Construction and Properties of Brownian Motion

Stephen Brady

November 6, 2006

1 Brownian Motion

Brownian motion refers to the mathematical models that are used to describe the random movement of particles immersed in a fluid. This random movement was discovered by botanist Robert Brown, in 1827, while studying pollen grains in water under the microscope. One of the first to describe Brownian motion was Thorvald N. Thiele, in 1880, in a paper on the method of least squares. In 1900, Louis Bachelier, in his PhD thesis applied Brownian motion to the stock and option market fluctuations. In 1905, Albert Einstein, unaware of the earlier work on the subject, discussed Brownian motion in his paper on the molecular kinetic theory of heat.

Brownian motion has been used to model thermal noise in electrical circuits, limiting behavior in queueing systems, and random fluctuations in many physical, biological, and economic systems. It is widely used in part because it is among the simplest stochastic processes on a continuous domain. It is also the limit of other stochastic processes. However, it should be noted that it is often convenience rather than accuracy that motivates its use.

This paper will give an existence proof of Brownian motion, and then outline some properties of Brownian motion.

We begin with a brief review of the normal random variable. For a fixed $x \in \mathbb{R}^d$ and $t > 0$, suppose that $Y_{t,x}$ is an $\mathbb{R}^d$-valued normal random variable, i.e. the distribution measure $\mu_{Y_{t,x}}$ has a density (where $dy = dy_1 \cdots dy_d$)

$$d\mu_{Y_{t,x}} = p(t,x,y)\,dy := (2\pi t)^{-d/2} \exp\left(-\tfrac{1}{2}\left\|y-x\right\|^2/t\right)\,dy. \tag{1}$$

The characteristic function of Y is

$$\phi_{Y_{t,x}}(\lambda) = e^{i\lambda\cdot x} e^{-\frac{1}{2}t\|\lambda\|^2}.$$

If Y_{s,x_1} and Y_{t,x_2} are independent, then

$$\begin{aligned}\phi_{\left(Y_{s,x_1}+Y_{t,x_2}\right)}(\lambda) &= \phi_{Y_{s,x_1}}(\lambda)\phi_{Y_{t,x_2}}(\lambda) = e^{-i\lambda\cdot x_1}e^{-\frac{1}{2}s\|\lambda\|^2}e^{-i\lambda\cdot x_2}e^{-\frac{1}{2}t\|\lambda\|^2} \\ &= e^{-i\lambda\cdot(x_1+x_2)}e^{-\frac{1}{2}(s+t)\|\lambda\|^2} = \phi_{Y_{s+t,x_1+x_2}}(\lambda).\end{aligned}$$

Since $\phi_{X_1} = \phi_{X_2} \Rightarrow \mu_{X_1} = \mu_{X_2}$ (see [La], Corollary 1, p. 63), it follows that

$$\begin{aligned} d\mu_{\left(Y_{s,x_1}+Y_{t,x_2}\right)} &= d\mu_{Y_{s+t,x_1+x_2}} \\ &= \left(2\pi\left(s+t\right)\right)^{-d/2} \exp\left(-\tfrac{1}{2}\left\|y-\left(x_1+x_2\right)\right\|^2 / \left(s+t\right)\right) dy \\ &= p(s+t, x_1+x_2, y)\, dy. \end{aligned}$$

Proposition 1 *We have*

$$\int_{\mathbb{R}^d} p\left(s,x,y\right) p\left(t,y,z\right)\, dy = p\left(s+t,x,z\right).$$

Proof. $d\mu_{X+Y} = \left(\int_{-\infty}^{\infty} p_X\left(x\right) p_Y\left(z-x\right)\, dx\right) dz$ (see (20), p. 29), so we get

$$\begin{aligned} p(s+t, x_1+x_2, y)\, dy &= d\mu_{\left(Y_{s,x_1}+Y_{t,x_2}\right)} = \left(\int_{\mathbb{R}^d} p_{Y_{s,x_1}}\left(z\right) p_{Y_{t,x_2}}\left(z-y\right)\, dz\right) dy \\ &= \left(\int_{\mathbb{R}^d} p\left(s,x_1,z\right) p\left(t,x_2,z-y\right)\, dz\right) dy. \end{aligned}$$

