1

INTRODUCTORY PROBLEMS

1. Every living person has shaken hands with a certain number of other persons. Prove that a count of the number of people who have shaken hands an odd number of times must yield an even number.

2. In chess, is it possible for the knight to go (by allowable moves) from the lower left-hand corner of the board to the upper right-hand corner and in the process to light exactly once on each square?

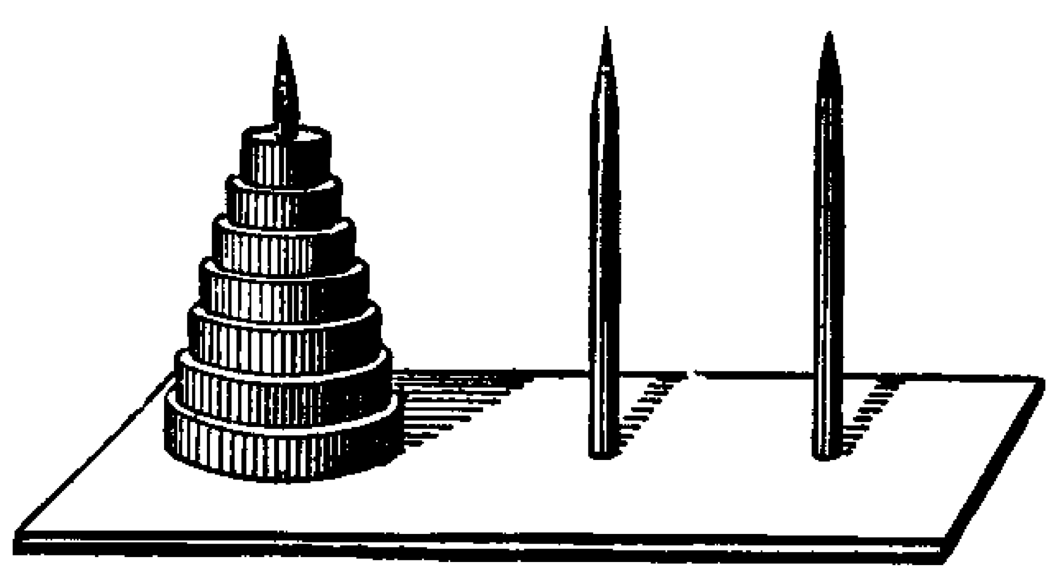

Figure 1

3. (a) N rings having different outer diameters are slipped onto an upright peg, the largest ring on the bottom, to form a pyramid (Figure 1). We wish to transfer all the rings, one at a time, to a

second peg, but we have a third (auxiliary) peg at our disposal. During the transfers it is not permitted to place a larger ring on a smaller one. What is the smallest number, k, of moves necessary to complete the transfer to peg number 2?†

(b)* A brain-teaser called the game of Chinese Rings is constructed as follows: n rings of the same size are each connected to a plate by a series of wires, all of which are the same length (see Figure 2). A thin, doubled rod is slipped through the rings in such a way that all the wires are inside the U-opening of the rod. (The wires are free to slide in holes in the plate, as shown.) The problem consists of removing all the rings from the rod. What is the least number of moves necessary to do this?

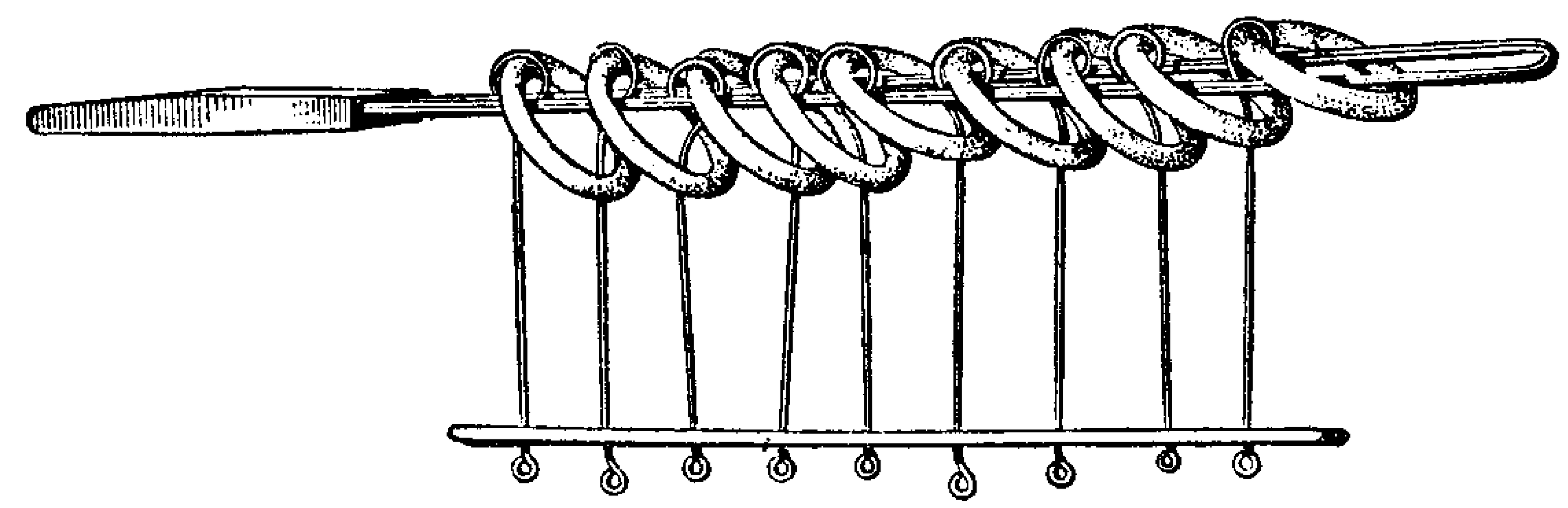

Figure 2

4. (a) We are given 80 coins of the same denomination; we know that one of them is counterfeit and that it is lighter than the others. Locate the counterfeit coin by using four weighings on a pan balance.

(b) It is known that there is one counterfeit coin in a collection of n similar coins. What is the least number of weight trials necessary to identify the counterfeit?

5. Twenty metal blocks are of the same size and external appearance; some are aluminum, and the rest are duraluminum, which is heavier. Using at most eleven weighings on a pan balance, how can we determine how many blocks are aluminum?

6. (a)* Among twelve similar coins there is one counterfeit. It is not known whether the counterfeit coin is lighter or heavier than a genuine one (all genuine coins weigh the same). Using three weighings on a pan balance, how can the counterfeit be identified and in the process determined to be lighter or heavier than a genuine coin?

† This is sometimes referred to as the Tower of Hanoi problem [*Editor*].

(b)** There is one counterfeit coin among 1000 similar coins. It is not known whether the counterfeit coin is lighter or heavier than a genuine one. What is the least number of weighings, on a pan balance, necessary to locate the counterfeit and to determine whether it is light or heavy?

Remark: Using the conditions of problem (a) it is possible to locate, in three weighings, one counterfeit out of thirteen coins, but we cannot determine whether it is light or heavy. For fourteen coins, four weighings are necessary. It would be interesting to determine the least number of weighings necessary to locate one counterfeit out of 1000 coins if we are relieved of the necessity of determining whether it is light or heavy.

7. (a) A traveler having no money, but owning a gold chain having seven links, is accepted at an inn on the condition that he pay one link per day for his stay. If the traveler is to pay daily, but may take change in the form of links previously paid, and if he remains seven days, what is the least number of links that must be cut out of the chain? (*Note*: A link may be taken from any part of the chain.)

(b) A chain consists of 2000 links. What is the least number of links that must be disengaged from the chain in order that any specified number of links, from 1 to 2000, may be gathered together from the parts of the chain thus formed?

8. Two-hundred students are positioned in 10 rows, each containing 20 students. From each of the 20 columns thus formed the shortest student is selected, and the tallest of these 20 (short) students is tagged A. These students now return to their initial places. Next the tallest student in each row is selected, and from these 10 (tall) students the shortest is tagged B. Which of the two tagged students is the taller (if they are different people)?

9. Given thirteen gears, each weighing an integral number of grams. It is known that any twelve of them may be placed on a pan balance, six on each pan, in such a way that the scale will be in equilibrium. Prove that all the gears must be of equal weight.

10. Refer to the following number triangle.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 3 & 2 & 1 & & \\
 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Each number is the sum of three numbers of the previous row: the number immediately above it and the numbers immediately to the right and left of that one. If no number appears in one or more of these locations, the number zero is used. Prove that every row, beginning with the third row, contains at least one even number.

11. Twelve squares are laid out in a circular pattern [as on the circumference of a circle]. Four different colored chips, red, yellow, green, blue, are placed on four consecutive squares. A chip may be moved in either a clockwise or a counterclockwise direction over four other squares to a fifth square, provided that the fifth square is not occupied by a chip. After a certain number of moves the same four squares will again be occupied by chips. How many permutations (rearrangements) of the four chips are possible as a result of this process?

12. An island is inhabited by five men and a pet monkey. One afternoon the men gathered a large pile of coconuts, which they proposed to divide equally among themselves the next morning. During the night one of the men awoke and decided to help himself to his share of the nuts. In dividing them into five equal parts he found that there was one nut left over. This one he gave to the monkey. He then hid his one-fifth share, leaving the rest in a single pile. Later during the night another man awoke with the same idea in mind. He went to the pile, divided it into five equal parts, and found that there was one coconut left over. This he gave to the monkey, and then he hid his one-fifth share, restoring the rest to one pile. During the same night each of the other three men arose, one at a time, and in ignorance of what had happened previously, went to the pile, and followed the same procedure. Each time one coconut was left over, and it was given to the monkey. The next morning all five men went to the diminished nut pile and divided it into five equal parts, finding that one nut remained over. What is the least number of coconuts the original pile could have contained?

13. Two brothers sold a herd of sheep which they owned. For each sheep they received as many rubles as the number of sheep originally in the herd. The money was then divided in the following manner. First, the older brother took ten rubles, then the younger brother took ten rubles, after which the older brother took another ten rubles, and so on. At the end of the division the younger brother, whose turn it was, found that there were fewer than ten rubles left, so he took what remained. To make the division just, the older brother gave the younger his penknife. How much was the penknife worth?

14.* (a) On which of the two days of the week, Saturday or Sunday, does New Year's Day fall more often?

(b) On which day of the week does the thirtieth of the month most often fall?

2

ALTERATIONS OF DIGITS IN INTEGERS

15. Which integers have the following property? If the final digit is deleted, the integer is divisible by the new number.

16. (a) Find all integers with initial digit 6 which have the following property, that if this initial digit is deleted, the resulting number is reduced to $\frac{1}{25}$ its original value.

(b) Prove that there does not exist any integer with the property that if its first digit is deleted, the resulting number is $\frac{1}{35}$ the original number.

17.* An integer is reduced to $\frac{1}{9}$ its value when a certain one of its digits is deleted, and the resulting number is again divisible by 9.

(a) Prove that division of this resulting integer by 9 results in deleting an additional digit.

(b) Find all integers satisfying the conditions of the problem.

18. (a) Find all integers having the property that when the third digit is deleted the resulting number divides the original one.

(b)* Find all integers with the property that when the second digit is deleted the resulting number divides the original one.

19. (a) Find the smallest integer whose first digit is 1 and which has the property that if this digit is transferred to the end of the number the number is tripled. Find all such integers.

(b) With what digits is it possible to begin a (nonzero) integer such that the integer will be tripled upon the transfer of the initial digit to the end? Find all such integers.

20. Prove that there does not exist a natural number which, upon transfer of its initial digit to the end, is increased five, six, or eight times.

21. Prove that there does not exist an integer which is doubled when the initial digit is transferred to the end.

22. (a) Prove that there does not exist an integer which becomes either seven times or nine times as great when the initial digit is transferred to the end.

(b) Prove that no integer becomes four times as great when its initial digit is transferred to the end.

23. Find the least integer whose first digit is seven and which is reduced to $\frac{1}{3}$ its original value when its first digit is tranferred to the end. Find all such integers.

24. (a) We say one integer is the "inversion" of another if it consists of the same digits written in reverse order. Prove that there exists no natural number whose inversion is two, three, five, seven, or eight times that number.

(b) Find all integers whose inversions are four or nine times the original number.

25. (a) Find a six-digit number which is multiplied by a factor of 6 if the final three digits are removed and placed (without changing their order) at the beginning.

(b) Prove that there cannot exist an eight-digit number which is increased by a factor of 6 when the final four digits are removed and placed (without changing their order) at the beginning.

26. Find a six-digit number whose product by 2, 3, 4, 5, or 6 contains the same digits as did the original number (in different order, of course).

3

THE DIVISIBILITY OF INTEGERS

27. Prove that for every integer n:
(a) $n^3 - n$ is divisible by 3;
(b) $n^5 - n$ is divisible by 5;
(c) $n^7 - n$ is divisible by 7;
(d) $n^{11} - n$ is divisible by 11;
(e) $n^{13} - n$ is divisible by 13.

Note: Observe that $n^9 - n$ is not necessarily divisible by 9 (for example, $2^9 - 2 = 510$ is not divisible by 9).

Problems (a-e) are special cases of a general theorem; see problem 240.

28. Prove the following:
(a) $3^{6n} - 2^{6n}$ is divisible by 35, for every positive integer n;
(b) $n^5 - 5n^3 + 4n$ is divisible by 120, for every integer n;
(c)* for all integers m and n, $mn(m^{60} - n^{60})$ is divisible by the number 56,786,730.

29. Prove that $n^2 + 3n + 5$ is never divisible by 121 for any posi-

† For a discussion of the general concepts involved in the solution of the majority of the problems in this section, see the book by B. B. Dynkin and V. A. Uspensky, *Mathematical Conversations*, Issue 6, Section 2, "Problems in Number Theory," Library of the USSR Mathematical Society.

tive integer n.

30. Prove that the expression

$$m^5 + 3m^4n - 5m^3n^2 - 15m^2n^3 + 4mn^4 + 12n^5$$

cannot have the value 33, regardless of what integers are substituted for m and n.

31. What remainders can result when the 100th power of an integer is divided by 125?

32. Prove that if an integer n is relatively prime to 10, the 101st power of n ends with the same three digits as does n. (For example, 1233^{101} ends with the digits 233, and 37^{101} ends with the digits 037.)

33. Find a three-digit number all of whose integral powers end with the same three digits as does the original number.

34. Let N be an even number not divisible by 10. What digit will be in the tens place of the number N^{20}, and what digit will be in the hundreds place of N^{200}?

35. Prove that the sum

$$1^k + 2^k + 3^k + \cdots + n^k ,$$

where n is an arbitrary integer and k is odd, is divisible by $1 + 2 + 3 + \cdots + n$.

36. Give a criterion that a number be divisible by 11.

37. The number 123456789(10)(11)(12)(13)(14) is written in the base 15—that is, the number is equal (in the base 10) to

$$14 + (13)\cdot 15 + (12)\cdot 15^2 + (11)\cdot 15^3 + \cdots + 2\cdot 15^{12} + 15^{13} .$$

What is the remainder upon dividing the number by 7?

38. Prove that 1, 3, and 9 are the only numbers K having the property that if K divides a number N, it also divides every number obtained by permuting the digits of N. (For $K = 1$, the condition given is trivial; for $K = 3$, or 9, the condition follows from the well-known fact that a number is divisible by 3, or 9, if and only if the sum of its digits is divisible by 3, or 9.)

39. Prove that $27{,}195^8 - 10{,}887^8 + 10{,}152^8$ is exactly divisible by 26,460.

40. Prove that $11^{10} - 1$ is divisible by 100.

41. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

42. Prove thas a number consisting of 3^n identical digits is divisible by 3^n. (For example, the number 222 is divisible by 3, the number 777,777,777, is divisible by 9, and so on).

43. Find the remainder upon dividing the following number by 7:

$$10^{10} + 10^{(10^2)} + \cdots + 10^{(10^{10})} .$$

44. (a) Find the final digit of the numbers $9^{(9^9)}$ and $2^{(3^4)}$.
(b) Find the final two digits of the numbers 2^{999} and 3^{999}.
(c)* Find the final two digits of the number $14^{(14^{14})}$.

45. (a) What is the final digit of the number

$$(\ldots(((7^7)^7)^7)^{\cdots 7})$$

(where the 7th power is taken 1000 times)? What are the final two digits?

