TOPICS IN MATHEMATICS (MATH 100)

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Abstract. We describe here in general terms the idea of the course *Topics in Mathematics*. The intention is not to give a master syllabus for the course, but rather to provide an understanding of the nature of the course which should influence the design of specific syllabi. It is a college level liberal arts course. We want students to learn about the nature of mathematics, our mathematical heritage, and how mathematics influences most every aspect of modern life. At the same time the course is designed to satisfy the symbolic reasoning requirement of our undergraduate students.

1. Introduction

The Department of Mathematics of the University of Hawaii at Manoa and most other campuses within the University of Hawaii system offer the course *Topics in Mathematics*; for short *Math 100*. By design, it is a *college level liberal arts course*, which covers a number of elementary albeit challenging topics in mathematics. At the same time, the course is designed to satisfy the university's symbolic reasoning requirement, as embodied in six hallmarks that we list below.

Math 100 is a terminal course; it is not a prerequisite for any other mathematics course and is not designed to prepare students for other courses.

This is no attempt to dictate a syllabus. We want to share our understanding of the concept of Math 100 and what we hope the students experience in the course. After reading these few pages we hope that other instructors are influenced in the way they design their own Math 100.

2. The liberal arts aspect of *Topics in Mathematics*

We would like students to learn, mostly through a guided direct experience with mathematics, about

- The nature of mathematics as a human activity.
- The role of mathematics in the world, from intellectual history to modern technology.

This is essential for any educated person. The former will require experience with mathematical experimentation, the development of formal mathematical systems and their analysis using rigorous mathematical logic. The latter involves more of this, and includes the development, analysis and use to

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algorithms, the key tools of mathematical applications in technology. Here technology refers not just to engineering but also to the mathematical modelling behind, for example, the design of insurance and pension plans, the tax code, investment vehicles, and political campaign strategies.

Students should, with respect to mathematics,

(1) Learn something of their intellectual heritage and how it has shaped their world.
(2) Discover in themselves a capacity for mathematical thought.
(3) Understand the enormous impact of mathematics in the world today in technology and other braches of human activity.

3. Choice of Topics and Pedagogy

So much for goals. Let us look at the course from a more practical point of view. First of all, the specific course content of Math 100 will depend on many variables, such as the class size, the background and interests of the instructor and the students, and where the course is taught. Many different choices of topics are acceptable, and we will look closely at some below.

As the title of the course indicates, we study a variety of topics. This allows us to illustrate that there is a great diversity of mathematical thought, which goes far beyond arithmetic, geometry and logic. Illustrating this diversity is one of the important features of the course. With the start of each new topic the students have a chance to make a fresh start themselves. This feature of the course helps to keep the class together and to prevent students from falling hopelessly behind.

In the discussion of each individual topic we generally follow the path of encounter, exploration, formulation, struggle, solution, application and generalization. Mathematical theory is based on and motivated by encounters in real life. There is a period of contemplation and exploration before the problem can be clearly formulated. This goes hand in hand with the struggle to solve the problem and the search for a solution. Many solutions can be applied in situations well beyond the problem which motivated the study. In real life, these periods are not distinct; there is typicaly is a high degree of interaction between these different aspects.

This approach is preferable not only for pedagogical reasons. In the process of developing a topic we can practice reasoning and developing ideas. The most important part of the lesson may not be the final result, but rather the process of finding it.

Through our choice of topics we want to expose the students to a broad view of mathematics. The ideal topic is both appealing in itself and of real significance, and yet it must be possible to discuss the topic with few assumptions about the students' mathematical proficiency. We shall list a few topics and explain in what way they meet our standards.

4. The decimal system

As a possible topic, let us look at the way in which the number system developed. The simplest way to write down a natural number is a tally. Essentially we can count, and make this more efficient by gathering groups into one symbol. We can use a fifth line diagonally through the first four to indicate five, or we can introduce a new symbol, like a V in the Roman system. More sophisticated versions were developed in many cultures. These ways of writing down numbers allow us to count, and to some extent to add and subtract. It is a great intelectual achievement to use numerals and the position of the numerals to express a number. The introduction of the zero is of great importance. It is an abstract idea, because a zero is not relevant for the counting process, but only for writing down numbers in an abstract way.

By this time, numbers are not only reflections of quantities anymore, numbers have become objects in their own right. It becomes possible to multiply and divide, and based on the way numbers are written these operations are simple. Having the addition and multiplication tables for one digit numbers allows us the add and subtract numbers with many digits. For the Babylonians simple algebraic operations with rational numbers were difficult, and could only be performed by highly educated people. Adding Roman numerals is hard, and multiplication is essentially impossible. With the modern system of writing numbers every child learns to add, subtract, multiply, and divide in elementary school. The ingenuity of expressing numbers has turned arithmetic into an easy task. Discovering this way of expressing numbers is a great achievment and took humankind thousands of years. Without this method of writing numbers modern science is unthinkable.

We are so proud of this achievement that we encode it in the messages that we transmit into outer space. If there is intelligent life on another planet, then we expect those beings to be able to decipher our way of writing numbers. It is that basic, universal, and beautiful and logically perfect. It is one of the few messages which we send out, and we think of it as one of the great triumphs of the human intellect.

How do we teach the subject to Math 100 students? We explore the subject from a historical perspective, and we see the problem by understanding the limitations imposed on older cultures by their lack of an efficient number system. We see the struggle to solve the problem of writing down a number in the various improvements which were made over the centuries. We explain the workings of the decimal system by working with some other basis. This teaches the students the principles. It also gives us time to practice algorithms like basic arithmetic using the one-digit addition and multiplication tables. People invented a way of expressing numbers which transgresses languages and cultures, and works anywhere where numbers are used on our planet and possibly anywhere in the universe.