Setting $x_1 = x$, $x_2 = 0$, we obtain

$$p(s+t,x,y) = \int_{\mathbb{R}^d} p\left(s,x,z\right) p\left(t,0,z-y\right)\, dz = \int_{\mathbb{R}^d} p\left(s,x,z\right) p\left(t,z,y\right)\, dz.$$

We can also get this "directly" from the Convolution Theorem (see (21), p. 29) via the computation

$$\begin{aligned} \left(p\left(s,x,\cdot\right) * p\left(s,0,\cdot\right)\right)^{\wedge}\left(\lambda\right) &= \left(2\pi\right)^{d/2} p\left(s,x,\cdot\right)^{\wedge}\left(\lambda\right) p^{\wedge}\left(s,0,\cdot\right)\left(\lambda\right) \\ &= \left(2\pi\right)^{d/2}\left(2\pi\right)^{-d/2} e^{-i\lambda\cdot x} e^{-\frac{1}{2}t\|\lambda\|^2}\left(2\pi\right)^{-d/2} e^{-\frac{1}{2}s\|\lambda\|^2} \\ &= \left(2\pi\right)^{-d/2} e^{-i\lambda\cdot x} e^{-\frac{1}{2}(t+s)\|\lambda\|^2} \\ &= \left(p\left(s+t,0,\cdot-x\right)\right)^{\wedge}\left(\lambda\right) \end{aligned}$$

Thus, taking the inverse transform, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} p\left(s,x,y\right) p\left(t,y,z\right)\, dy &= \int_{\mathbb{R}^d} p\left(s,x,y\right) p\left(t,0,z-y\right)\, dy \\ &= \left(p\left(s,x,\cdot\right) * p\left(s,0,\cdot\right)\right)\left(z\right) = p\left(s+t,0,z-x\right) \\ &= p\left(s+t,x,z\right). \end{aligned}$$

■

Let Ω be the set of all continuous functions $\omega : [0,\infty) \to \mathbb{R}^d$, and let $X_t : \Omega \to \mathbb{R}^d$ be given by $X_t\left(\omega\right) = \omega(t)$. We let Σ be the smallest collection of subsets of Ω which contains $X_t^{-1}(A)$ for all Borel subsets A of $\mathbb{R}^d$ and all $t \in [0,\infty)$.

Definition 2 *For $x \in \mathbb{R}^d$, suppose that P_x is a measure on Σ such that for each finite sequence $0 < t_1 < t_2 < \cdots < t_n$, the joint distribution of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ is given by*

$$\begin{aligned} & d\mu_{\left(X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right)}\left(x_1, \ldots, x_n\right) \\ = \; & p\left(t_1, x, x_1\right) p\left(t_2 - t_1, x_1, x_2\right) \cdots p\left(t_n - t_{n-1}, x_{n-1}, x_n\right) \, dx_1 \cdots dx_n \end{aligned}$$

*on $\left(\mathbb{R}^d\right)^n$. Then the triple $(\Sigma, P_x, \{X_t : t \in [0, \infty)\})$ is called **Brownian motion in $\mathbb{R}^d$ starting at** x.*

Intuitively,

$$p\left(t_1, x, x_1\right) p\left(t_2 - t_1, x_1, x_2\right) \cdots p\left(t_n - t_{n-1}, x_{n-1}, x_n\right) \, dx_1 \cdots dx_n$$

is the probability that a particle ω starting at x is in a volume element dx_1 about x_1 at time t_1, in a volume element dx_2 about x_2 at time t_2, ..., and in a volume element dx_n about x_n at time t_n.

2 Construction of Brownian Motion

To construct Brownian motion, we will apply Kolmogorov Extension. The theorem is stated here without proof. For a proof, see [La], p. 13–18.