(b) What is the final digit of the number

$$7^{\left(7^{\cdot^{\cdot^{\cdot^{\left(7^{(7^7)}\right)}}}}\cdots\right)} ,$$

which contains 1001 sevens, as does the number given in problem (a), but with the exponents used differently? What are the final two digits of this number?

46.* Determine the final five digits of the number

$$N = 9^{\left(9^{\cdot^{\cdot^{\cdot^{\left(9^{(9^9)}\right)}}}}\cdots\right)} ,$$

which contains 1001 nines, positioned as shown.

47.* Find the last 1000 digits of the number

$$N = 1 + 50 + 50^2 + 50^3 + \cdots + 50^{999} .$$

48. How many zeros terminate the number which is the product of all the integers from 1 to 100, inclusive?

Here we may use the following well-known notation:

$$1\cdot 2\cdot 3\cdot 4\cdots(n-1)\cdot n = n!$$

(called factorial n). The problem can then be stated more succinctly: How many zeros are at the end of 100!?

49. (a) Prove that the product of n consecutive integers is divisible by $n!$.

(b) Prove that if $a + b + \cdots + k \leqq n$, then the fraction

$$\frac{n!}{a!b!\cdots k!}$$

is an integer.

(c) Prove that $(n!)!$ is divisible by $n!^{(n-1)!}$.

(d)* Prove that the product of the n integers of an arithmetic progression of n terms, where the common difference is relatively prime to $n!$, is divisible by $n!$.

Note: Problem 49 (d) is a generalization of 49 (a).

50. Is the number, C_{1000}^{500}, of combinations of 1000 elements, taken 500 at a time, divisible by 7?†

51. (a) Find all numbers n between 1 and 100 having the property that $(n-1)!$ is not divisible by n.

(b) Find all numbers n between 1 and 100 having the property that $(n-1)!$ is not divisible by n^2.

52.* Find all integers n which are divisible by all integers not exceeding $\sqrt{n}$.

53. (a) Prove that the sum of the squares of five consecutive integers cannot be the square of any integer.

(b) Prove that the sum of even powers of three consecutive numbers cannot be an even power of any integer.

(c) Prove that the sum of the same even power of nine consecutive integers, the first of which exceeds 1, cannot be any integral power of any integer.

54. (a) Let A and B be two distinct seven-digit numbers, each of which contains all the digits from 1 to 7. Prove that A is not divisible by B.

(b) Using all the digits from 1 to 9, make up three, three-digit numbers which are related in the ratio $1:2:3$.

55. Which integers can have squares that end with four identical digits?

56. Prove that if two adjacent sides of a rectangle and its diagonal can be expressed in integers, then the area of the rectangle is divisible by 12.

† More "standard" notations for this are $C(1000, 500)$ or C_{500}^{1000} or $\binom{1000}{500}$. However, retention of the notation used in the original will cause no difficulty [*Editor*].

57. Prove that if all the coefficients of the quadratic equation

$$ax^2 + bx + c = 0$$

are odd integers, then the roots of the equation cannot be rational.

58. Prove that if the sum of the fractions

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$$

(where n is a positive integer) is put in decimal form, it forms a nonterminating decimal of deferred periodicity.‡

59. Prove that the following numbers (where m and n are natural numbers) cannot be integers:

(a) $$M = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n};$$

(b) $$N = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m};$$

(c) $$K = \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}.$$

60.** (a) Prove that if p is a prime number greater than 3, then the numerator of the (reduced) fraction

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}$$

is divisible by p^2. For example,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12},$$

the numerator of which is 5^2.

(b) Prove that if p is a prime number exceeding 3, then the numerator of the (reduced) fraction which is the sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(p-1)^2}$$

is divisible by p. For example,

‡ Deferred periodicity means that the periodic portion is preceded by one or more nonrepeating digits. The criterion is whether the denominator of the (reduced) fraction has a common factor with 10 [*Editor*].

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} = \frac{205}{144}$$

has a numerator which is divisible by 5.

61. Prove that the expression

$$\frac{a^3 + 2a}{a^4 + 3a^2 + 1},$$

where a is any positive integer, is a fraction in lowest terms.

62.* Let $a_1, a_2, \cdots, a_n$ be n distinct integers. Show that the product of all the fractions of form $\dfrac{a_k - a_l}{k - l}$, where $n \geqq k > l$, is an integer.

63. Prove that all numbers made up as follows,

$$10001,\ 100010001,\ 1000100010001,\ \cdots$$

(three zeros between the ones), are composite numbers.

64. (a) Divide $a^{128} - b^{128}$ by

$$(a + b)(a^2 + b^2)(a^4 + b^4)(a^8 + b^8)(a^{16} + b^{16})(a^{32} + b^{32})(a^{64} + b^{64})\,.$$

(b) Divide $a^{2^{k+1}} - b^{2^{k+1}}$ by

$$(a + b)(a^2 + b^2)(a^4 + b^4)(a^8 + b^8)\cdots(a^{2^{k-1}} + b^{2^{k+1}})(a^{2^k} + b^{2^k})\,.$$

65. Prove that any two numbers of the following sequence are relatively prime:

$$2 + 1,\ 2^2 + 1,\ 2^4 + 1,\ 2^8 + 1,\ 2^{16} + 1,\ \cdots,\ 2^{2^n} + 1,\ \cdots.$$

Remark: The result obtained here proves that there is an infinite number of primes (see also problems 159 and 253).

66. Prove that if one of the numbers $2^n - 1$ and $2^n + 1$ is prime, where $n > 2$, then the other number is composite.

67. (a) Prove that if p and $8p - 1$ are both prime, then $8p + 1$ is composite.

(b) Prove that if p and $8p^2 + 1$ are both prime, then $8p^2 - 1$ is also prime.

68. Prove that the square of every prime number greater than 3 yields a remainder of 1 when divided by 12.

69. Prove that if three prime numbers, all greater than 3, form an arithmetic progression, then the common difference of the progression is divisible by 6.

70.* (a) Ten primes, each less than 3000, form an arithmetic progression. Find these prime numbers.

(b) Prove that there do not exist eleven primes, all less than 20,000, which can form an arithmetic progression.

71. (a) Prove that, given five consecutive positive integers, it is always possible to find one which is relatively prime to all the rest.

(b) Prove that among sixteen consecutive integers it is always possible to find one which is relatively prime to all the rest.

4

SOME PROBLEMS FROM ARITHMETIC

72. The integer A consists of 666 threes, and the integer B has 666 sixes. What digits appear in the product $A \cdot B$?

73. What quotient and what remainder are obtained when the number consisting of 1001 sevens is divided by the number 1001?

74. Find the least square which commences with six twos.

75. Prove that if the number α is given by the decimal $0.999\ldots$, where there are at least 100 nines, then $\sqrt{\alpha}$ also has 100 nines at the beginning.

76. Adjoin to the digits 523... three more digits such that the resulting six-digit number is divisible by 7, 8, and 9.

77. Find a four-digit number which, on division by 131, yields a remainder of 112, and on division by 132 yields a remainder of 98.

78. (a) Prove that the sum of all the n-digit integers ($n > 2$) is equal to

$$494\underbrace{99\cdots9}_{(n-3)\text{ nines}}55\underbrace{00\cdots0}_{(n-2)\text{ zeros}}.$$

(For example, the sum of all three-digit numbers is equal to 494,550,

and the sum of all six-digit numbers is 494,999,550,000.)

(b) Find the sum of all the four-digit even numbers which can be written using $0, 1, 2, 3, 4, 5$ (and where digits can be repeated in a number).

79. How many of each of the ten digits are needed in order to write out all the integers from 1 to 100,000,000 inclusive?

80. All the integers beginning with 1 are written successively (that is, $1234567891011121314\cdots$). What digit occupies the 206,788th position?

81. Does the number $0.1234567891011121314\cdots$, which is obtained by writing successively all the integers, represent a rational number (that is, is it a periodic decimal)?

82. We are given 27 weights which weigh, respectively, $1^2, 2^2, 3^2, \cdots, 27^2$ units. Group these weights into three sets of equal weight.

83. A regular polygon is cut from a piece of cardboard. A pin is put through the center to serve as an axis about which the polygon can revolve. Find the least number of sides which the polygon can have in order that revolution through an angle of $25\frac{1}{2}$ degrees will put it into coincidence with its original position.

84. Using all the digits from 1 to 9, make up three, three-digit numbers such that their product will be:

(a) least; (b) greatest.

85. The sum of a certain number of consecutive positive integers is 1000. Find these integers.

86. (a) Prove that any number which is not a power of 2 can be represented as the sum of at least two consecutive positive integers, but that such a representation is impossible for powers of 2.

(b) Prove that any composite odd number can be represented as a sum of some number of consecutive odd numbers, but that no prime number can be represented in this form. Which even numbers can be represented as the sum of consecutive odd numbers?

(c) Prove that every power of a natural number n ($n > 1$) can be represented as the sum of n positive odd numbers.

87. Prove that the product of four consecutive integers is one less than a perfect square.

88. Given $4n$ positive integers such that if any four distinct integers are taken, it is possible to form a proportion from them. Prove that at least n of the given numbers are identical.

89.* Take four arbitrary natural numbers, A, B, C, and D. Prove that if we use them to find the four numbers A_1, B_1, C_1, and D_1, which are equal, respectively, to the differences between A and B, B and C, C and D, D and A (taking the positive difference each time), and then we repeat this process with A_1, B_1, C_1 and D_1 to obtain four other numbers A_2, B_2, C_2, and D_2, and so on, we eventually must obtain four zeros.

For example, if we begin with the numbers 32, 1, 110, 7, we obtain the following pattern:

32,	1,	110,	7,
31,	109,	103,	25,
78,	6,	78,	6,
72,	72,	72,	72,
0,	0,	0,	0.

90.* (a) Rearrange the integers from 1 to 100 in such an order that no eleven of them appear in the rearrangement (adjacently or otherwise) in either ascending or descending order.

(b) Prove that no matter what rearrangement is made with the integers from 1 to 101 it will always be possible to choose eleven of them which appear (adjacently or otherwise) in the arrangement in either an ascending or a descending order.

91. (a) From the first 200 natural numbers, 101 of them are arbitrarily chosen. Prove that among the numbers chosen there exists a pair of numbers such that one of them is divisible by the other.

(b) From the first 200 natural numbers select a set of 100 numbers such that no one of them is divisible by any other.

(c) Prove that if one of 100 numbers taken from the first 200 natural numbers is less than 16, then one of those 100 numbers is divisible by another.

92. (a) Prove that, given any 52 integers, there exist two of them whose sum, or else whose difference, is divisible by 100.

(b) Prove that out of any 100 integers, none divisible by 100, it is always possible to find two or more integers whose sum is divisible by 100.

93.* A chess master who has eleven weeks to prepare for a tournament decides to play at least one game every day, but in order not to tire himself he agrees to play not more than twelve games

during any one week. Prove that there exists a succession of days during which the master will have played exactly twenty games.

94. Let N be an arbitrary natural number. Prove that there exists a multiple of N which contains only the digits 0 and 1. Moreover, if N is relatively prime to 10 (that is, is not divisible by 2 or 5), then some multiple of N consists entirely of ones. (If N is not relatively prime to 10, then, of course, there exists no number of form $11 \cdots 1$ which is divisible by N.)

95.* Given the sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots,$$

where each number, beginning with the third, is the sum of the two preceding numbers (this is called a Fibonacci sequence). Does there exist, among the first 100,000,001 numbers of this sequence, a number terminating with four zeros?

96.* Let α be an arbitrary irrational number. Clearly, no matter which integer n is chosen, the fraction taken from the sequence $\frac{0}{n} = 0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots$, and which is closest to α, differs from α by no more than half of $1/n$. Prove that there exist n's such that the fraction closest to α differs from α by not more than $0.001\left(\frac{1}{n}\right)$.

97. Let m and n be two relatively prime natural numbers. Prove that if the $m + n - 2$ fractions

$$\frac{m+n}{m}, \quad \frac{2(m+n)}{m}, \quad \frac{3(m+n)}{m}, \quad \cdots, \quad \frac{(m-1)(m+n)}{m},$$
$$\frac{m+n}{n}, \quad \frac{2(m+n)}{n}, \quad \frac{3(m+n)}{n}, \quad \cdots, \quad \frac{(n-1)(m+n)}{n}$$

are points on the real-number axis, then precisely one of these fractions lies inside each one of the intervals $(1, 2), (2, 3), (3, 4), \cdots, (m + n - 2, m + n - 1)$ (see Figure 3, in which $m = 3$, $n = 4$).

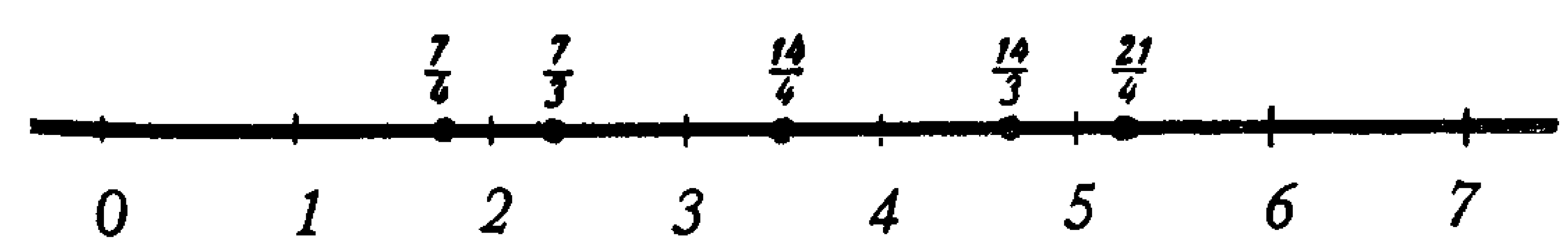

Figure 3

98.* Let $a_1, a_2, a_3, \cdots, a_n$ be n natural numbers, each less than

1000, but where the least common multiple of any two of the numbers exceeds 1000. Prove that the sum of the reciprocals of these numbers is less than 2.

99.* The fraction q/p, where $p \neq 5$ is an odd prime, is expanded as a (periodic) decimal fraction. Prove that if the number of digits appearing in the period of the decimal is even, then the arithmetic mean of these digits is 9/2 (that is, coincides with the arithmetic mean of the digits $0, 1, 2, \cdots, 9$ (this shows that the "greater" and the "lesser" digits of the period appear "equally often"). If the number of digits in the period is odd, then the arithmetic mean of these digits is different from 9/2.

100.* Prove that if the numbers of the following sequence are written as decimals,

$$\frac{a_1}{p}, \frac{a_2}{p^2}, \frac{a_3}{p^3}, \cdots, \frac{a_n}{p^n}, \cdots,$$

(where p is a prime different from 2 or 5, and where $a_1, a_2, \cdots, a_n$ are all relatively prime to p), then some (perhaps only one) of the first few decimal fractions may contain the same number of digits in their periods, but the subsequent decimal fractions of the sequence will all have p times as many digits in their periods as has the preceding term.

For example: $\frac{1}{3} = 0.\overline{3}$; $\frac{4}{9} = 0.\overline{4}$, $\frac{10}{27} = 0.\overline{370}$; $\frac{80}{81} = 0.\overline{987654320}$; $\frac{116}{243}$ has 27 digits in its period; $\frac{653}{729}$ has 81 digits in its period; and so on.

Remark: By "the greatest integer in x" we shall mean the greatest integer not exceeding x (that is, to the left of x on the number axis if x is not a whole number). This concept will be designated by the use of brackets, that is, by writing $[x]$. For example: $[2.5] = 2$, $[2] = 2$, $[-2.5] = -3$.

101. Prove the following properties of the greatest integer in a number.

(1) $[x + y] \geqq [x] + [y]$.

(2) $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$, where n is an integer.

(3) $[x] + \left[x + \frac{1}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx]$.

102.* Prove that if p and q are relatively prime natural numbers, then

$$\left[\frac{p}{q}\right] + \left[\frac{2p}{q}\right] + \left[\frac{3p}{q}\right] + \cdots + \left[\frac{(q-1)p}{q}\right]$$
$$= \left[\frac{q}{p}\right] + \left[\frac{2q}{p}\right] + \left[\frac{3q}{p}\right] + \cdots + \left[\frac{(p-1)q}{p}\right] = \frac{(p-1)(q-1)}{2}.$$

103. (a) Prove that

$$t_1 + t_2 + t_3 + \cdots + t_n = \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots + \left[\frac{n}{n}\right],$$

where t_n is the number of divisors of the natural number n. [*Note*: 1 and n are always counted as divisors.]