5. The Hallmarks and Beyond

Let us recall the hallmarks before we analyze another possible topic. They are

(1) Students will be exposed to the beauty, power, clarity and precision of formal systems.
(2) Instructors will help students understand the concept of proof as a chain of inferences.
(3) Instructors will teach students how to apply formal rules or algorithms.
(4) Students will be required to use appropriate symbolic techniques in the context of problem solving, and in the presentation and critical evaluation of evidence.
(5) The course will not focus solely on computational skills.
(6) Instructors will build a bridge from theory to practice and show the students how to traverse this bridge.

We confess to being not comfortable with a few of these hallmarks; for example the fourth seems to reflect too narrowly the nature of a specific course (symbolic logic, taught by the department of philosophy). Nevertheless if these hallmarks are interpreted in a way that is meaningful for the actual practice of mathematics, they do indicate some of what we want in a Math 100 course, and a well-designed such course should routinely be regarded as satisfying them.

So, what do we want from Math 100? Merchants worked with numbers in commerce and generals counted their soldiers, priests calculated distances and created a calender. But mathematics is much more. Mathematics is a great human endavor, it is a way of thinking. It is a human creation like music, art, poetry, or literature. We start with real problems, we explore them and seek to understand them. We disect the problems, we analyse them, we look for the underlying structure. We formulate our findings as propositions, and we prove them based on the rules of logic. To understand the essence it is useful to strip away the specifics, to abstract. After solving a problem we don't hesitate to generalize and explore the limits of our knowledge.

We want the students to experience mathematics beyond the numbers, to explore, discover, and reason. We want discuss topics in depth and expose their beauty, and still not be overwhelming. We want to share some of the excitement of mathematics.

6. The Fundamental Theorem of Arithmetic

The Fundamental Theorem of Arithmetic asserts that the prime factorization of a natural number is unique. This classical result, essentially proven by the Greek mathematician Euclid, illustrates in a striking and beautiful way that the primes are the multiplicative building blocks for the natural numbers. One may use the historical development as one motivation for the discussion. Prime factorizations are also used in the determination of least common multiples, which are used in the addition of fractions. This second motivation is at the same time a practical application.

The remarkable observation is how addition and multiplications are related. The greatest common divisor (a, b) of two two natural numbers a and b can be written in the form

$$(a, b) = na + mb \tag{6.1}$$

for two integers n and m. The Euclidean division algorithm provides the method for finding (a, b) and it provides a method for finding the integers m and n. This provides us with an opportunity to introduce and use an algorithm for an important computation. The equation in (6.1) allows us the prove that if a prime divides a product of natural numbers, then it divides one of the factors. A fairly short proof (chain of inferences) deduces the Fundamental Theorem of Arithmetic.

In this program one uses concepts of divisibility. There are a few simple rules. These rules are clear to the students, and under the guidance of the instructor the students can deduce the facts needed to justify the steps required in the program outlined in the previous paragraph.

All of the hallmarks are met in this one topic of discussion. The Fundamental Theorem of Arithmetic is a beautiful and powerful result. The instructor guides the students through its ingeneous and still rather elementary proof. Along the way the students are familiarized with the ideas by applying algorithms in computational form. In a very limited setting, divisibility, the students learn how to apply reasoning techniques, i.e. logic, to derive statements needed in the proof. Students have applied the theorem in practice since they found least common multiples when they added fractions. We can only mention the importance of the techniques in other fields, like cryptography. Important coding methods are based on prime factorizations, and the opponents technical inability to work them out.

7. Euler Circuits

For a given network (consisting of vertices and edges) one may ask whether it is possible to travel along each edge exactly once. It is said that Euler invented this question when he asked whether it is possible to take a walk around his hometown Königsberg and cross each of the bridges exactly once. He is also said to have developed the method for solving his specific and the general problem one Sunday afternoon when he had his traditional walk around town.

The key idea, used in the proofs of the relevant propositions is that whenever you enter a vertex, then you must be able to leave it again (unless your route starts or ends at the vertex). We have one argument, and this is used in quite a few settings. One remarkable observation is that a problem which looks like a global one is really a strictly local one. We can tell exactly when it is possible to travel along each edge exactly once, and exactly which verices can serve as starting or stopping points for such routes.

Why does this problem lend itself to be discussed in Math 100? It is concrete and the students can experiment with many specific examples to gather experience. There are some key ideas to be discovered which help us to solve the problem. There is a very small toolbox which allows us to prove all relevant assertions. The instructor may have to demonstrate one proof, and this will be sufficiently instructive so that the students can do similar ones. We can develop algorithms which will find the routes which travel over each edge exactly once. We can push the students to develop the algorithms and carry them out. The problems are concrete, so that there is one bridge to practice. These and similar circuits have occurred in technical applications, like the design of computer memory and the routing of long distance calls.

8. Reasoning

Formally speaking, the reasoning techniques referred to in the topics above are neither symbolic nor formal. Students encounter proofs. Sometimes the instructor leads them through a proof, sometimes students are asked to invent a proof similar to one they have seen before. The topics are chosen so that few prerequisites are needed. Rather simple arguments are used to create a chain of inference which then proves the desired result. The setting is fairly concrete, so that correctness can be checked in concrete situations. In this course a lot of logic is applied, and many observations are proved. But logic itself, like propositional logic, is not a topic of the course.

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