Theorem 3 ***Kolmogorov Extension Theorem***

Let U be a complete, separable metric space. For each finite sequence $S_n = (i_1, \ldots, i_n)$ of distinct elements in an index set I, let μ_{S_n} be a measure on U^n. Let p be a permutation of $1, 2, \ldots, n$. Define $p \cdot S = \left(i_{p(1)}, \ldots, i_{p(n)}\right)$ and let $p_{U^n} : U^n \to U^n$ be defined by

$$p_{U^n}\left(x_1, \ldots x_n\right) = \left(x_{p(1)}, \ldots x_{p(n)}\right)$$

For integers $N > n > 0$, let $f_{N,n} : U^N \to U^n$ be the projection given by

$$f_{N,n}(x_1, ..., x_n, ..., x_N) = (x_1, ..., x_n)$$

Assume that for all such finite sequences $S_n = (i_1, i_2, \ldots, i_n)$ and $S_N = (i_1, i_2, \ldots, i_N)$ with $N > n$, and all permutations p of $\{1, 2, \ldots, n\}$, we have

$$\begin{aligned} &(a) \quad \mu_{p \cdot S_n}(A) = \mu_{S_n}\left(p_{U^n}^{-1}(A)\right) \quad \textit{and} \\ &(b) \quad \mu_{S_n}(A) = \mu_{S_N}\left(f_{N,n}^{-1}(A)\right) \quad \textit{for all } A \in \mathcal{B}(U^n). \end{aligned}$$

Then there is a measure μ on U^I defined on the sigma field $\sigma(\mathcal{F})$ generated by the field $\mathcal{F}$ of cylinder sets, such that

$$\mu\left(\pi_{S_n}^{-1}(A)\right) = \mu_{S_n}(A) \;\; \textit{for all } A \in \mathcal{B}(U^n).$$

We are concerned with the case $U = \mathbb{R}^d$, $I = [0, \infty)$ and for $S_n = (t_1, \ldots, t_n)$, with $t_1 \leq \ldots \leq t_n$,we take μ_{S_n} to be the measure on U^n with density

$$d\mu_{S_n}(x_1, \ldots, x_n) = p(t_1, x, x_1) \, p(t_2 - t_1, x_1, x_2) \cdots p(t_n - t_{n-1}, x_{n-1}, x_n) \; dx_1 \cdots dx_n$$

where $x_i \in \mathbb{R}^d$. If it is not the case that $t_1 \leq \ldots \leq t_n$, then we define μ_{S_n} to be $\mu_{\sigma \cdot S_n}$ where σ is chosen so that $t_{\sigma(1)} \leq \ldots \leq t_{\sigma(n)}$. Then the consistency condition (a) in Theorem 3 is automatic. For condition (b), we need

$$\mu_{S_n}(A) = \mu_{S_N}\left(f_{N,n}^{-1}(A)\right) \tag{2}$$

for $A \in \mathcal{B}(U^n)$, where for integers $N > n > 0$, recall that $f_{N,n} : U^N \to U^n$ is the projection given by

$$f_{N,n}(x_1, \ldots, x_n, \ldots, x_N) = (x_1, \ldots, x_n).$$

What (2) means is that if $(x_1, \ldots, x_n)$ is a subsequence of $(y_1, \ldots, y_N)$, say $(x_1, \ldots, x_n) = (y_{i_1}, \ldots, y_{i_n})$, and $y_{j_1}, \ldots, y_{j_{N-n}}$ is the complementary subsequence, then

$$\begin{aligned} & p(t_1, x, x_1) \, p(t_2 - t_1, x_1, x_2) \cdots p(t_n - t_{n-1}, x_{n-1}, x_n) \\ = & \int p(t_1, x, y_1) \, p(t_2 - t_1, y_1, y_2) \cdots p(t_N - t_{N-1}, y_{N-1}, y_N) \; dy_{j_1} \cdots dy_{j_{N-n}}, \end{aligned}$$

but this follows from repeated application of Proposition 1, namely

$$\int_{\mathbb{R}^d} p(s, x, y) \, p(t, y, z) \; dy = p(s + t, x, z)$$

in the form

$$\int_{\mathbb{R}^d} p(u - t, x, y) \, p(t - s, y, z) \; dy = p(u - s, x, z).$$

Thus, by Theorem 3, there is a measure μ on $\left(\mathbb{R}^d\right)^{[0,\infty)}$ defined on the sigma field $\sigma(\mathcal{F})$ generated by the field $\mathcal{F}$ of cylinder sets, such that

$$\mu\left(\pi_{S_n}^{-1}(A)\right) = \mu_{S_n}(A) \text{ for all } A \in \mathcal{B}(U^n).$$