(b) Prove that

$$s_1 + s_2 + s_3 + \cdots + s_n = \left[\frac{n}{1}\right] + 2\left[\frac{n}{2}\right] + 3\left[\frac{n}{3}\right] + \cdots + n\left[\frac{n}{n}\right],$$

where s_n is the sum of the divisors of the integer n.

104. Does there exist a natural number n such that the fractional part of the number $(2 + \sqrt{2})^n$, that is, the difference

$$(2 + \sqrt{2})^n - [(2 + \sqrt{2})^n],$$

exceeds 0.999999?

105.* (a) Prove that for any natural number n, the integer $[(2 + \sqrt{3})^n]$ is odd.

(b) Find the highest power of 2 which divides the integer $[(1 + \sqrt{3})^n]$.

106. Prove that if p is an odd prime, it divides the difference

$$[(2 + \sqrt{5})^p] - 2^{p+1}.$$

107.* Prove that if p is a prime number, the difference

$$C_n^p - \left[\frac{n}{p}\right]$$

is divisible by p. (C_n^p is the number of combinations of n elements taken p at a time, where n is a natural number not less than p.)

For example,

$$C_{11}^5 = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 462;$$

$$C_{11}^5 - \left[\frac{11}{5}\right] = 462 - 2,$$

which is divisible by 5.

108.* Prove that if the positive numbers α and β have the property that among the numbers

$$[\alpha], [2\alpha], [3\alpha], \cdots; \qquad [\beta], [2\beta], [3\beta], \cdots$$

every natural number appears exactly once, then α and β are irrational numbers such that $1/\alpha + 1/\beta = 1$. Conversely, if α and β are irrational numbers with the property that $1/\alpha + 1/\beta = 1$, then every natural number N appears precisely once in the sequence

$$[\alpha], [2\alpha], [3\alpha], \cdots; \qquad [\beta], [2\beta], [3\beta], \cdots.$$

We shall designate by (a) the whole number nearest a. If a lies exactly between two integers, then (a) will be defined to be the larger integer. For example: $(2.8) = 3$; $(4) = 4$; $(3.5) = 4$.

109.* Prove that in the equality

$$N = \frac{N}{2} + \frac{N}{4} + \frac{N}{8} + \cdots + \frac{N}{2^n} + \cdots$$

(where N is an arbitrary natural number) every fraction may be replaced by the nearest whole number:

$$N = \left(\frac{N}{2}\right) + \left(\frac{N}{4}\right) + \left(\frac{N}{8}\right) + \cdots + \left(\frac{N}{2^n}\right) + \cdots.$$

5

EQUATIONS HAVING INTEGER SOLUTIONS

110. (a) Find a four-digit number which is an exact square, and such that its first two digits are the same and also its last two digits are the same.

(b) When a certain two-digit number is added to the two-digit number having the same digits in reverse order, the sum is a perfect square. Find all such two-digit numbers.

111. Find a four-digit number equal to the square of the sum of the two two-digit numbers formed by taking the first two digits and the last two digits of the original number.

112. Find all four-digit numbers which are perfect squares and are written:

(a) with four even integers;
(b) with four odd integers.

113. (a) Find all three-digit numbers equal to the sum of the factorials of their digits.

(b) Find all integers equal to the sum of the squares of their digits.

114. Find all integers equal to:

(a) the square of the sum of the digits of the number;

(b) the sum of the digits of the cube of the number.

115. Solve, in whole numbers, the following equations.
(a) $1! + 2! + 3! + \cdots + x! = y^2$.
(b) $1! + 2! + 3! + \cdots + x! = y^z$.

116. In how many ways can 2^n be expressed as the sum of four squares of natural numbers?

117. (a) Prove that the only solution in integers of the equation

$$x^2 + y^2 + z^2 = 2xyz$$

is $x = y = z = 0$.
(b) Find integers x, y, z, v such that

$$x^2 + y^2 + z^2 + v^2 = 2xyzv .$$

118.* (a) For what integral values of k is the following equation possible (where x, y, z are natural numbers)?

$$x^2 + y^2 + z^2 = kxyz.$$

(b) Find (up to numbers less than 1000) all possible triples of integers the sum of whose squares is divisible by their product.

119.** Find (within the first thousand) all possible pairs of relatively prime numbers such that the square of one of the integers when increased by 125 is divisible by the other.

120.* Find four natural numbers such that the square of each of them, when added to the sum of the remaining numbers, again yields a perfect square.

121. Find all integer pairs having the property that the sum of the two integers is equal to their product.

122. The sum of the reciprocals of three natural numbers is equal to one. What are the numbers?

123. (a) Solve, in integers (positive and negative),

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{14} .$$

(b)* Solve, in integers,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

(write a formula which gives all solutions.)

124. (a) Find all distinct pairs of natural numbers which satisfy the equation

$$x^y = y^x .$$

(b) Find all positive rational number pairs, not equal, which satisfy the equation

$$x^y = y^x$$

(write a formula which gives all solutions).

125. Two seventh-grade students were allowed to enter a chess tournament otherwise composed of eighth-grade students. Each contestant played once against each other contestant. The two seventh graders together amassed a total of 8 points, and each eighth grader scored the same number of points as his classmates. (In the tournament, a contestant received 1 point for a win and $\frac{1}{2}$ point for a tie.) How many eighth graders participated?

126. Ninth- and tenth-grade students participated in a tournament. Each contestant played each other contestant once. There were ten times as many tenth-grade students, but they were able to win only four-and-a-half times as many points as ninth graders. How many ninth-grade students participated, and how many points did they collect?

127.* An *integral triangle* is defined as a triangle whose sides are measurable in whole numbers. Find all integral triangles whose perimeter equals their area.

128.* What sides are possible in:
(a) a right-angled integral triangle;
(b) an integral triangle containing a 60° angle;
(c) an integral triangle containing a 120° angle?
(Write a formula giving all solutions.)

Remark: It can be shown that an integral triangle cannot have a rational angle (that is, an angle whose degree measure is a rational number) other than one of 90°, 60°, or 120°.

129.* Find the lengths of the sides of the smallest integral triangle for which:
(a) one of the angles is twice another;
(b) one of the angles is five times another;
(c) one angle is six times another.

130.** Prove that if the legs of right-angle triangle are expressible as the squares of integers, the hypotenuse cannot be an integer.

6

EVALUATING SUMS AND PRODUCTS

131. Prove that

$$(n+1)(n+2)(n+3)\cdots(2n-1)2n = 2^n\cdot 1\cdot 3\cdot 5\cdots(2n-3)(2n-1)\,.$$

132. Calculate the following sums.

(a) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{(n-1)n}$;

(b) $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{1}{2\cdot 3\cdot 4}+\dfrac{1}{3\cdot 4\cdot 5}+\cdots+\dfrac{1}{(n-2)(n-1)n}$;

(c) $\dfrac{1}{1\cdot 2\cdot 3\cdot 4}+\dfrac{1}{2\cdot 3\cdot 4\cdot 5}+\dfrac{1}{3\cdot 4\cdot 5\cdot 6}+\cdots+\dfrac{1}{(n-3)(n-2)(n-1)n}$.

133. Prove that

(a) $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$;

(b) $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\cdots+n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$;

(c) $1\cdot 2\cdot 3\cdots p+2\cdot 3\cdots p(p+1)+\cdots+n(n+1)+\cdots+(n+p-1)=\dfrac{n(n+1)(n+2)\cdots(n+p)}{p+1}$

for any p.

134. Calculate the following sums.

(a) $1^2+2^2+3^2+\cdots+n^2$;

(b) $1^3+2^3+3^3+\cdots+n^3$;

(c) $1^4+2^4+3^4+\cdots+n^4$;

(d) $1^3+3^3+5^3+\cdots+(2n-3)^3$.

135. Prove the identity

$$a+b(1+a)+c(1+a)(1+b)+d(1+a)(1+b)(1+c)+\cdots+l(1+a)(1+b)\cdots(1+k)=(1+a)(1+b)(1+c)\cdots(1+l)-1\,.$$

Investigate the case in which $a=b=c=\cdots=l$.

136. Calculate the following.

(a) $1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+n\cdot n!$;

(b) $C^1_{n+1}+C^2_{n+2}+C^3_{n+3}+\cdots+C^k_{n+k}$.

137. Prove that

$$\frac{1}{\log_2 N}+\frac{1}{\log_3 N}+\frac{1}{\log_4 N}+\cdots+\frac{1}{\log_{100} N}=\frac{1}{\log_{100!} N},$$

where 100! is the product $1\cdot 2\cdot 3\cdots 100$.

138. Given n positive numbers $a_1, a_2, \cdots, a_n$. Find the sum of all the fractions

$$\frac{1}{a_{k_1}(a_{k_1}+a_{k_2})(a_{k_1}+a_{k_2}+a_{k_3})\cdots(a_{k_1}+a_{k_2}+\cdots+a_{k_n})},$$

where the set $k_1, k_2, \cdots, k_n$ of indices runs through all possible permutations of $1, 2, \cdots, n$ (of which there are $n!$).

139. Simplify the following expressions.

(a) $\left(1+\dfrac{1}{3}\right)\left(1+\dfrac{1}{9}\right)\left(1+\dfrac{1}{81}\right)\left(1+\dfrac{1}{3^8}\right)\cdots\left(1+\dfrac{1}{3^{2^n}}\right)$;

(b) $\cos\alpha\cos 2\alpha\cos 4\alpha\cdots\cos 2^n\alpha$.

140. How many digits are there in the integer 2^{100} after it has been "multiplied out"?

141. (a) Prove that

$$\frac{1}{15} < \frac{1}{10\sqrt{2}} < \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{99}{100} < \frac{1}{10}.$$

(b) Prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{99}{100} < \frac{1}{12}.$$

Remark: The result of problem (b) is evidently a refinement of that of problem (a).

142. Prove that

$$\frac{2^{100}}{10\sqrt{2}} < C_{100}^{50} < \frac{2^{100}}{10}.$$

(C_{100}^{50} is the number of combinations of one-hundred elements taken fifty at a time.)

143. Which is larger, $99^n + 100^n$ or 101^n (where n is a natural number)?

144. Which is larger, 100^{300} or $300!$?

145. Prove that, for any natural number n, the following is true:

$$2 < \left(1 + \frac{1}{n}\right)^n < 3.$$

146. Which is larger, $(1.000001)^{1,000,000}$ or 2?

147. Which is larger, 1000^{1000} or 1001^{999}?

148. Prove that for any integer $n > 6$

$$\left(\frac{n}{2}\right)^n > n! > \left(\frac{n}{3}\right)^n.$$

149.* Prove that if $m > n$ (where m, n are natural numbers):

(a) $\left(1 + \frac{1}{m}\right)^m > \left(1 + \frac{1}{n}\right)^n$.

For example,

$$\left(1 + \frac{1}{2}\right)^2 = \frac{9}{4} = 2\frac{1}{4}, \text{ and } \left(1 + \frac{1}{3}\right)^3 = \frac{64}{27} = 2\frac{10}{27} > 2\frac{1}{4}.$$

(b) $\left(1 + \frac{1}{m}\right)^{m+1} < \left(1 + \frac{1}{n}\right)^{n+1} \qquad (n \geqq 2)$.

For example,

$$\left(1 + \frac{1}{2}\right)^3 = \frac{27}{8} = 3\frac{3}{8}, \text{ and } \left(1 + \frac{1}{3}\right)^4 = \frac{256}{81} = 3\frac{13}{81} < 3\frac{3}{8}.$$

From problem (a) it follows that in the sequence of numbers $(1 + 1/2)^2$, $(1 + 1/3)^3, \cdots, (1 + 1/n)^n, \cdots$, each is greater than that preceding. Since, on the other hand, no member of the sequence exceeds 3 (see problem 145), it follows that if $n \to \infty$, the magnitude of $(1 + 1/n)^n$ approaches some definite limit (which is evidently a number between 2 and 3). This limiting number is designated by e. It is equal, approximately, to $2.7182818284590 45\cdots$.

Analogously, problem 149 (b) shows that in the sequence $(1 + 1/2)^3, (1 + 1/3)^4, (1 + 1/4)^5, \cdots, (1 + 1/n)^{n+1}, \cdots$ every number is less than that preceding. Since every number of the sequence exceeds 1, the magnitude $(1 + 1/n)^{n+1}$, where n increases without bound, tends toward some limiting number. The numbers of the second sequence then become successively closer and closer to the numbers of the first [that is, the ratios $(1 + 1/n)^{n+1} : (1 + 1/n)^n = 1 + 1/n$ become closer and closer to 1]. Hence, the limiting number must, in the second case, also be equal to e. This number, e, plays a very important role in higher mathematics, and is encountered in a wide variety of problems (see, for example, problems 156 and 159).

150. Prove that, for any integer n, the following inequality holds,

$$\left(\frac{n}{e}\right)^n < n! < n\left(\frac{n}{e}\right)^n,$$

where $e = 2.71828\cdots$ is the limit of $(1 + 1/n)^n$ as $n \to \infty$.

This result is an extension of the result of problem 148. It follows, in particular, that for any two numbers, α_1 and α_2, such that $\alpha_1 < e < \alpha_2$ (for example, for $a_1 = 2.7$ and $a_2 = 2.8$; for $a_1 = 2.71$ and $a_2 = 2.72$; for $a_1 = 2.718$ and $a_2 = 2.719$, and so on) for all integers n which are "large enough" (greater than some integer N, where the magnitude of N depends on what α_1 we consider), the following inequality holds:

$$\left(\frac{n}{a_1}\right)^n > n! > \left(\frac{n}{a_2}\right)^n.$$

Thus, the number e is that limitig number which separates the numbers a for which $(n/a)^n$ exceeds, or "dominates," $n!$ from those numbers a for which the $(n/a)^n$ are "dominated" by $n!$. (The existence of such a limiting number follows from problem 148.)

Actually, $(n/a_2)^n < n!$ for every n exceeding 6 [if $\alpha_2 > e$, and if $n > 6$, in view of problem 150, $n! > (n/e)^n$]. Further, from the results of problems 145 and 149, it follows that, for $n \geqq 3$, the following inequalities hold:

$$n > e > \left(1 + \frac{1}{n}\right)^n = \frac{(n+1)^n}{n^n},$$
$$n^{n+1} > (n+1)^n,$$
$$\sqrt[n]{n} > \sqrt[n+1]{n+1};$$

consequently, for $n \geqq 3$, $\sqrt[n]{n}$ diminishes as n increases. It is readily seen that if n becomes very large, $\sqrt[n]{n}$ approaches as close to unity as we wish. It follows, for example, that $\log \sqrt[10^k]{10^k} = k/10^k$, for sufficiently large k, can be made as small as we wish. Let us now select an N such that the inequality $\sqrt[N]{N} < e/a_1$ holds. Then for $n > N$ the approximation $\sqrt[n]{n}. < e/a_1$ is still more improved, and from problem 150 it follows that

$$n! < \left(\frac{n/e}{\sqrt[n]{n}}\right)^n < \left(\frac{n}{a_1}\right)^n.$$

The inequality of problem 150 admits a great deal of precision. It is possible to show that for sufficiently large n the number $n!$ is approximated by $C\sqrt{n}\,(n/e)^n$, where C is a constant equal to $\sqrt{2\pi}$:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \dagger$$

[more precisely, it is possible to prove that if n increases without bound, ratio

$$\frac{n!}{\sqrt{2\pi n}\,(n/e)^n}$$

tends to unity. (See the book by A. M. Yaglom and E. M. Yaglom, *Non-elementary Problems Treated by Elementary Means*, Library of the Mathematical Society, Volume 5)].

151. Prove that

$$\frac{1}{k+1}n^{k+1} < 1^k + 2^k + 3^k + \cdots + n^k$$
$$< \left(1 + \frac{1}{n}\right)^{k+1} \frac{1}{k+1} n^{k+1}$$

(n and k are arbitrary integers).

Remark: A particular consequence of problem 151 is the following:

$$\lim_{n\to\infty} \frac{1^k + 2^k + 3^k + \cdots + n^k}{n^{k+1}} = \frac{1}{k+1}.$$

(See also problem 316.)

152. Prove that for all integers $n > 1$:

† The approximation given for $n!$ is usually referred to as *Stirling's formula* [*Editor*].

(a) $\dfrac{1}{2} < \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{2n} < \dfrac{3}{4}$;

(b) $1 < \dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{3n+1} < 2$.

