_n; moreover, by the bound $a_n = \Omega(n^{(k-1)/k})$, both bounds have absolutely value strictly less than that of b_n for n sufficiently large. Consequently, for n large,

$$|e_1| \leq |b_n|.$$

We now work on e_2. By Taylor's theorem with remainder applied to $(1+x)^m$ for $x > 0$ and $0 < m < 1$,

$$\begin{aligned} 1 + mx &\geq (1+x)^m \\ &\geq 1 + mx + \frac{m(m-1)}{2} x^2. \end{aligned}$$

The "main term" of $L((n+1)^{k/(k+1)} - n^{k/(k+1)})$ is $L\frac{k}{k+1} n^{-1/(k+1)}$. To make this coincide with $L^{-1/k} n^{-1/(k+1)}$, we take

$$L = \left(\frac{k}{k+1}\right)^{k/(k+1)}.$$

We then find that

$$|e_2| = O(n^{-2}),$$

and because $b_{n+1} = e_1 + e_2$, we have $|b_{n+1}| \leq |b_n| + |e_2|$. Hence

$$|b_n| = O\left(\sum_{i=1}^{n} i^{-2}\right) = O(n^{-1}),$$

and so

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k} = L^{k+1} = \left(\frac{k}{k+1}\right)^k.$$

Remark: One can make a similar argument for any sequence given by $a_{n+1} = a_n + f(a_n)$, when f is a *decreasing* function.

Remark: Richard Stanley suggests a heuristic for determining the asymptotic behavior of sequences of this type: replace the given recursion

$$a_{n+1} - a_n = a_n^{-1/k}$$

by the differential equation

$$y' = y^{-1/k}$$

and determine the asymptotics of the latter.