In other words,

$$\begin{aligned} & \mu\left(\left\{\omega \in \left(\mathbb{R}^d\right)^{[0,\infty)} : (\omega(t_1), \ldots, \omega(t_n)) \in A\right\}\right) \\ = & \int_A p(t_1, x, x_1) \, p(t_2 - t_1, x_1, x_2) \cdots p(t_n - t_{n-1}, x_{n-1}, x_n) \; dx_1 \cdots dx_n. \end{aligned} \tag{3}$$

For the remainder of this section our goal is to show that from the measure μ we can construct Brownian motion (Ω, P_x, Σ) as in Definition 2.

Lemma 4 *Let $X_1, \ldots, X_n$ be independent $\mathbb{R}^d$-valued random variables on a probability space with measure ν, such that X_j and $-X_j$ are identically distributed. Let $S_j = X_1 + \cdots + X_j$ for $j = 1, \ldots, n$. Then for any $a \geq 0$*

$$\nu\left(\max_{1\leq j\leq n}(\|S_j\|) > a\right) \leq 2\nu\left(\|S_n\| > a\right).$$

Proof. Let $A_j := \{\|S_j\| > a \text{ and } \|S_i\| \leq a \text{ for } 0 \leq i < j\}$. Note that the $A_1, \ldots, A_n$ are disjoint and $A_1 \cup \cdots \cup A_n = \{\max_{1\leq j\leq n}(\|S_j\|) > a\}$. We have

$$\begin{aligned} 0 &\leq \|S_n - S_j\|^2 = \|S_n\|^2 + \|S_j\|^2 - 2S_n \cdot S_j = \|S_n\|^2 - \|S_j\|^2 + 2\|S_j\|^2 - 2S_n \cdot S_j \\ &= \|S_n\|^2 - \|S_j\|^2 - 2(S_n - S_j) \cdot S_j. \end{aligned}$$

Thus,

$$2(S_n - S_j) \cdot S_j \geq 0 \Rightarrow \|S_n\|^2 \geq \|S_j\|^2 \Rightarrow \|S_n\| \geq \|S_j\|.$$

Hence,

$$\|S_j\| > a \text{ and } 2(S_n - S_j) \cdot S_j \geq 0 \Rightarrow \|S_n\| > a,$$

and so

$$A_j \cap \{2(S_n - S_j) \cdot S_j \geq 0\} \subset A_j \cap \{\|S_n\| > a\}. \tag{4}$$

Note that since $\{X_{j+1}, \cdots, X_n\}$ and $\{X_1, \cdots, X_j\}$ are independent sets of random variables, we have that $(S_n - S_j) = (X_{j+1} + \cdots + X_n)$ and $S_j = (X_1 + \cdots + X_j)$ are independent when restricted to A_j (which is determined by $\{X_1, \cdots, X_j\}$). Moreover, since $(X_1 + \cdots + X_j)$ is distributed the same as $-(X_1 + \cdots + X_j) = -S_j$, we have

$$\begin{aligned} \nu\left(A_j \cap \{(S_n - S_j) \cdot S_j > 0\}\right) &= \nu\left(A_j \cap \{(S_n - S_j) \cdot -S_j > 0\}\right) \\ &= \nu\left(A_j \cap \{(S_n - S_j) \cdot S_j < 0\}\right). \end{aligned}$$