153.* (a) Calculate the whole part of the number

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{1{,}000{,}000}}.$$

(b) Calculate the sum

$$\frac{1}{\sqrt{10{,}000}} + \frac{1}{\sqrt{10{,}001}} + \frac{1}{\sqrt{10{,}002}} + \cdots + \frac{1}{\sqrt{1{,}000{,}000}}$$

to within a tolerance (allowable error) of 1/50.

154.* Find the whole part of the number

$$\frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}}\,\frac{1}{\sqrt[3]{6}} + \cdots + \frac{1}{\sqrt[3]{1{,}000{,}000}}.$$

155. (a) Determine the sum

$$\frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2} + \cdots + \frac{1}{1000^2}$$

to a tolerance of 0.006.

(b) Determine the sum

$$\frac{1}{10!} + \frac{1}{11!} + \frac{1}{12!} + \cdots + \frac{1}{1000!}$$

to a tolerance of 0.000000015.

156. Prove that the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is greater than any previously selected number N, if n is taken sufficiently great.

Remark: The calculation of this sum can be made very precise. It is possible to show that the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n},$$

for large n, is very close to the value of $\log n$ (this logarithm taken to the base $e = 2.718\cdots$). In every case, it can be shown that for any n the difference

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$$

does not exceed unity (see the reference following problem 150 to the book by A. M. Yaglom and E. M. Yaglom).

157. Prove that if in the summation

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

we throw out every term which contains the digit 9 in its denominator, then the sum of the remaining terms, for any n, will be less than 80.

158. (a) Prove that, for any n, the following holds:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots + \frac{1}{n^2} < 2\,.$$

(b) Prove that for all n

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} < 1\frac{3}{4}\,.$$

It is evident that the inequality of problem (b) is a refinement of problem (a). An even more precise bound in given by problem 233. That problem shows that the sum

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

is less than $\pi^2/6 = 1.6449340668\cdots$ (but for any number less than $\pi^2/6$, for instance for $N = 1.64$ or for $N = 1.644934$, it is possible to find an n such that the sum

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

will exceed N).

159.* Consider the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \cdots + \frac{1}{p}\,,$$

in which the denominators run through the prime numbers from 2 to some prime number p. Prove that this sum becomes greater than any preassigned number N, provided the prime p is taken sufficiently great.

Remark: The summation of the series in this problem can be found with great accuracy. For large p, the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{p}$$

differs relatively little from $\log \log p$ (where the logarithms are taken to the base $e = 2.718\cdots$), and the differences

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{p} - \log \log p$$

never exceed 15 (refer to the book by A. M. Yaglom and E. M. Yaglom). Comparison of the results of this problem with those of problems 157 and 158 emphasizes that among the prime numbers may be found arbitrarily large integers (this problem reaffirms that there are infinitely many). It is possible, for example, to say that the primes are "more numerous" in the sequence of natural numbers than either squares or numbers failing to contain the digit 9, inasmuch as the sum of the reciprocals of all the squares, as well as the sum of all those reciprocals of whole numbers not containing the digit 9, are bounded (by $1\frac{3}{4}$ and by 80, respectively), whereas the sum of the reciprocals of all the primes becomes arbitrarily great.

7

MISCELLANEOUS PROBLEMS FROM ALGEBRA

160. If $a+b+c=0$, what does the following expression equal?

$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right).$$

161. Prove that if $a+b+c=0$, then

$$a^3+b^3+c^3=3abc\,.$$

162. Factor the following:

(a) $a^3+b^3+c^3-3abc$;

(b) $(a+b+c)^3-a^3-b^3-c^3$.

163. Rationalize the denominator:

$$\frac{1}{\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}}\,.$$

164. Prove that

$$(a+b+c)^{333}-a^{333}-b^{333}-c^{333}$$

is divisible by

$$(a+b+c)^3-a^3-b^3-c^3\,.$$

165. Factor the following expression:

$$a^{10}+a^5+1\,.$$

166. Prove that the polynomial

$$x^{9999}+x^{8888}+x^{7777}+\cdots+x^{2222}+x^{1111}+1$$

is divisible by

$$x^9+x^8+x^7+\cdots+x^2+x+1\,.$$

167. Using the result of problem 162 (a), find the general formula for the solution of the cubic equation

$$x^3+px+q=0\,.$$

Remark: This result enables us to solve any equation of the third degree. Let

$$x^3+Ax^2+Bx+C=0$$

be any cubic equation (the coefficient of x^3 is taken as 1, since in any other case we can divide through by the coefficient of x^3). We make the substitution

$$x=y+c\,,$$

and we obtain

$$y^3+3cy^2+3c^2y+c^3+A(y^2+2cy+c^2)+B(y+c)+C=0\,,$$

or

$$y^3+(3c+A)y^2+(3c^2+2Ac+B)y+(c^3+Ac^2+Bc+C)=0\,.$$

From this, if we set $c=-A/3$ (that is, $x=y-A/3$), we arrive at

$$y^3+\left(\frac{3A^2}{9}-\frac{2A^2}{9}+B\right)y+\left(-\frac{A^3}{27}+\frac{A^3}{9}-\frac{AB}{3}+C\right)=0\,,$$

which has the same forms as that of the given problem:

$$y^3+py+q=0\,,$$

where

$$p=-\frac{A^2}{3}+B \quad\text{and}\quad q=\frac{2A^3}{27}-\frac{AB}{3}+C\,.$$

168. Solve the equation

$$\sqrt{a-\sqrt{a+x}}=x\,.$$

169.* Find the real roots of the equation

$$x^3+2ax+\frac{1}{16}=-a+\sqrt{a^2+x-\frac{1}{16}}\quad\left(0<a<\frac{1}{4}\right).$$

170. (a) Find the real roots of the equation

$$\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\cdots+2\sqrt{x+2\sqrt{3x}}}}}}_{\{n \text{ radicals}\}} = x$$

(all roots are considered positive).

(b) Solve the equation

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdot_{\cdot_{\cdot_{1+\frac{1}{x}}}}}}} = x\,.$$

(In the expression on the left the fraction designation is repeated n times.)

171. Find the real roots of the equation

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1\,.$$

(All square roots are to be taken as positive.)

172. Solve the equation

$$|x+1|-|x|+3|x-1|-2|x-2|=x+2\,.$$

173. A system of two second-degree equations

$$\begin{cases} x^2-y^2=0\,, \\ (x-a)^2+y^2=1 \end{cases}$$

has, in general, four solutions. For what values of a is the number of solutions of this system decreased to three or to two?

174. (a) Solve the system of equations

$$\begin{cases} ax+y=a^2\,, \\ x+ay=1\,. \end{cases}$$

For what values of a does this system fail to have solutions, and for what values of a are there infinitely many solutions?

(b) Answer the above question for the system

$$\begin{cases} ax+y=a^3\,, \\ x+ay=1\,. \end{cases}$$

(c) Answer the above question for the system

$$\begin{cases} ax+y+z=1\,, \\ x+ay+z=a\,, \\ x+z+az=a^2\,. \end{cases}$$

175. Find the conditions which must be satisfied by the numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that the following system of six equations in four unknowns has a solution:

$$\begin{cases} x_1+x_2=\alpha_1\alpha_2\,, \\ x_1+x_3=\alpha_1\alpha_3\,, \\ x_1+x_4=\alpha_1\alpha_4\,, \\ x_2+x_3=\alpha_2\alpha_3\,, \\ x_2+x_4=\alpha_2\alpha_4\,, \\ x_3+x_4=\alpha_3\alpha_4\,. \end{cases}$$

Find the values of the unknowns x_1, x_2, x_3, x_4.

176. How many real solutions has the following system?

$$\begin{cases} x+y=2\,, \\ xy-z^2=1\,. \end{cases}$$

177. (a) How many roots has the following equation?

$$\sin x = \frac{x}{100}\,.$$

(b) How many roots has the following equation?

$$\sin x = \log x\,.$$

(*Note*: $\log x \equiv \log_{10} x$.)

178.* Prove that if x_1 and x_2 are roots of the equation $x^2-6x+1=0$, then $x_1^n+x_2^n$ is, for any natural number n, an integer not divisible by 5.

179. Is it possible for the expression

$$(a_1+a_2+\cdots+a_{999}+a_{1000})^2 = a_1^2+a_2^2+\cdots+a_{999}^2+a_{1000}^2 + 2a_1a_2+2a_1a_3+\cdots+2a_{999}a_{1000}$$

(where some of the numbers $a_1, a_2, \cdots, a_{999}, a_{1000}$ are positive and the rest are negative) to contain the same number of positive and negative terms in a_ia_j?

Investigate the analogous problem for the expression

$$(a_1+a_2+\cdots+a_{9999}+a_{10.000})^2\,.$$

180. Prove that any integral power of the number $\sqrt{2}-1$ can be expressed in the form $\sqrt{N}-\sqrt{N-1}$, where N is an integer (for example: $(\sqrt{2}-1)^2 = 3-2\sqrt{2} = \sqrt{9}-\sqrt{8}$, and $(\sqrt{2}-1)^3 = 5\sqrt{2}-7 = \sqrt{50}-\sqrt{49}$.

181. Prove that the number $99{,}999 + 111{,}111\sqrt{3}$ cannot be written in the form $(A+B\sqrt{3})^2$, where A and B are integers.

182. Prove that $\sqrt[3]{2}$ cannot be represented in the form $p+q\sqrt{r}$, where p, q, r are rational numbers.

183. (a) Which of the following two expressions is greater?

$$\frac{2.00000000004}{(1.00000000004)^2 + 2.00000000004}\,;$$
$$\frac{2.00000000002}{(1.00000000002)^2 + 2.00000000002}\,.$$

(b) Let $a > b > 0$. Which of the following is greater?

$$\frac{1+a+a^2+\cdots+a^{n-1}}{1+a+a^2+\cdots+a^n}\,;$$
$$\frac{1+b+b^2+\cdots+b^{n-1}}{1+b+b^2+\cdots+b^n}\,.$$

184. Given n numbers $a_1, a_2, \cdots, a_n$. Find the number x such that the sum

$$(x-a_1)^2 + (x-a_2)^2 + \cdots + (x-a_n)^2$$

has the least possible value.

185. (a) Given four distinct numbers $a_1 < a_2 < a_3 < a_4$. Put these numbers in such an order, $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}$ (i_1, i_2, i_3, i_4 being some rearrangement of $1, 2, 3, 4$) that the sum

$$\Phi = (a_{i_1}-a_{i_2})^2 + (a_{i_2}-a_{i_3})^2 + (a_{i_3}-a_{i_4})^2 + (a_{i_4}-a_{i_1})^2$$

has the least possible value.

(b)* Given n real distinct numbers $a_1, a_2, \cdots, a_n$. Put these numbers in such an order $a_{i_1}, a_{i_2}, \cdots, a_{i_n}$ that the sum

$$\Phi = (a_{i_1}-a_{i_2})^2 + (a_{i_2}-a_{i_3})^2 + \cdots + (a_{i_{n-1}}-a_{i_n})^2 + (a_{i_n}-a_{i_1})^2$$

has the least possible value.

186. (a) Prove that, regardless of what numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$ are taken, the following relation always holds:

$$\sqrt{a_1^2+b_1^2} + \sqrt{a_2^2+b_2^2} + \cdots + \sqrt{a_n^2+b_n^2} \geqq \sqrt{(a_1+a_2+\cdots+a_n)^2 + (b_1+b_2+\cdots+b_n)^2}\,.$$

Under what conditions does the equality hold?

(b) A pyramid is called *a right pyramid* if, when a circle is inscribed in its base, the altitude of the pyramid falls on the center of the circle. Prove that a right pyramid has less lateral surface area than any other pyramid of the same altitude and base area and having the same perimeter.

Remark: The inequality of problem (a) is a special case of what is called the Inequality of Minkowski (see problem 308).

187.* Prove that for any real numbers $a_1, a_2, \cdots, a_n$ the following inequality holds:

$$\sqrt{a_1^2+(1-a_2)^2} + \sqrt{a_2^2+(1-a_3)^2} + \cdots + \sqrt{a_{n-1}^2+(1-a_n)^2} + \sqrt{a_n^2+(1-a_1)^2} \geqq \frac{n\sqrt{2}}{2}\,.$$

For what values of the numbers is the left member equal to the right member?

188. Prove that if the numbers x_1 and x_2 do not exceed 1 in absolute value, then

$$\sqrt{1-x_1^2} + \sqrt{1-x_2^2} \leqq 2\sqrt{1-\left(\frac{x_1+x_2}{2}\right)^2}\,.$$

For what numbers x_1 and x_2 does the equality hold?

189. Which is greater, $\cos(\sin x)$ or $\sin(\cos x)$?

190. Prove, without using logarithm tables, that:

(a) $\dfrac{1}{\log_2 \pi} + \dfrac{1}{\log_5 \pi} > 2\,;$

(b) $\dfrac{1}{\log_2 \pi} + \dfrac{1}{\log_\pi 2} > 2\,.$

191. Prove that if α and β are acute angles, with $\alpha < \beta$, then

(a) $\alpha - \sin\alpha < \beta - \sin\beta\,;$

(b) $\tan\alpha - \alpha < \tan\beta - \beta\,.$

192.* Prove that if α and β are acute angles and $a < \beta$, then

$$\frac{\tan \alpha}{\alpha} < \frac{\tan \beta}{\beta} .$$

193. Find the relationship between $\arcsin [\cos (\arcsin x)]$ and $\arccos [\sin (\arccos x)]$.

194. Prove that for arbitrary coefficients $a_{31}, a_{30}, \cdots, a_2, a_1$, the sum

$$\cos 32x + a_{31} \cos 31x + a_{30} \cos 30x + \cdots + a_2 \cos 2x + a_1 \cos x$$

cannot take on only positive values for all x.

195. Let some of the numbers $a_1, a_2, \cdots, a_n$ be $+1$ and the rest be -1. Prove that

$$2 \sin \left(a_1 + \frac{a_1 a_2}{2} + \frac{a_1 a_2 a_3}{4} + \cdots + \frac{a_1 a_2 \cdots a_n}{2^{n-1}} \right) 45^\circ = a_1 \sqrt{2 + a_2 \sqrt{2 + a_3 \sqrt{2 + \cdots + a_n \sqrt{2}}}} .$$

For example, let $a_1 = a_2 = \cdots = a_n = 1$:

$$2 \sin \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} \right) 45^\circ = 2 \cos \frac{45^\circ}{2^{n-1}} = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}} .$$

8

THE ALGEBRA OF POLYNOMIALS

196. Find the sum of the coefficients of the polynomial obtained after expanding and collecting the terms of the product

$$(1 - 3x + 3x^2)^{743} (1 + 3x - 3x^2)^{744} .$$

197. Which of the expressions,

$$(1 + x^2 - x^3)^{1000} \quad \text{or} \quad (1 - x^2 + x^3)^{1000} ,$$

will have the larger coefficient for x^{20} after expansion and collecting of terms?

198. Prove that in the product

$$(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100}) ,$$

after multiplying and collecting terms, there does not appear a term in x of odd degree.

199. Find the coefficient of x^{50} in the following polynomials:

(a) $(1 + x)^{1000} + x(1 + x)^{999} + x^2(1 + x)^{998} + \cdots + x^{1000}$;

(b) $(1 + x) + 2(1 + x)^2 + 3(1 + x)^3 + \cdots + 1000(1 + x)^{1000}$.

200.* Find the coefficient of x^2 upon the expansion and collecting of terms in the expression

$$\underbrace{((((x-2)^2-2)^2-2)^2-\cdots-2)^2}_{n \text{ times}} .$$

201. Find the remainders upon dividing the polynomial $x + x^3 + x^9 + x^{27} + x^{81} + x^{243}$

(a) by $x - 1$;
(b) by $x^2 - 1$.

202. An unknown polynomial yields a remainder of 2 upon division by $x - 1$, and a remainder of 1 upon division by $x - 2$. What remainder is obtained if this polynomial is divided by $(x - 1)(x - 2)$?

203. If the polynomial $x^{1951} - 1$ is divided by $x^4 + x^3 + 2x^2 + x + 1$, a quotient and remainder are obtained. Find the coefficient of x^{14} in the quotient.

204. Find an equation with integral coefficients whose roots include the numbers

(a) $\sqrt{2} + \sqrt{3}$,
(b) $\sqrt{2} + \sqrt[3]{3}$.

