SIXTY-EIGHTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 1, 2007 **Examination A**

Problem A1

Find all values of α for which the curves $y = \alpha x^2 + \alpha x + \frac{1}{24}$ and $x = \alpha y^2 + \alpha y + \frac{1}{24}$ are tangent to each other.

Problem A2

Find the least possible area of a convex set in the plane that intersects both branches of the hyperbola $xy = 1$ and both branches of the hyperbola $xy = -1$. (A set S in the plane is called *convex* if for any two points in S the line segment connecting them is contained in S.)

Problem A3

Let k be a positive integer. Suppose that the integers $1, 2, 3, \ldots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

Problem A4

A *repunit* is a positive integer whose digits in base 10 are all ones. Find all polynomials f with real coefficients such that if n is a repunit, then so is $f(n)$.

Problem A5

Suppose that a finite group has exactly n elements of order p, where p is a prime. Prove that either $n = 0$ or p divides $n+1$.

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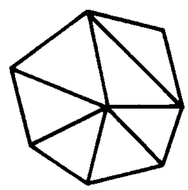
Problem A6

A *triangulation* $\mathcal{T}$ of a polygon P is a finite collection of triangles whose union is P, and such that the intersection of any two triangles is either empty, or a shared vertex, or a shared side. Moreover, each side of P is a side of exactly one triangle in $\mathcal{T}$. Say that $\mathcal{T}$ is *admissible* if every internal vertex is shared by 6 or more triangles. For example



Prove that there is an integer M_n, depending only on n, such that any admissible triangulation of a polygon P with n sides has at most M_n triangles.

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Saturday, December 1, 2007 **Examination B**

Problem B1

Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n=1$.

Problem B2

Suppose that $f:[0,1]\to\mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx=0$.

Prove that for every $\alpha\in(0,1)$,

$$\left|\int_0^\alpha f(x)\,dx\right|\le\frac{1}{8}\max_{0\le x\le 1}|f'(x)|.$$

Problem B3

Let $x_0=1$ and for $n\ge 0$, let $x_{n+1}=3x_n+\left\lfloor x_n\sqrt{5}\right\rfloor$. In particular, $x_1=5$, $x_2=26$, $x_3=136$, $x_4=712$. Find a closed-form expression for x_{2007}. ($\lfloor a\rfloor$ means the largest integer $\le a$.)

Problem B4

Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2+(Q(X))^2=X^{2n}+1$$

and $\deg P>\deg Q$.

Problem B5

Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n),\ldots,P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor\frac{n}{k}\right\rfloor^k=P_0(n)+P_1(n)\left\lfloor\frac{n}{k}\right\rfloor+\cdots+P_{k-1}(n)\left\lfloor\frac{n}{k}\right\rfloor^{k-1}.$$

($\lfloor a\rfloor$ means the largest integer $\le a$.)

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Examination B

Problem B6

For each positive integer n, let $f(n)$ be the number of ways to make $n!$ cents using an unordered collection of coins, each worth $k!$ cents for some k, $1 \le k \le n$. Prove that for some constant C, independent of n,

$$n^{n^2/2 - Cn} e^{-n^2/4} \le f(n) \le n^{n^2/2 + Cn} e^{-n^2/4}.$$