Thus,

$$\begin{aligned} \nu(A_j) &= 2\nu\left(A_j \cap \{(S_n - S_j) \cdot S_j > 0\}\right) + \nu\left(A_j \cap \{(S_n - S_j) \cdot S_j = 0\}\right) \\ &\leq 2\nu\left(A_j \cap \{(S_n - S_j) \cdot S_j > 0\}\right) + 2\nu\left(A_j \cap \{(S_n - S_j) \cdot S_j = 0\}\right) \\ &= 2\nu\left(A_j \cap \{2(S_n - S_j) \cdot S_j \geq 0\}\right) \\ &\leq 2\nu\left(A_j \cap \{\|S_n\| > a\}\right) \quad \text{by (4).} \end{aligned}$$

Hence,

$$\begin{aligned} \nu(\max_{1\leq j\leq n}(\|S_j\|) &> a) = \nu(A_1 \cup \cdots \cup A_n) = \sum\nolimits_{j=1}^{n} \nu(A_j) \\ &\leq 2\sum\nolimits_{j=1}^{n} \nu\left(A_j \cap \{\|S_n\| > a\}\right) \leq 2\nu\left(\|S_n\| > a\right). \end{aligned}$$

■

Let μ be the measure on $\left(\mathbb{R}^d\right)^{[0,\infty)}$ constructed before Lemma 4 and for each $t \geq 0$, let $X_t : \left(\mathbb{R}^d\right)^{[0,\infty)} \to \mathbb{R}^d$ be given by $X_t(\omega) = \omega(t)$.

Proposition 5 *$X_t - X_s$ has density $p(t-s,\cdot)$; i.e.,*

$$d\left(\mu_{X_t - X_s}\right)(y) = p(t-s,0,y)\,dy.$$

Proof. By (3), for $0 \le s < t$ the pair $(X_t,\, X_s)$ has joint density

$$\left(d\mu_{(X_t,X_s)}\right)(x_1,x_2) = p(s,x,x_1)\,p(t-s,x_1,x_2)\ \ dx_1 dx_2.$$

Then,

$$\begin{aligned}
& P_x(X_t - X_s \le b) \\
= & \iint_{x_1 - x_2 \le b} p(s,x,x_1)\,p(t-s,x_1,x_2)\ \ dx_1 dx_2 \\
= & \int_{-\infty}^{b}\int_{-\infty}^{\infty} p(s,x,x_1)\,p(t-s,x_1,z+x_1)\ \ dx_1 dz \quad \text{where } z = x_2 - x_1 \\
= & \int_{-\infty}^{b}\int_{-\infty}^{\infty} p(s,x,x_1)\,p(t-s,0,z)\ \ dx_1 dz \\
= & \int_{-\infty}^{b}\left(\int_{-\infty}^{\infty} p(s,x,x_1)\ \ dx_1\right) p(t-s,0,z)\ dz = \int_{-\infty}^{b} p(t-s,0,z)\ dz.
\end{aligned}$$

■

Remark 6 *It is easy to show that*

$$\int_0^{\infty} \rho^m e^{-\frac{1}{2}\rho^2} d\rho = 2^{(m-1)/2}\Gamma\left(\tfrac{m+1}{2}\right),\ \textit{for } m > -1.$$

Substitute $x = \frac{1}{2}\rho^2$ in the integral for the gamma function to get (for $t > 0$)

$$\Gamma(t) := \int_0^{\infty} e^{-x}x^{t-1}dx = \int_0^{\infty} e^{-\frac{1}{2}\rho^2}\left(\tfrac{1}{2}\rho^2\right)^{t-1}\rho\,d\rho = \tfrac{1}{2^{t-1}}\int_0^{\infty} e^{-\frac{1}{2}\rho^2}\rho^{2t-1}d\rho.$$

Let $m = 2t-1$, or $t = (m+1)/2$, and note that $2^{t-1} = 2^{(m+1)/2-1} = 2^{(m-1)/2}$.