205. Prove that if α and β are the roots of the equation

$$x^2 + px + 1 = 0 ,$$

and if γ and δ are the roots of the equation

$$x^2 + qx + 1 = 0 ,$$

then

$$(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = q^2 - p^2 .$$

206. Let α and β be the roots of the equation

$$x^2 + px + q = 0 ,$$

and γ and δ be the roots of the equation

$$x^2 + Px + Q = 0 .$$

Express the product

$$(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta)$$

in terms of the coefficients of the given equations.

207. Given the two polynomials

$$x^2 + ax + 1 = 0 ,$$
$$x^2 + x + a = 0 .$$

Determine all values of the coefficient a for which these equations have at least one common root.

208. Find an integer a such that $(x - a)(x - 10) + 1$ can be written as a product $(x + b)(x + c)$ of two factors with integers b and c.

209. Find (nonzero) distinct integers a, b, and c such that the following fourth-degree polynomial with integral coefficients, can be written as the product of two other polynomials with integral coefficients:

$$x(x - a)(x - b)(x - c) + 1$$

210. For what integers $a_1, a_2, \cdots, a_n$, where these are all distinct, are the following polynomials with integral coefficients expressible as the product of two polynomials with integral coefficients?

(a) $(x - a_1)(x - a_2)(x - a_3)\cdots(x - a_n) - 1$;
(b) $(x - a_1)(x - a_2)(x - a_3)\cdots(x - a_n) + 1$.

211.* Prove that if the integers $a_1, a_2, \cdots, a_n$ are all distinct, then the polynomial

$$(x - a_1)^2(x - a_2)^2\cdots(x - a_n)^2 + 1$$

cannot be written as a product of two other polynomials with integral coefficients.

212. Prove that if the polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n ,$$

with integral coefficients, takes on the value 7 for four integral values of x, then it cannot have the value 14 for any integral value of x.

213. Prove that if the polynomial

$$a_0x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5x^2 + a_6x + a_7 ,$$

of seventh degree, with integral coefficients, has for seven integral values of x the value $+1$ or -1, then it cannot be factored as the product of two polynomials with integral coefficients.

214. Prove that if the polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n ,$$

with integral coefficients, has odd values for $x = 0$ and $x = 1$, then the equation $P(x) = 0$ cannot have integral roots.

215.* Prove that if the polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n ,$$

with integral coefficients, is equal in absolute value to 1 for tw integers $x = p$ and $x = q$ ($p > q$), and if the equation $P(x) = 0$ ha rational roots a, then $p - q$ is equal to 1 or 2, and $a = \dfrac{p+q}{2}$.

216.* Prove that neither of the following polynomials can be wri ten as a product of two polynomials with integral coefficients:

(a) $x^{2222} + 2x^{2220} + 4x^{2218} + 6x^{2216} + 8x^{2214} + \cdots + 2218x^4 + 2220x^2 + 2222$

(b) $x^{250} + x^{249} + x^{248} + \cdots + x^2 + x + 1$.

217. Prove that if the product of two polynomials with integr coefficients is a polynomial with even coefficients, not all of whic are divisible by 4, then in one of the polynomials all coefficien must be even, and in the other not all coefficients will be even.

218. Prove that all the rational roots of the polynomial

$$P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n ,$$

with integral coefficients and with leading coefficient 1, are integer

219.* Prove that there does not exist a polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

such that $P(0), P(1), P(2), \cdots$ are all prime numbers.

Remark: The proposition stated in this problem was first proven by t mathematician L. Euler. Also credited to him are polynomials whose valu for many consecutive integers are prime numbers. For example, for the pol nomial $P(x) = x^2 - 79x + 1601$, the 80 numbers $P(0) = 1$, $P(1) = 1523$, $P($ $P(3), \cdots, P(79)$ are all primes.

220. Prove that if the polynomial

$$P(x) = x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_{n-1} + A_n$$

assumes integral values for all integral values for x, then it is po sible to represent it as a sum of polynomials

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{x(x-1)}{1\cdot 2},\ \cdots,\ P_n(x) = \frac{x(x-1)(x-2)\cdots(x-n+1)}{1\cdot 2\cdot 3\cdots n} ,$$

having the same property [in view of problem 49 (a)] and having integral coefficients.

221. (a) Prove that if the nth degree polynomial $P(x)$ has integral values for $x = 0, 1, 2, \cdots, n$, then it has integral values for all integral values of x.

(b) Prove that if a polynomial $P(x)$ of degree n has integral values for $n + 1$ successive integers x, then it is integral valued for all integers x.

(c) Prove that if the polynomial $P(x)$ of degree n has integral values for $x = 0, 1, 4, 9, 16, \cdots, n^2$, then it has integral values for all integers x which are perfect squares (but this does not necessarily follow for all integers x).

Give an example of a polynomial which assumes integral values for all integers x which are perfect squares, but which for some other value of x yields a rational (not whole) number.

9

COMPLEX NUMBERS

In many of the problems in this section the following formulas are useful.

(1) The formula for the product of complex numbers in trigonometric form:

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)\,.$$

(2) De Moivre's formula:

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

(where n is a natural number), which is an n-fold application of the previous formula.

(3) The formula for the roots of complex numbers:

$$\sqrt[n]{\cos\alpha + i\sin\alpha} = \cos\frac{\alpha + 360°\cdot k}{n} + i\sin\frac{\alpha + 360°\cdot k}{n} \qquad (k = 0, 1, 2, \cdots, n-1)\,,$$

which is an extended form of De Moivre's theorem.

In particular, a large role is played in the following problems by the formula for the nth rooths of unity, that is, the roots of the nth-degree equation

$$x^n - 1 = 0\,,$$

which are given by the following formulas:

$$\sqrt[n]{1} = \sqrt[n]{\cos 0 + i\sin 0} = \cos\frac{360°\cdot k}{n} + i\sin\frac{360°\cdot k}{n} \qquad (k = 0, 1, 2, \cdots, n-1)\,,$$

The following observation will often be useful in solving the problems of this section. Let the equation of degree n,

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0\,,$$

have the n roots $x_1, x_2, \cdots, x_{n-1}, x_n$. Then the left member of the equation is divisible by $(x-x_1)(x-x_2)\cdots(x-x_n)$; that is,

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = (x-x_1)(x-x_2)\cdots(x-x_{n-1})(x-x_n)\,.$$

If we multiply out the second member of this equation and equate coefficients of like powers of x from both members, we obtain the following formulas giving relationships between the coefficients on the left and the roots of the equation (Vieta's formulas).

$$\begin{aligned}
a_1 &= -(x_1 + x_2 + \cdots + x_{n-1} + x_n)\,,\\
a_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n\,,\\
a_3 &= -(x_1x_2x_3 + \cdots + x_{n-2}x_{n-1}x_n)\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,,\\
a_{n-1} &= (-1)^{n-1}(x_1x_2\cdots x_{n-1} + x_1x_2\cdots x_{n-2}x_n + \cdots + x_2x_3\cdots x_n)\,,\\
a_n &= (-1)^n x_1x_2x_3\cdots x_n\,.
\end{aligned}$$

222. (a) Prove that

$$\cos 5\alpha = \cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha\,,$$
$$\sin 5\alpha = \sin^5\alpha - 10\sin^3\alpha\cos^2\alpha + 5\sin\alpha\cos^4\alpha\,.$$

(b) Prove that for integers n

$$\cos n\alpha = \cos^n\alpha - C_n^2\cos^{n-2}\alpha\sin^2\alpha + C_n^4\cos^{n-4}\alpha\sin^4\alpha - C_n^6\cos^{n-6}\alpha\sin^6\alpha + \cdots\,,$$
$$\sin n\alpha = C_n^1\cos^{n-1}\alpha\sin\alpha - C_n^3\cos^{n-3}\alpha\sin^3\alpha + C_n^5\cos^{n-5}\alpha\sin^5\alpha - \cdots\,,$$

where the terms designated by $\cdots$, which are readily identified from the given terms, are continued while they preserve the sense of the binomial coefficients.

Remark: Problem (b) is, of course, a generalization of problem (a).

223. Express $\tan 6\alpha$ in terms of $\tan\alpha$.

224. Prove that if $x + \dfrac{1}{x} = 2\cos\alpha$, then

$$x^n + \frac{1}{x^n} = 2\cos n\alpha\,.$$

225. Prove that

$$\sin\varphi + \sin(\varphi+\alpha) + \sin(\varphi+2\alpha) + \cdots + \sin(\varphi+n\alpha) = \frac{\sin\dfrac{(n+1)\alpha}{2}\sin\left(\varphi + \dfrac{n\alpha}{2}\right)}{\sin\dfrac{\alpha}{2}}\,,$$

and that

$$\cos\varphi + \cos(\varphi+\alpha) + \cos(\varphi+2\alpha) + \cdots + \cos(\varphi+n\alpha) = \frac{\sin\dfrac{(n+1)\alpha}{2}\cos\left(\varphi+\dfrac{n\alpha}{2}\right)}{\sin\dfrac{\alpha}{2}}.$$

226. Find the value of

$$\cos^2\alpha + \cos^2 2\alpha + \cdots + \cos^2 n\alpha,$$

and of

$$\sin^2\alpha + \sin^2 2\alpha + \cdots + \sin^2 n\alpha.$$

227. Evaluate

$$\cos\alpha + C_n^1\cos 2\alpha + C_n^2\cos 3\alpha + \cdots + C_n^{n-1}\cos n\alpha + \cos(n+1)\alpha$$

and

$$\sin\alpha + C_n^1\sin 2\alpha + C_n^2\sin 3\alpha + \cdots + C_n^{n-1}\sin n\alpha + \sin(n+1)\alpha.$$

228. Prove that if m, n, and p are arbitrary integers, then

$$\sin\frac{m\Pi}{p}\sin\frac{n\Pi}{p} + \sin\frac{2m\Pi}{p}\sin\frac{2n\Pi}{p} + \sin\frac{3m\Pi}{p}\sin\frac{3n\Pi}{p} + \cdots + \sin\frac{(p-1)m\Pi}{p}\sin\frac{(p-1)n\Pi}{p}\Big]$$

$$= \begin{cases} -\dfrac{p}{2}, & \text{if } m+n \text{ is divisible by } 2p \text{ and } m-n \text{ is not divisible by } 2p; \\ \dfrac{p}{2}, & \text{if } m-n \text{ is divisible by } 2p \text{ and } m+n \text{ is not divisible by } 2p; \\ 0, & \text{if } m+n \text{ and } m-n \text{ are both divisible by } 2p, \text{ or if neither is divisible by } 2p. \end{cases}$$

229. Prove that

$$\cos\frac{2\Pi}{2n+1} + \cos\frac{4\Pi}{2n+1} + \cos\frac{6\Pi}{2n+1} + \cdots + \cos\frac{2n\Pi}{2n+1} = -\frac{1}{2}.$$

230. Construct an equation whose roots are the numbers:

(a) $\sin^2\dfrac{\Pi}{2n+1},\ \sin^2\dfrac{2\Pi}{2n+1},\ \sin^2\dfrac{3\Pi}{2n+1},\ \ldots,\ \sin^2\dfrac{n\Pi}{2n+1};$

(b) $\cot^2\dfrac{\Pi}{2n+1},\ \cot^2\dfrac{2\Pi}{2n+1},\ \cot^2\dfrac{3\Pi}{2n+1},\ \ldots,\ \cot^2\dfrac{n\Pi}{2n+1}.$

231. Find the following sums.

(a) $$\cot^2\frac{\Pi}{2n+1} + \cot^2\frac{2\Pi}{2n+1} + \cot^2\frac{3\Pi}{2n+1} + \cdots + \cot^2\frac{n\Pi}{2n+1}.$$

(b) $$\csc^2\frac{\Pi}{2n+1} + \csc^2\frac{2\Pi}{2n+1} + \csc^2\frac{3\Pi}{2n+1} + \cdots + \csc^2\frac{n\Pi}{2n+1}.$$

232. Calculate the following products.

(a) $$\sin\frac{\Pi}{2n+1}\sin\frac{2\Pi}{2n+1}\sin\frac{3\Pi}{2n+1}\cdots\sin\frac{n\Pi}{2n+1},$$

and

$$\sin\frac{\Pi}{2n}\sin\frac{2\Pi}{2n}\sin\frac{3\Pi}{2n}\cdots\sin\frac{(n-1)\Pi}{n}.$$

(b) $$\cos\frac{\Pi}{2n+1}\cos\frac{2\Pi}{2n+1}\cos\frac{3\Pi}{2n+1}\cdots\cos\frac{n\Pi}{2n+1},$$

and

$$\cos\frac{\Pi}{2n}\cos\frac{2\Pi}{2n}\cos\frac{3\Pi}{2n}\cdots\cos\frac{(n-1)\Pi}{2n}.$$

233. Using the results of problems 231 (a) and (b), show that for any natural number n the sum

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

lies between the values

$$\left(1-\frac{2}{n+1}\right)\left(1-\frac{2}{2n+1}\right)\frac{\Pi^2}{6}$$

and

$$\left(1-\frac{1}{2n+1}\right)\left(1+\frac{1}{2n+1}\right)\frac{\Pi^2}{6}.$$

Remark: A particular result which follows from problem 233 is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6},$$

where the summation on the left is the limit to which $1 + 1/2^2 + \cdots + 1/n$ tends as $n \to \infty$.

234. (a) On a circle which circumscribes an n-sided polygo $A_1A_2\cdots A_n$, a point M is taken. Prove that the sum of the square of the distances from this point to all the vertices of the polygon i a number independent of the position of the point M on the circle and that this sum is equal to $2nR^2$, where R is the radius of th circle.

(b) Prove that the sum of the squares of the distances fro an arbitrary point M, taken in the plane of a regular n-sided polygo $A_1A_2\cdots A_n$ to all the vertices of the polygon, depends only upon th distance l of the point M from the center O of the polygon, and i equal to $n(R^2 + l^2)$, where R is the radius of the circle circumscribin the regular n-sided polygon.

(c) Prove that statement (b) remains correct even when poin M does not lie in the plane of the n-sided polygon $A_1A_2\cdots A_n$.

235. Let M be a point on the circle circumscribing a regular n sided polygon $A_1A_2\cdots A_n$. Prove the following.

(a) If n is even, then the sum of the squares of the distance from M to the vertices indicated by even-numbered subscripts (fo example, A_2, A_4, and so on) is equal to the sum of the squared di tances to the vertices having odd subscripts.

(b) If n is odd, then the sum of the distances from the poin M to the vertices of the polygon which are even-numbered is equa to the sum of the distances to those which are odd-numbered.

236. The radius of a circle which circumscribes a regular n-side polygon $A_1A_2\cdots A_n$ is equal to R. Prove the following.

(a) The sum of the squares of all the sides and all th diagonals is equal to n^2R^2.

(b) The sum of all the sides and all the diagonals of th polygon is equal to $n \cot \dfrac{\pi}{2n} R$.

(c) The product of all the sides and all the diagonals of th polygon is equal to $n^{n/2}R^{[n(n-1)]/2}$.

237.* Find the sum of the 50th powers of all the sides and a the diagonals of the regular 100-sided polygon inscribed in a circl of radius R.

238.* Prove that in a triangle whose sides have integral lengt it is not possible to find angles differing from 60°, 90°, and 120°, and commensurable with a right angle.

239.* (a) Prove that for any odd integer $p > 1$ the angle arc $\cos \dfrac{1}{p}$ cannot contain a rational number of degrees.

(b) Prove that an angle arc $\tan \dfrac{p}{q}$, where p and q are distinct positive integers, cannot contain a rational number of degrees.

10

SOME PROBLEMS OF NUMBER THEORY

These problems are concerned with that division of mathematics treati properties of integers, Elementary Number Theory. Many of the problems other sections of this book also deal with number theory—particularly Sectio 3, 4, and 5. Several of the following theorems, stated here as problems, pl an important role in number theory (see, for example, problems 240, 241, 24 247, 249, 253). Clearly, these problems do not pretend to explore with a completeness the rich variety of methods and ideas that have permeated t discipline, which is at once one of the most fruitful and one of the most d ficult of all mathematical endeavors. A good systematic account of some pa of number theory is given in the book by B. B. Dynkin and V. A. Uspensk *Mathematical Conversations*, Issue 6, Library of the USSR Mathematical Socie There the reader will find alternate solutions to some of the problems of t section. An excellent condensed treatment is the article by A. Y. Khinch "Elementary Number Theory," appearing in the *Encyclopedia of Elementa Mathematics*, Government Technical Publishing House, Moscow, 1951, whi contains, as an appendix, an extensive bibliography covering the topics touch on in the article.

240. *Fermat's Theorem.* Prove that if p is a prime number, the the difference $a^p - a$ is, for any integer a, divisible by p.

Remark: Problems 27 (a)-(d) are special cases of this theorem.

241. *Euler's Theorem.* Let N be any natural number and let

be the number of integers in the sequence $1, 2, 3, \cdots, N-1$ which are relatively prime to N. Prove that if a is any integer which is relatively prime to N, then $a^r - 1$ is divisible by N.

Remark: If the number N is prime, then all the integers of the sequence are, of course, relatively prime to N; that is, $r = N - 1$. In this case, Euler's theorem assumes the form $a^{N-1} - 1$ is divisible by N, if N is prime. It is clear that Fermat's theorem (problem 240) can be considered a special case of Euler's theorem.

If $N = p^n$, where p is a prime number, then of the first $N - 1 = p^n - 1$ positive integers, those not relatively prime to $N = p^n$ will be $p, 2p, 3p, \cdots, N - p = (p^{n-1} - 1)p$. Therefore, we have $r = (p^n - 1) - (p^{n-1} - 1) = p^n - p^{n-1}$, and Euler's theorem provides the following corollary: The difference $a^{p^n - p^{n-1}} - 1$, where p is prime and a is not divisible by p, is divisible by p^n.

If $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \cdots, p_k$ are distinct primes, then the number r of prime numbers less than N and relatively prime to N is given by the formula

$$r = N\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) x \cdots \left(1 - \frac{1}{p_k}\right).$$

(See, for example, the article by A. Y. Khinchin, referred to above.) If $N = p^n$ is a power of the prime p, this formula yields

$$r = p^n\left(1 - \frac{1}{p}\right) = p^n - p^{n-1},$$

which is the result obtained previously.

242.* According to Euler's theorem, the difference $2^k - 1$, where $k = 5^n - 5^{n-1}$, is divisible by 5^n (see problem 241, and the remark following it). Prove that there exists no k less than $5^n - 5^{n-1}$ such that $2^k - 1$ is divisible by 5^n.

243. Let us write, in order, the consecutive powers of the number 2: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, $\cdots$. Note that in this sequence the final digits periodically repeat with a period of 4:

$$2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6, \cdots.$$

Prove that, if we begin at a suitable point of the sequence, the last ten digits of the numbers of the sequence will also repeat periodically. Find the length of the period and the number of integers in the sequence for which this observed periodicity occurs.

244.* Prove that there exists some power of 2 whose final 1000 digits are all ones and twos.

245. *Wilson's Theorem.* Prove that: if the integer p is prime, then the number $(p-1)!+1$ is divisible by p; if p is composit then $(p-1)!+1$ is not divisible by p.

246.* Let p be a prime number which yields the remainder 1 upo division by 4. Prove that there exists an integer x such that $x^2 +$ is divisible by p.

247.** Prove the following.

(a) If each of the two integers A and B can be represente as the sum of two squares, then their product $A \cdot B$ can also be r presented in this manner.

(b) All prime numbers of form $4n+1$ can be written the sum of two squares, and no number of form $4n+3$ can be expressed.

(c) A composite number N can be written as the sum two squares if and only if all its prime factors of form $4n+3$ o cur an even number of times.

For example, the numbers $10{,}000 = 2^4 \cdot 5^4$ and $2430 = 2^2 \cdot 3^2 \cdot 5 \cdot 13$ ca be represented as the sum of the squares of two integers (in the fir number there are no factors of form $4n+3$, and in the seco number there is one such factor, 3, which occurs twice); the numb $2002 = 2 \cdot 7 \cdot 11 \cdot 13$ cannot be represented as the sum of two squar (the factors 7 and 11, of form $4n+3$, appear once).

248. Prove that, for any prime p, it is possible to find integers and y such that $x^2 + y^2 + 1$ is divisible by p.

249.** Prove the following.

(a) If each of two numbers A and B can be written as t sum of the squares of four integers, then their product $A \cdot B$ ca also be represented in this manner.

(b) Every natural number can be written as the sum of n more than four squares.

example, $35 = 25 + 9 + 1 = 5^2 + 3^2 + 1^2$; $60 = 49 + 9 + 1 + 1 = 7^2$ $3^3 + 1^2 + 1^2$; $1000 = 900 + 100 = 30^2 + 10^2$, and so on.

250. Prove that no number of the form $4^n(8k-1)$, where n and are integers, that is, no number belonging to the geometric progre sions

$$\begin{array}{rrrrl}
7, & 28, & 112, & 448, & \cdots, \\
15, & 60, & 240, & 960, & \cdots, \\
23, & 92, & 368, & 1472, & \cdots, \\
31, & 124, & 496, & 1984, & \cdots.
\end{array}$$

can be a square or the sum of two squares or three squares of integers.

Remark: It has been shown that every integer which *cannot* be written in form $4^n(8k-1)$ is representable as the sum of three or fewer squares. However, the proof is very complicated.

251.** Prove that every positive integer can be written as the sum of not more than 53 fourth powers of integers.

Remark: Experimental trials indicate that integers of moderate size are representable as the sum of far fewer fourth powers than 53. To the present time, no integer has been produced which cannot be given as the sum of 19, or fewer, fourth powers. (Of the numbers less than 100, only one—the number 79—requires as many as 19 fourth powers; that is four terms of 2^4 and 15 units). It has been conjectured that 19 fourth powers suffice for every integer, but no proof of this has as yet appeared. The best result in this direction has been the proof that every natural number can be written as the sum of not more than 21 fourth powers. This is a substantial improvement over the proposition given as problem 251, but the proof of it involves considerable higher mathematics.

In problem 239 (b) it was stated that every integer can be written as the sum of not more than four squares. It has also been shown that every integer can be written as the sum of not more than nine cubes.

All these propositions are embraced by the following remarkable theorem: *For every positive integer k there exists a positive integer N (depending, of course, on k) such that every integer may be written as the sum of not more than N kth powers of positive integers.* This theorem has been provided with several different proofs, but only recently has a proof been given which does not require considerable higher mathematics. In 1942 the Soviet mathematician U. V. Linnik gave the elementary proof. This proof is presented in the popular little book by A. Y. Khinchin, *Three Pearls of Number Theory*, Government Technical Publishing House, Moscow, 1949.† Although Linnik's proof is elementary, it is not easy reading. Khinchin himself remarks that almost anybody can understand it with "only two or three weeks work with pencil and paper."

252.** Prove that every positive rational number (in particular, every positive integer) can be written as the sum of three cubes of positive rational numbers.

Remark: Not all positive rational numbers can be represented as the sum of two cubes of positive rational numbers. Consider, for example, the number 1. The equation

† An English translation has been published by Graylock Press, Rochester. N. Y., 1952, 64 pp., $2.00 [*Editor*].

$$1 = \left(\frac{m}{n}\right)^3 + \left(\frac{p}{q}\right)^3$$

can be written

$$(nq)^3 = (mq)^3 + (np)^3 ,$$

where m, n, p, and q are integers. But it is known that no solution in integers exists for the equation

$$x^3 + y^3 = z^3$$

(a proof of this may be found in most standard texts on number theory).

253. Prove that there exists an infinite number of prime numbers.

254. (a) Prove that among the numbers of the arithmetic progressions $3, 7, 11, 15, 19, 23, \cdots$ and $5, 11, 17, 23, 29, 35, \cdots$ there are an infinite number of primes.

(b)* Prove that there are an infinite number of primes in the arithmetic progression

$$5, 9, 13, 17, 21, 25, \cdots .$$

(c)* Prove that there are an infinite number of primes in the arithmetic progression

$$11, 21, 31, 41, 51, 61, \cdots .$$

Remark: The following more general theorem holds: *If the first term of an infinite arithmetic progression of integers is relatively prime to the common difference, the progression contains an infinite number of primes.* However, the proof of this theorem is quite complicated. (It is interesting that an elementary, albeit very difficult, proof of this classical theorem of number theory was published for the first time only in 1952 by the Danish mathematician Selberg. Prior to this the only known proofs involved higher mathematics.)

11

SOME DISTINCTIVE INEQUALITIES

This section presents several problems relating to inequalities stemming from two important inequalities which play a major role in mathematical analysis and in geometry. These are the theorem relating arithmetic and geometric means (problem 268), and the so-called Cauchy-Buniakowski inequality (problem 289). Many problems on inequalities, not related to these two but of importance in other applications, appear in other sections of this book (see, in particular, Sections 6 and 7).

A great many interesting inequalities may be found in the *Problem Book in Algebra*, by V. A. Kretchmer, Government Technical Publishing House, Moscow, 1950, where an entire chapter is devoted to inequalities. That book offers alternative proofs of several of the inequalities presented here. There is also much interesting material in the books by P. P. Korovkin, *Inequalities* (Government Technical Publishing House, Moscow, 1951), by G. L. Nevyashy, *Inequalities* (Pedagogical Publishing House, Moscow, 1947), and particularly that by Hardy, Littlewood, and Polya, *Inequalities*, (Government Technical Publishing House, Moscow, 1949).†

The initial chapters of the last book may be read by persons not acquainted with higher mathematics.

The following problems are not presented in order of increasing difficulty.

† The last book was originally written in English. It is published by Cambridge University Press, revised edition, [*Editor*].

The ordering is such that in some instances the result of one problem will b useful in solving the next; in other instances problems conceptually related ar grouped together. The simplest properties of inequalities are assumed known

In all the problems of this section, small English letters designate rea numbers.

Theorems on Arithmetic and Geometric Means and Their Applications

We know, from formal mathematics courses, that the geometric mean of tw positive numbers a and b is less than, or equal to, their arithmetic mean,

$$\sqrt{ab} \leqq \frac{a+b}{2}, \qquad (1$$

and the equality holds only if $a = b$. This is proved as follows.

If we square both members of the inequality and clear of fractions, we arriv at

$$4ab \leqq (a+b)^2 .$$

Expanding the right member, transposing $4ab$ to the right side, and so on, w obtain

$$0 \leqq a^2 - 2ab + b^2 = (a-b)^2 ,$$

which clearly is true for all numbers a and b, since the square of any re number is nonnegative.

Hence, inequality (*1*) holds for every real number. Moreover, it is evide that $(a-b)^2$ can be zero only if $a = b$; that is, the last inequality reduces t the equality only for $a = b$. Therefore, this criterion must hold also for i equality (*1*).

Inequality (*1*) may be rewritten in the following equivalent form, which w shall use hereafter:

$$\left(\frac{a+b}{2}\right)^2 \leqq \frac{a^2+b^2}{2} . \qquad (1$$

If we expand the left member of (*1'*), clear of fractions and put all terms i the right member, we obtain

$$0 \leqq 2a^2 + 2b^2 - (a^2 + 2ab + b^2) = (a-b)^2 .$$

Use of inequalities (*1*) and (*1'*) simplifies the solution of the first of the pro lems which follow. These two forms of the inequality are useful in the der vation of many generalizations, the most important of which are the propositio of problems 268 and 283.

The arithmetic mean of n positive numbers $a_1, a_2, \cdots, a_n$ is defined by th following expression:

$$A_n(a) = \frac{a_1 + a_2 + \cdots + a_n}{n} .$$

The geometric mean of n positive numbers $a_1, a_2, \cdots, a_n$ is defined as the nth root of their product:

$$\Gamma_n(a) = \sqrt[n]{a_1 a_2 \cdots a_n} .$$

Finally, *the harmonic mean* of n positive numbers is the number $H(a)$ such that

$$\frac{1}{H(a)} = \frac{1/a + 1/a_2 + \cdots + 1/a_n}{n}$$

(the reciprocal of the harmonic mean of n numbers is the arithmetic mean of the numbers inverse to the given ones). In particular, the harmonic mean of two numbers a and b is determined by the equation

$$\frac{1}{c} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right),$$

from which $c = 2ab/(a+b)$.

255. (a) Prove that, of all rectangles having the same given perimeter P, the square encloses the greatest area.

(b) Prove that, of all rectangles having the same given area S, that of smallest perimeter is the square.

256. Prove that the sum of the legs of a right triangle never exceeds $\sqrt{2}$ times the hypotenuse of the triangle.

257. Prove that for every acute angle α

$$\tan\alpha + \cot\alpha \geqq 2 .$$

258. Prove that if $a + b = 1$, where a and b are positive numbers, then

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geqq \frac{25}{2} .$$

Determine for what values of a and b the equality holds.

259. Prove, given any three positive numbers a, b, and c, the following inequality holds:

$$(a+b)(b+c)(c+a) \geqq 8abc .$$

Show that the equality holds only for $a = b = c$.

260. For what values of x does the following fraction have the least value?

$$\frac{a + bx^4}{x^2} \qquad (a \text{ and } b \text{ positive}).$$

261. A butcher has an inaccurate balance scale (its beams are of unequal length). Knowing that it is inaccurate, and being an honest merchant, he weighs his meat as follows. He takes half of it and places it on one pan, and he places the weights on the other pan;

then he weighs the other half of the meat by reversing this procedure, that is, by removing the weights and placing the meat on that pan. Thus, the butcher believes he is giving honest weight. Is his assumption correct?

262. (a) Prove that the geometric mean of two positive numbers is equal to the geometric mean of their arithmetic and harmonic means.

(b) Prove that the harmonic mean of two positive numbers a and b does not exceed the geometric mean, and that the equality holds only if $a = b$.

263.* Prove that the arithmetic mean of three positive numbers is not less than their geometric mean, that is,

$$\frac{a+b+c}{3} \geqq \sqrt[3]{abc},$$

and that the equality holds only if $a = b = c$.

264. Prove that, of all triangles with the same given perimeter, the greatest area is enclosed by the equilateral triangle.

265. Given a three-faced pyramid having a right trihedral angle at the vertex. Designate the edges from the vertex by x, y, and z. For what x, y, and z is the volume of the pyramid a maximum if it is known that

$$x + y + z = a?$$

266. Given six positive numbers $a_1, a_2, a_3, b_1, b_2, b_3$. Prove that the following inequality holds:

$$\sqrt[3]{(a_1+b_1)(a_2+b_2)(a_3+b_3)} \geqq \sqrt[3]{a_1a_2a_3} + \sqrt[3]{b_1b_2b_3}.$$

267. *A Special Case of the Theorem Concerning the Arithmetic and Geometric Means.* Given 2^m positive numbers $a_1, a_2, \cdots, a_{2^m}$. Prove the inequality

$$\Gamma_{2^m} \leqq A_{2^m}(a),$$

that is,

$$\sqrt[2^m]{a_1a_2\cdots a_{2^m}} \leqq \frac{a_1+a_2+\cdots+a_{2^m}}{2^m},$$

and that the equality holds only if all the numbers $a_1, a_2, \cdots, a_{2^m}$ are equal.

268.* *Theorem of the Arithmetic and Geometric Means for n Numbers.* Prove that for any n positive numbers $a_1, a_2, \cdots, a_n$

$$\Gamma_n(a) \leqq A_n(a),$$

that is,

$$\sqrt[n]{a_1a_2\cdots a_n} \leqq \frac{a_1+a_2+\cdots+a_n}{n},$$

and that the equality holds only if $a_1 = a_2 = \cdots = a_n$.

269. (a) Consider all sets of n positive numbers whose sum is a given number k. Prove that the maximum product of the numbers of any such set is attained when all the numbers are equal.

(b) Given n positive numbers $a_1, a_2, \cdots, a_n$. Prove that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_1} \geqq n.$$

270. Prove that for n positive numbers $a_1, a_2, \cdots, a_n$ the following inequality holds,

$$H(a) \leqq \Gamma(a),$$

that is,

$$\frac{n}{\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)} \leqq \sqrt[n]{a_1a_2\cdots a_n},$$

and that the equality is obtained only if $a_1 = a_2 = \cdots = a_n$.

271. Prove that for two positive numbers a and b

$$\sqrt[n+1]{ab^n} \leqq \frac{a+nb}{n+1},$$

and that equality can hold only if $a = b$.

272. Prove that for any set of positive numbers $a_1, a_2, \cdots, a_n$

$$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right) \geqq n^2.$$

When does the equality hold?