Lemma 7 *For each $\delta > 0$, we have*

$$\lim_{n\to\infty} 2^n \mu\left(\|X_{2^{-n}} - x\| > \delta\right) = 0.$$

As usual, $\mu(\|X_{2^{-n}} - x\| > \delta)$ is short for

$$\mu\left(\left\{\omega \in \left(\mathbb{R}^d\right)^{[0,\infty)} : \|X_{2^{-n}}(\omega) - x\| > \delta\right\}\right).$$

Proof. In the following, the integrals are over subsets of $\mathbb{R}^d$, and we use the shorthand $dy = dy^1 \cdots dy^d$, $d\zeta = d\zeta^1 \cdots d\zeta^d$, etc.. For $t > 0$,

$$
\begin{aligned}
& \mu\left(\|X_t - x\| > \delta\right) = \int_{\|y-x\|>\delta} p\left(t,x,y\right)\, dy \\
= & \int_{\|y-x\|>\delta} \left(2\pi t\right)^{-d/2} e^{-\frac{1}{2}\|y-x\|^2/t}\, dy \\
& = \left(2\pi\right)^{-d/2} \int_{\|z\|>\delta} e^{-\frac{1}{2}\|z\|^2/t}\, t^{-d/2} dz \quad \text{with } z = y - x \\
& = \left(2\pi\right)^{-d/2} \int_{\|\zeta\|>t^{-\frac{1}{2}}\delta} e^{-\frac{1}{2}\|\zeta\|^2}\, d\zeta \quad \text{with } \zeta = z/\sqrt{t} \\
\leq & \left(2\pi\right)^{-d/2} \int_{\|\zeta\|>t^{-\frac{1}{2}}\delta} \left(\frac{\|\zeta\|}{t^{-\frac{1}{2}}\delta}\right)^2 e^{-\frac{1}{2}\|\zeta\|^2}\, d\zeta \\
= & \; t\delta^{-2} \left(2\pi\right)^{-d/2} \int_{\|\zeta\|>t^{-\frac{1}{2}}\delta} \|\zeta\|^2 e^{-\frac{1}{2}\|\zeta\|^2}\, d\zeta
\end{aligned} \tag{5}
$$

For $t = 2^{-n}$, we then get

$$
\mu\left(\|X_{2^{-n}} - x\| > \delta\right) = 2^{-n}\delta^{-2} \left(2\pi\right)^{-d/2} \int_{\|\zeta\|>2^{n/2}\delta} \|\zeta\|^2 e^{-\frac{1}{2}\|\zeta\|^2}\, d\zeta
$$

and so

$$
\lim_{n\to\infty} 2^n \mu\left(\|X_{2^{-n}} - x\| > \delta\right) = \delta^{-2} \left(2\pi\right)^{-d/2} \lim_{n\to\infty} \int_{\|\zeta\|>2^{n/2}\delta} \|\zeta\|^2 e^{-\frac{1}{2}\|\zeta\|^2}\, d\zeta = 0.
$$

Note that

$$
\int_{\mathbb{R}^d} \|\zeta\|^2 e^{-\frac{1}{2}\|\zeta\|^2}\, d\zeta < \infty, \tag{6}
$$

since for any integer $k > 0$,

$$
\begin{aligned}
\|\zeta\|^2 e^{-\frac{1}{2}\|\zeta\|^2} & < \frac{\|\zeta\|^2}{1 + \frac{1}{2}\|\zeta\|^2 + \cdots + \frac{1}{(k+1)!}\left(\frac{1}{2}\|\zeta\|^2\right)^{k+1}} < \frac{\|\zeta\|^2}{\frac{1}{(k+1)!}\left(\frac{1}{2}\|\zeta\|^2\right)^{k+1}} \\
& = \frac{(k+1)!}{2^{-(k+1)}} \|\zeta\|^{-2k},
\end{aligned}
$$

which is integrable over $\|\zeta\| \geq 1$ for $2k > d$. More precisely, let ρ be the "spherical coordinate" giving the distance from the origin in $\mathbb{R}^d$, and let

$$
\omega_{d-1} := \frac{\pi^{d/2}}{\frac{1}{2}\Gamma\left(\frac{d}{2}\right)}
$$

be the $(d-1)$-measure of the unit sphere $\|\zeta\| = 1$ in $\mathbb{R}^d$ (e.g., $\omega_1 = 2\pi, \omega_2 = 4\pi$). Then for any $R > 0$ and any $k \in \mathbb{R}$ with $2k > d$, we have