273. Prove that for any integer $n > 1$

$$n! < \left(\frac{n+1}{2}\right)^n.$$

274. Prove that the following inequality holds for any four positive numbers a_1, a_2, a_3, a_4:

$$a_1a_2^2a_3^3a_4^4 \leqq \left(\frac{a_1 + 2a_2 + 3a_3 + 4a_4}{10}\right)^{10}.$$

275. Prove the following.

(a) $1 \cdot \frac{1}{2^2} \cdot \frac{1}{3^3} \cdot \frac{1}{4^4} \cdots \frac{1}{n^n} < \left[\frac{2}{n+1}\right]^{[n(n+1)]/2}$;

(b) $1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdots n^n < \left[\frac{2n+1}{3}\right]^{[n(n+1)]/2}$

([a] means "the largest integer in a").

276. Let $a_1, a_2, \cdots, a_n$ be positive numbers, and let

$$s = a_1 + a_2 + \cdots + a_n .$$

Prove that

$$(1 + a_1)(1 + a_2)\cdots(1 + a_n) \leqq 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots + \frac{s^n}{n!}.$$

277. Prove that for every integer n

$$\sqrt{2}\,\sqrt[4]{4}\,\sqrt[8]{8}\cdots\sqrt[2^n]{2^n} \leqq n + 1 .$$

278. For which value of x is the product

$$(1 - x)^5(1 + x)(1 + 2x)^2$$

a maximum, and what is this value?

279.* Inscribe between a given segment of a circle and the arc of the circle the rectangle of greatest area.

280. From a square piece of cardboard measuring $2a$ on each side a box with no top is to be formed by cutting out from each corner a square with sides b and bending up the flaps, as shown in Figure 4. For what value of b will the box contain the greatest volume?

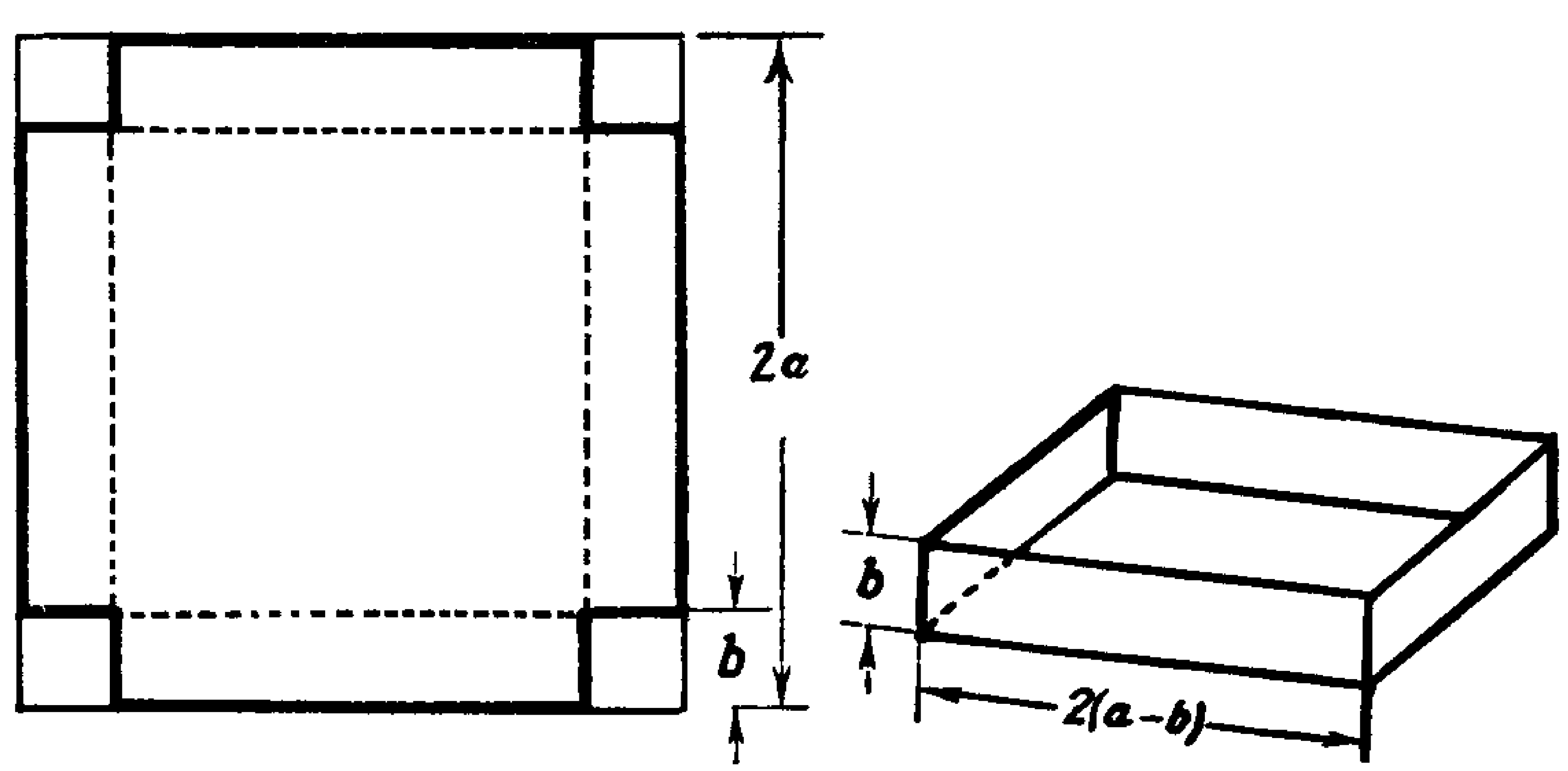

Figure 4

Two Generalizations of the Theorem Concerning Arithmetic and Geometric Means

The *power mean* of order α of n positive numbers $a_1, a_2, \cdots, a_n$ is defined to be the number

$$S_\alpha(a) = \left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}\right)^{1/\alpha};$$

in particular, if $\alpha = k$ is a whole number, we obtain

$$S_k(a) = \sqrt[k]{\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}}.$$

It is easy to see that $S_1(a) = A(a)$ and $S_{-1}(a) = H(a)$.

If $\alpha = 0$, the expression for S_α is meaningless. On the other hand, it can be proved that if $\alpha \to 0$, then $S_\alpha(a)$ tends to the geometric mean $\Gamma(a)$.† Therefore, it is convenient to define $S_0(a) = \Gamma(a)$. (An additional justification for this definition is given in problem 282.) The power mean of order 2 is referred to as the *quadratic mean*.

Inequality (*1'*) (see the remark at the beginning of this section) can now be stated as follows: *The arithmetic mean of two numbers does not exceed their quadratic mean* (and the equality holds only if the numbers are equal).

281.* (a) Prove that the arithmetic mean of n positive numbers does not exceed the quadratic mean:

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^2 \leqq \frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}.$$

The equality holds only if the numbers are all equal.

(b) Let k be any integer greater than 1. Prove that the arithmetic mean of n positive numbers does not exceed their power mean of order k:

$$\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^k \leqq \frac{a_1^k + a_2^k + \cdots + a_n^k}{n}.$$

The equality holds only if all the numbers are equal.

† That is,

$$\lim_{\alpha\to 0}\left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}\right)^{1/\alpha} = \sqrt[n]{a_1a_2\cdots a_n}.$$

See V. E. Levine, "Elementary proof of one theorem of the theory of means," Math. Educa., Issue 3, pp. 177-181. Moscow, 1958.

282. Prove that the power mean of order α of n positive numbers, for $\alpha > 0$, is not less than the geometric mean, and, for $\alpha < 0$, is not greater than the geometric mean (equality holds only if all n numbers are equal.)

Remark: The theorems of problems 268 and 270 are particular cases of this proposition.

283.* *Theorem of Power Means.* Prove that if $\alpha < \beta$, then the power mean of order α does not exceed the power mean of order β:

$$\left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}\right)^{1/\alpha} \leqq \left(\frac{a_1^\beta + a_2^\beta + \cdots + a_n^\beta}{n}\right)^{1/\beta}.$$

The equality holds only if $a_1 = a_2 = \cdots = a_n$.

284. (a) The sum of three positive numbers is equal to 6. What is the smallest value which the sum of their squares can have? What is the smallest value which the sum of their cubes can have?

(b) The sum of the squares of three positive numbers is equal to 18. What is the smallest possible value for the sum of the cubes of these numbers? What is the smallest possible value for the sum of these numbers?

The symmetric mean of order k of n positive numbers $a_1, a_2, \cdots, a_n$ (where k is a natural number not exceeding n) is defined to be the kth root of the sum of all possible products of these n numbers taken k at a time:

$$\sum_k(a) = \sqrt[k]{\frac{a_1a_2\cdots a_k + a_1a_2\cdots a_{k-1}a_{k+1} + \cdots + a_{n-k+1}a_{n-k+2}\cdots a_n}{C_n^k}}.$$

It is clear that $\sum_1(a) = A(a)$, $\sum_n(a) = \Gamma(a)$.

285. Prove that

$$(\textstyle\sum_k)^{2k} \geqq (\sum_{k+1})^{k+1}\cdot(\sum_{k-1})^{k-1}.$$

286. *Theorem of the Symmetric Mean.* Prove that if $k > l$, then

$$\textstyle\sum_k(a) \leqq \sum_l(a).$$

The equality holds only if $a_1 = a_2 = \cdots = a_n$.

287. Given that the sum of all six possible pairwise products of four numbers is equal to 24. What is the smallest value possible for the sum of the four numbers? What is the greatest possible value for the product of the numbers?

288. Let $\alpha + \beta + \gamma = \pi$.

(a) Find the smallest possible value for

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2}.$$

(b) Find the largest possible value for

$$\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2}.$$

The Cauchy–Buniakowski Inequality

The following elementary inequality is readily verified:

$$a_1b_1 + a_2b_2 \leqq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$$

or,

$$(a_1b_1 + a_2b_2)^2 \leqq (a_1^2 + a_2^2)(b_1^2 + b_2^2). \qquad (1)$$

Expanding both sides and collecting all terms on one side we obtain

$$(a_1b_2 - a_2b_1)^2 \geqq 0.$$

It follows that inequality (*1*) becomes an equality if

$$a_1b_2 = a_2b_1,$$

that is, if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

Inequality (*1*) yields a significant generalization which is important in inequality theory and has useful applications in mathematics and physics.†

289. *The Cauchy–Buniakowski Inequality.* Prove that for any $2n$ real numbers $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ the following inequality holds:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leqq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

The equality holds only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

290. Use the Cauchy-Buniakowski inequality to derive the results of problem 272.

291. Use the Cauchy-Buniakowski inequality to obtain the theorem of problem 281 (a).

292. Prove that if $\alpha + \beta + \gamma = \Pi$, then

† This inequality is sometimes referred to, in other texts, as the Cauchy-Schwarz inequality [*Editor*].

$$\tan^2\frac{\alpha}{2}+\tan^2\frac{\beta}{2}+\tan^2\frac{\gamma}{2}\geqq 1\,.$$

293. Prove for any positive numbers $x_1, x_2, \cdots, x_n;\ y_1, y_2, \cdots, y_n$:

$$\sqrt{(x_1+y_1)^2+(x_2+y_2)^2+\cdots+(x_n+y_n)^2} \leqq \sqrt{x_1^2+x_2^2+\cdots+x_n^2}+\sqrt{y_1^2+y_2^2+\cdots+y_n^2}\,.$$

294. Let Q be the sum of all the possible pairwise products of the n positive numbers $a_1, a_2, \cdots, a_n$, and let P be the sum of their squares. Prove that

$$Q\leqq\frac{n-1}{2}P\,.$$

295. Prove that, given $2n$ positive numbers $p_1, p_2, \cdots, p_n;\ x_1, x_2, \cdots, x_n$, the following inequality holds:

$$(p_1x_1+p_2x_2+\cdots+p_nx_n)^2 \leqq (p_1+p_2+\cdots+p_n)(p_1x_1^2+p_2x_2^2+\cdots+p_nx_n^2)\,.$$

296. Verify that for any three arbitrary numbers x_1, x_2, x_3 the following inequality holds:

$$\left(\frac{1}{2}x_1+\frac{1}{3}x_2+\frac{1}{6}x_3\right)^2\leqq\frac{1}{2}x_1^2+\frac{1}{3}x_2^2+\frac{1}{6}x_3^2\,.$$

297. Prove that if $x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n$ are positive numbers, then

$$\sqrt{x_1y_1}+\sqrt{x_2y_2}+\cdots+\sqrt{x_ny_n} \leqq \sqrt{x_1+x_2+\cdots+x_n}\cdot\sqrt{y_1+y_2+\cdots+y_n}\,.$$

298. Let $a_1, a_2, \cdots, a_n;\ b_1, b_2, \cdots, b_n;\ c_1, c_2, \cdots, c_n;\ d_1, d_2, \cdots, d_n$ be four sequences of positive numbers. Prove the inequality

$$(a_1b_1c_1d_1+a_2b_2c_2d_2+\cdots+a_nb_nc_nd_n)^4 \leqq (a_1^4+a_2^4+a_3^4+\cdots+a_n^4)(b_1^4+b_2^4+b_3^4+\cdots+b_n^4) \times (c_1^4+c_2^4+c_3^4+\cdots+c_n^4)(d_1^4+d_2^4+d_3^4+\cdots+d_n^4)\,.$$

299.* The Cauchy-Buniakowski inequality (problem 289) verifies that the relationship

$$\frac{(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2)}{(a_1b_1+a_2b_2+\cdots+a_nb_n)^2}\,,$$

where $a_1, a_2, \cdots, a_n;\ b_1, b_2, \cdots, b_n$ are two sequences of positive numbers, is greater than or equal to 1 $\left(\text{and is equal to 1 only if } \frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}\right)$. Prove that this value is always included between 1 and the quantity

$$1+\left(\frac{\sqrt{M_1M_2/m_1m_2}-\sqrt{m_1m_2/M_1M_2}}{2}\right)^2 \left[=\left(\frac{\sqrt{M_1M_2/m_1m_2}+\sqrt{m_1m_2/M_1M_2}}{2}\right)^2,\right]$$

where M_1 and m_1 are, respectively, the greatest and the least of the numbers $a_1, a_2, \cdots, a_n$, and M_2 and m_2 are, respectively, the greatest and least of the numbers $b_1, b_2, \cdots, b_n$. In which case does the value exactly equal the following?

$$1+\left(\frac{\sqrt{M_1M_2/m_1m_2}-\sqrt{m_1m_2/M_1M_2}}{2}\right)^2.$$

Some Additional Inequalities

300. *Chebycheff's Inequality.* Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be two nonincreasing sequences of numbers. The following inequality holds:

$$\frac{a_1+a_2+\cdots+a_n}{n}\cdot\frac{b_1+b_2+\cdots+b_n}{n}\leqq\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n}\,,$$

the equality holding only if $a_1=a_2=\cdots=a_n$ and $b_1=b_2=\cdots=b_n$.

Remark: It is possible to show that if $a_1, a_2, \cdots, a_n$ is a nonincreasing sequence of numbers, and if $b_1, b_2, \cdots, b_n$ is nondecreasing, then

$$\frac{a_1+a_2+\cdots+a_n}{n}\cdot\frac{b_1+b_2+\cdots+b_n}{n}\geqq\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n}\,.$$

The proof of this proposition is left to the reader.

301. Let p and q be positive rational numbers for which

$$\frac{1}{p}+\frac{1}{q}=1\,.$$

Prove that for any positive numbers x and y the following inequality holds:

$$xy\leqq\frac{1}{p}x^p+\frac{1}{q}y^q\,.$$

Remark: For $p = q = 2$ we obtain the Elementary Theorem of the Arithmetic and Geometric Means.

302. Let α and β be positive rational numbers, where $\alpha + \beta = 1$. Prove that for any positive numbers $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$ the following inequality holds:

$$a_1^\alpha b_1^\beta + a_2^\alpha b_2^\beta + \cdots + a_n^\alpha b_n^\beta \leqq (a_1 + a_2 + \cdots + a_n)^\alpha (b_1 + b_2 + \cdots + b_n)^\beta .$$

Remark: If $\alpha = \beta = \frac{1}{2}$, it is readily seen that we obtain the inequality of problem 297, which is equivalent to the Cauchy-Buniakowski inequality.

303. *Holder's Inequality*. Let p and q be positive rational numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1 .$$

Prove that for any positive numbers $x_1, x_2, \cdots, x_n$; $y_1, y_2, \cdots, y_n$ the following inequality holds:

$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \leqq (x_1^p + x_2^p + \cdots + x_n^p)^{1/p} (y_1^q + y_2^q + \cdots + y_n^q)^{1/q} .$$

Remark: If $p = q = 2$, this inequality becomes that of Cauchy-Buniakowski (problem 289), which, in turn, is a special case of Hölder's inequality.

304. Let $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$; $\cdots$; $l_1, l_2, \cdots, l_n$ be k sequences of positive numbers, and $\alpha, \beta, \cdots, \lambda$ be k positive number such that

$$\alpha + \beta + \cdots + \lambda = 1 .$$

Prove that

$$a_1^\alpha b_1^\beta \cdots l_1^\lambda + a_2^\alpha b_2^\beta \cdots l_2^\lambda + \cdots + a_n^\alpha b_n^\beta \cdots l_n^\lambda \leqq (a_1 + a_2 + \cdots + a_n)^\alpha (b_1 + b_2 + \cdots + b_n)^\beta \cdots (l_1 + l_2 + \cdots + l_n)^\lambda .$$

305. Let $a_1, a_2, \cdots, a_n$ be n positive numbers, and let g be their geometric mean ($g = \sqrt[n]{a_1 a_2 \cdots a_n}$). Prove that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geqq (1 + g)^n .$$

306. Prove that if $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$; $\cdots$; $l_1, l_2, \cdots, l_n$ are k sequences of positive numbers, then the following holds:

$$\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} + \cdots + \sqrt[n]{l_1 l_2 \cdots l_n} \leqq \sqrt[n]{(a_1 + b_1 + \cdots + l_1)(a_2 + b_2 + \cdots + l_2)(a_n + b_n + \cdots + l_n)} .$$

307. Let x, y, and z be positive numbers for which

$$x + y + z = 1 .$$

Prove that

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right) \geqq 64 .$$

308. *Minkowski's Inequality*. Let $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$; $\cdots$; $l_1, l_2, \cdots, l_n$, be k sequences of positive numbers. Prove that

$$\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} + \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} + \cdots + \sqrt{l_1^2 + l_2^2 + \cdots + l_n^2}$$
$$\geqq \sqrt{(a_1 + b_1 + \cdots + l_1)^2 + (a_2 + b_2 + \cdots + l_2)^2 + \cdots + (a_n + b_n + \cdots + l_n)^2} .$$

Remark: The inequality of problem 308 [a generalization of the result of problem 186 (a)] can also be written in the form

$$S_2(a) + S_2(b) + \cdots + S_2(l) \geqq S_2(a + b + \cdots + l) ,$$

where S_2 is the quadratic mean of n numbers (see p. 67).

A more general formulation of Minkowski's inequality is as follows. If $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$; $\cdots$; $l_1, l_2, \cdots, l_n$, are k sequences of positive numbers, then

$$S_2(a) + S_2(b) + \cdots + S_2(l) \begin{cases} \geqq S_\alpha(a + b + \cdots + l) & \text{if } \alpha > 1 ; \\ \leqq S_\alpha(a + b + \cdots + l) & \text{if } \alpha < 1 . \end{cases}$$

In particular, the inequality of problem 306, which may be written

$$\Gamma(a) + \Gamma(b) + \cdots + \Gamma(l) \leqq \Gamma(a + b + \cdots + l)$$

or

$$S_0(a) + S_0(b) + \cdots + S_0(l) \leqq S_0(a + b + \cdots + l)$$

is a special case of Minkowski's inequality for $\alpha = 0$.

12

DIFFERENCE SEQUENCES AND SUMS

Consider the sequence of numbers

$$u_0, u_1, u_2, \cdots, u_n, \cdots .$$

The *first difference sequence* of this sequence is the set of numbers

$$u_0^{(1)} = u_1 - u_0 ;$$
$$u_1^{(1)} = u_2 - u_1 ;$$
$$u_2^{(1)} = u_3 - u_2 ;$$
$$\cdots\cdots\cdots ;$$
$$u_n^{(1)} = u_{n+1} - u_n , \cdots .$$

The *second difference sequence* is the difference sequence of the previous sequence:

$$u_0^{(2)} = u_1^{(1)} - u_0^{(1)} ;$$
$$u_1^{(2)} = u_2^{(1)} - u_1^{(1)} ;$$
$$u_2^{(2)} = u_3^{(1)} - u_2^{(1)} ;$$
$$\cdots\cdots\cdots ;$$
$$u_n^{(2)} = u_{n+1}^{(1)} - u_n^{(1)} .$$

Analogously, the sequence of *differences of* kth *order*, $u_0^{(k)}, u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$, is the sequence obtained by working on the $(k-1)$st sequence of differences, $u_0^{(k-1)}, u_1^{(k-2)}, u_2^{(k-3)}, \cdots$. For example, if the initial sequence of numbers is the arithmetic progression $1, 5, 9, 13, 17, \cdots$, then the first row of differences consists of the numbers $4, 4, 4, 4, \cdots$, and the differences of second order form

a sequence of zeros: $0, 0, 0, 0, \cdots$. If the initial sequence is the set of squares of integers, $1, 4, 9, 16, 25, 36, 49, \cdots$, then the differences of first order form the sequence of odd numbers: $3, 5, 7, 9, 11, 13, \cdots$; the differences of second order form the sequence $2, 2, 2, 2, \cdots$, and the third sequence (differences of third order) consists of zeros. [In the examples investigated we quickly arrived at a sequence of zeros, and this is related to the general proposition of 309 (b).]

The sequences of differences of a finite sequence of numbers can be conveniently written in ***triangular array***:

$$\begin{array}{ccccccccc}
u_0 & & u_1 & & u_2 & & \cdots & & u_n \\
& u_0^{(1)} & & u_1^{(1)} & & u_2^{(1)} & \cdots & u_{n-1}^{(1)} & \\
& & u_0^{(2)} & & u_1^{(2)} & & \cdots u_{n-2}^{(2)} & & \\
& & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
& & & & u_0^{(n)} & & & &
\end{array}$$

Here it is apparent that each number is the difference of the two adjacent numbers of the row above. For an infinite sequence of numbers the triangular (infinite) array has the form

$$\begin{array}{ccccccccccc}
u_0 & & u_1 & & u_2 & & \cdots & & u_n & & \cdots \\
& u_0^{(1)} & & u_1^{(1)} & & u_2^{(1)} & & \cdots & & u_n^{(1)} & & \cdots \\
& & u_0^{(2)} & & u_1^{(2)} & & u_2^{(2)} & & \cdots & & u_n^{(2)} & & \cdots \\
& & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
& & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots &
\end{array}$$

In a fashion analogous to finding the successive sequences of differences of a set of numbers we can also define sequences of sums. The sequence of ***sums of first order*** of the set of numbers $u_0, u_1, u_2, \cdots, u_n, \cdots$, which we shall designate by writing $\bar{u}_0^{(1)}, \bar{u}_1^{(1)}, \bar{u}_2^{(1)}, \cdots, \bar{u}_n^{(1)}, \cdots$, is defined by

$$\bar{u}_0^{(1)} = u_0 + u_1 ;$$
$$\bar{u}_1^{(1)} = u_1 + u_2 ;$$
$$\cdots\cdots\cdots\cdots ;$$
$$\bar{u}_n^{(1)} = u_n + u_{n+1} ;$$
$$\cdots\cdots\cdots\cdots ;$$

The sequence of ***sums of*** kth ***order*** of the numbers $u_0, u_1, \cdots, u_n, \cdots$ is obtained from the $(k-1)$st row of such sums. We shall designate the sums of kth order by $\bar{u}_0^{(k)}, \bar{u}_1^{(k)}, \cdots, \bar{u}_n^{(k)}, \cdots$.

For example, if the initial set is the sequence of ones, $1, 1, 1, 1, \cdots$, then the sequence of sums of first order consists of two's: $2, 2, 2, 2, \cdots$; the sequence of sums of second order is $4, 4, 4, 4, \cdots$; the third sequence is $8, 8, 8, 8, \cdots$; and so on. If the initial set is the sequence of natural numbers, $1, 2, 3, 4, 5, \cdots$, then each sequence of sums will form an arithmetic progression:

$$\begin{array}{ccccccccccccc} 3, & & 5, & & 7, & & 9, & & 11, & & 13, & & \cdots \\ & 8, & & 12, & & 16, & & 20, & & 24, & & \cdots & \\ & & 20, & & 28, & & 36, & & 44, & & \cdots, & & \end{array}$$

and so on.

Sequences of successive sums of a finite set $u_0, u_1, \cdots, u_n$ can be conveniently displayed in triangular array:

$$\begin{array}{ccccccccccc} u_0 & & u_1 & & u_2 & & \cdots & & u_{n-1} & & u_n \\ & \bar{u}_0^{(1)} & & \bar{u}_1^{(1)} & & \bar{u}_2^{(1)} & & \cdots & & \bar{u}_{n-1}^{(1)} & \\ & & \bar{u}_0^{(2)} & & \bar{u}_1^{(2)} & & \cdots & & \bar{u}_{n-2}^{(2)} & & \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & & \bar{u}_0^{(n-1)} & & \bar{u}_1^{(n-1)} & & & & \\ & & & & & \bar{u}_0^{(n)} & & & & & \end{array}$$

Here, each number is the sum of the two adjacent numbers in the row above. If we consider an infinite sequence of numbers $u_0, u_1, u_2, \cdots, u_n$, then the triangular array continues indefinitely.

We now consider a related concept, *Pascal's Triangle* (or the *Arithmetic Triangle*):

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Here, the rows are bordered on each end by ones, and the interior integers are obtained as the sum of the two adjacent numbers of the previous row.

For convenience we shall start the row enumeration of the Pascal triangle with the number zero; that is, the number one at the apex of the triangle will be thought of as the 0th row; the sequence 1, 1 constitutes the first row, and so on. We shall designate the $(k+1)$st element of the nth row as C_n^k (that is, in each row, too, we shall start counting from zero). Using this terminology, we have the following format for Pascal's triangle:

$$\begin{array}{ccccccccc} & & & & C_0^0 & & & & \\ & & & C_1^0 & & C_1^1 & & & \\ & & C_2^0 & & C_2^1 & & C_2^2 & & \\ & C_3^0 & & C_3^1 & & C_3^2 & & C_3^3 & \\ C_4^0 & & C_4^1 & & C_4^2 & & C_4^3 & & C_4^4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

A number of properties of the members in the Pascal triangle have been

developed in the book by B. B. Dynkin and V. A. Uspensky, *Mathematical Conversations*, Issue 6, Section 2, Chapter III, Library of the Mathematical Society. The material contained in the problems of this section are closely related to the material in the interesting popular book by A. E. Markuskevich, *Reflexive Series*, Government Technical Publishing House, Moscow, 1950.

The sequence of numbers obtained by successively substituting, for x in a polynomial $P(x) = a_0x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k$ the sequence of integers $1, 2, 3, \cdots, n$ [that is, the sequence $P(1), P(2), \cdots, P(n)$] will be called the kth *order sequence* of $P(x)$. A special case of a kth order sequence is the sequence $1^k, 2^k, 3^k, 4^k, \cdots, n^k, \cdots$ [that is, $P(x) = x^k$].

309. Let $u_0, u_1, u_2, \cdots, u_n$ be a sequence of kth order; that is, let $u_n = a_0n^k + a_1n^{k-1} + \cdots + a_k$.

(a) Prove that $u_n^{(1)}$ forms a sequence of $(k-1)$st order.

(b) Prove that the $(k+1)$st difference sequence of this sequence consists only of zeros.

310. Prove that if $u_n = a_0n^k + a_1n^{k-1} + \cdots + a^k$, then all the numbers of the kth row of the difference sequences $u_0, u_1, u_2, \cdots, u_n, \cdots$ are equal to $a_0k!$.

311. Prove that:

(a) $\bar{u}_n^{(k)} = C_k^0u_n + C_k^1u_{n+1} + C_k^2u_{n+2} + \cdots + C_k^ku_{n+k}$;

(b) $u_n^{(k)} = (-1)^kC_k^0u_n + (-1)^{k-1}C_k^1u_{n+1}$
$\qquad + (-1)^{k-2}C_k^2u_{n+2} + \cdots + C_k^ku_{n+k}$.

312. Prove that

$$C_n^k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \qquad (k > 0),$$

where $k! = 1\cdot2\cdot3\cdots k$.

313. Prove that

$$u_n = C_n^0u_0 + C_n^1u_0^{(1)} + C_n^2u_0^{(2)} + \cdots + C_n^ku_0^{(k)}.$$

314. Assume that the $(k+1)$st row of successive differences [differences of $(k+1)$st order] of some seqnence consists of zeros, but that the kth row consists of nonzero numbers. Prove that this sequence is a sequence of order k.

Remark: The theorem represented by this problem is the converse of the theorem of problem 309 (b). There we were to prove that the (k + 1)st row of the differences of a kth order sequence consists of zeros. Here we are to prove that if the $(k+1)$st differences of some sequence consists of zeros, then the sequence is of kth order.

315. Find the formula giving the sum of the series

$$1^4 + 2^4 + 3^4 + \cdots + n^4 .$$

316. (a) Prove that the sum $1^k + 2^k + 3^k + \cdots + n^k$ is a polynomial in n of degree $k + 1$.

(b) Calculate the coefficients of n^{k+1} and of n^k of this polynomial.

317. We say that a sequence of integers is divisible by a number d if every number of this sequence is divisible by d. [For example, the sequence of numbers $n^{13} - n$ is divisible by 13; the sequence of numbers $3^{6n} - 2^{6n}$ is divisible by 35; the sequence of numbers $n^5 - 5n^3 + 4n$ is divisible by 120. See problems 27 (d), 28 (a), (b)].

Let u_n be a kth order sequence, $u_n = a_0 n^k + a_1 n^{k-1} + \cdots + a_k$, where the coefficients $a_0, a_1, a_2, \cdots, a^k$ are relatively prime integers. Prove that if the sequence u_n is divisible by an integer d, then d is a divisor of $k!$.

318. Calculate $(C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 + \cdots + (C_n^n)^2$.

319. Using the result of problem 313, prove Newton's binomial formula:

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \cdots + \frac{k(k-1)\cdots 2\cdot 1}{k!}b^k .$$

320. Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$. Construct the successive-difference triangle:

$$\begin{array}{ccccccccccccc}
1 & & \frac{1}{2} & & \frac{1}{3} & & \frac{1}{4} & & \frac{1}{5} & & \frac{1}{6} & & \cdots \\
& -\frac{1}{2} & & -\frac{1}{6} & & -\frac{1}{12} & & -\frac{1}{20} & & -\frac{1}{30} & & \cdots & \\
& & \frac{1}{3} & & \frac{1}{12} & & \frac{1}{30} & & \frac{1}{60} & & \cdots & & \\
& & & -\frac{1}{4} & & -\frac{1}{20} & & -\frac{1}{60} & & \cdots & & & \\
& & & & \frac{1}{5} & & \frac{1}{30} & & \cdots & & & & \\
& & & & & -\frac{1}{6} & & & & & & &
\end{array}$$

Turn this triangle 60° clockwise such that the apex consists of the number 1:

$$\begin{array}{ccccccccccc}
& & & & & 1 & & & & & \\
& & & & -\frac{1}{2} & & \frac{1}{2} & & & & \\
& & & \frac{1}{3} & & -\frac{1}{6} & & \frac{1}{3} & & & \\
& & -\frac{1}{4} & & \frac{1}{12} & & -\frac{1}{12} & & \frac{1}{4} & & \\
& \frac{1}{5} & & -\frac{1}{20} & & \frac{1}{30} & & -\frac{1}{20} & & \frac{1}{5} & \\
-\frac{1}{6} & & \frac{1}{30} & & -\frac{1}{60} & & \frac{1}{60} & & -\frac{1}{30} & & \frac{1}{6}
\end{array}$$

....

Disregard the minus signs of this triangle, and divide through every row by the number at the end of that row to obtain

$$\begin{array}{ccccccccccc}
& & & & & 1 & & & & & \\
& & & & 1 & & 1 & & & & \\
& & & 1 & & \frac{1}{2} & & 1 & & & \\
& & 1 & & \frac{1}{3} & & \frac{1}{3} & & 1 & & \\
& 1 & & \frac{1}{4} & & \frac{1}{6} & & \frac{1}{4} & & 1 & \\
1 & & \frac{1}{5} & & \frac{1}{10} & & \frac{1}{10} & & \frac{1}{5} & & 1
\end{array}$$

....

Finally, substitute for each number its reciprocal (that is, replace a/b by b/a).

Prove that this end result gives Pascal's triangle.