$$\int_{\|\zeta\| \geq R} \|\zeta\|^2 e^{-\frac{1}{2}\|\zeta\|^2} d\zeta = \omega_{d-1} \int_R^\infty \rho^2 e^{-\frac{1}{2}\rho^2} \rho^{d-1} d\rho$$

$$\begin{aligned}
&\leq \omega_{d-1} \int_R^\infty \frac{(k+1)!}{2^{-(k+1)}} \rho^{-2k} \rho^{d-1} d\rho \\
&= \omega_{d-1} \frac{(k+1)!}{2^{-(k+1)}} \int_R^\infty \rho^{-(2k+1-d)} d\rho \\
&= \omega_{d-1} \frac{(k+1)!}{2^{-(k+1)}} \frac{1}{2k-d} R^{-2k+d},
\end{aligned}$$

which tends to 0 as $R = 2^{n/2}\delta \to \infty$.

Alternatively, (6) could be obtained from Remark 6. ■

Let S be the set of non-negative diadic rational numbers, namely

$$S = \left\{ k2^{-m} : k, m = 0, 1, 2, \ldots \right\}.$$

Define

$$U_n(\omega) := \sup \left\{ \|X_t(\omega) - x\| : t \in S \text{ and } t \leq 2^{-n} \right\}.$$

Lemma 8 *For any $\delta > 0$, we have*

$$\lim_{n \to \infty} 2^n \mu(U_n > \delta) = 0.$$

In other words,

$$\lim_{n \to \infty} 2^n \mu \left(\sup \left\{ \|X_t - x\| : t \in S \textit{ and } t \leq 2^{-n} \right\} > \delta \right) = 0.$$

Proof. For $m \geq n$, let

$$U_{nm} = \max_{1 \leq j \leq 2^{m-n}} \left\| X_{2^{-m}j} - x \right\|.$$

Note that

$$1 \leq j \leq 2^{m-n} \Rightarrow 2^{-m} j \leq 2^{-m} 2^{m-n} = 2^{-n}.$$

Thus,

$$\left\{ 2^{-m} j : 1 \leq j \leq 2^{m-n} \right\}$$

consists of all diadic rationals in the interval $[0, 2^{-n}]$ which are multiples of 2^{-m}. Thus, $U_{nm} \subseteq U_{n(m+1)}$ and

$$U_n = \cup_{m=n}^\infty U_{mn}.$$

Hence, we have

$$P(U_n > \delta) = \lim_{m\to\infty} \mu\left(U_{mn} > \delta\right).$$

Now by Lemma 4, applied to the random variables $X_{2^{-m}j} - x$ for $1 \leq j \leq 2^{m-n}$, we have

$$\mu\left(U_{mn} > \delta\right) = \mu\left(\max_{1\leq j\leq 2^{m-n}} \left\|X_{2^{-m}j} - x\right\| > \delta\right) \leq 2\mu\left(\left\|X_{2^{-n}} - X\left(0\right)\right\| > \delta\right).$$

By Lemma 7, we then have

$$\lim_{n\to\infty} 2^n \mu(U_n > \delta) = \lim_{n\to\infty}\lim_{m\to\infty} 2^n \mu\left(U_{mn} > \delta\right) \leq 2 \lim_{n\to\infty} 2^n \mu\left(\left\|X_{2^{-n}} - x\right\| > \delta\right) = 0.$$

■

For N and n positive integers, let

$$V_n := \sup\left\{\|X_t - X_s\| : s,t \in S\cap[0,N] \text{ and } |s-t| \leq 2^{-n}\right\}$$

Lemma 9

$$\mu\left(\lim_{n\to\infty} V_n = 0\right) = 1.$$

Proof. Let

$$V_{nm} = \sup\left\{\|X_t - X_{2^{-n}m})\| : t \in S\cap\left[2^{-n}m, 2^{-n}\left(m+1\right)\right]\right\}.$$

Note that $s,t\in S\cap[0,N]$ and $|s-t|\leq 2^{-n}$ (say $s\leq t$) implies that for some m we have $s\in S\cap[2^{-n}\left(m-1\right), 2^{-n}m]$ and $t\in S\cap[2^{-n}m, 2^{-n}\left(m+1\right)]$, in which case

$$\|X_t - X_s\| \leq \|X_t - X_{2^{-n}m}\| + \|X_s - Y_{2^{-n}m}\| \leq V_{n(m-1)} + V_{nm}.$$

Hence,

$$V_n \leq 2 \max_{0\leq m\leq 2^n N} V_{nm}.$$

As $\mu(V_{nm} > \delta) = \mu(U_n > \delta)$, Lemma 8 yields

$$\begin{aligned}\overline{\lim}_{n\to\infty}\mu(V_n &> 2\delta) \leq \overline{\lim}_{n\to\infty}\mu(\max_{0\leq m\leq 2^n N} V_{nm} > \delta) \leq 2^n N\mu(V_{nm} > \delta)\\ &= \lim_{n\to\infty} 2^n N\mu(U_n > \delta) = 0.\end{aligned}$$

Since V_n is nonincreasing, $\lim_{n\to\infty}\mu(V_n > 2\delta) = \overline{\lim}_{n\to\infty}\mu(V_n > 2\delta) = 0$ and

$$\mu\left(\lim_{n\to\infty} V_n \leq 2\delta\right) = 1.$$

Finally,

$$\mu\left(\lim_{n\to\infty} V_n = 0\right) = \mu\left(\cap_{k=1}^{\infty}\left(\lim_{n\to\infty} V_n \le k^{-1}\right)\right) = \lim_{k\to\infty}\mu\left(\lim_{n\to\infty} V_n \le k^{-1}\right) = 1.$$

■

Lemma 9 tells us that there is a subset of measure 1, say Ω', of $\left(\mathbb{R}^d\right)^{[0,\infty)}$, such that $\omega \in \Omega'$ implies that $X_t(\omega)$ is a uniformly continuous function of $t \in S$. For $\omega \in \Omega'$, the function $t \mapsto X_t(\omega) = \omega(t)$ extends to a *continuous* function on $[0,\infty)$ via

$$Y_{t'}(\omega) := \lim_{t\to t', t\in S} X_t(\omega),\ t' \in [0,\infty). \tag{7}$$

We denote this function by $\widetilde{\omega}$. Thus, we have a map $f : \Omega' \to \Omega \subset \left(\mathbb{R}^d\right)^{[0,\infty)}$ where Ω is the subset of continuous functions in $\left(\mathbb{R}^d\right)^{[0,\infty)}$. The induced measure μ_f on Ω is given by $\mu_f(A) = \mu(f^{-1}(A))$ where A is such that $f^{-1}(A)$ is measurable. We claim that μ_f serves as P_x in a Brownian motion $(\Sigma, P_x, \{Y_t : t \in [0,\infty)\})$ on $\mathbb{R}^d$ starting at $x \in \mathbb{R}^d$, where

$$\Sigma := \left\{A \subset \Omega : f^{-1}(A) \in \sigma(\mathcal{F})\right\}$$

with $\sigma(\mathcal{F})$ being the sigma field generated by the field $\mathcal{F}$ of cylinder sets.

The X_t have the joint distributions given by definition 2. However, it remains to show that the Y_t have the correct joint distributions. It must be shown that

$$\mu\left[Y_t(\omega) = X_t(\omega)\right] = 1$$

By (7), we need only consider $t \notin S$.

Lemma 10 *Let $s < t$. For each $\varepsilon > 0$, there exists $k_\varepsilon > 0$ s.t.*

$$\mu\left[\|X_t - X_s\| > \varepsilon\right] \le k_\varepsilon(t-s) \tag{8}$$

Proof.

$$\begin{aligned}
\mu\left[\|X_t - X_s\| > \varepsilon\right] &= \mu\left[\left\|X_{(t-s)} - x\right\| > \varepsilon\right] \text{ by Proposition 5} \\
&= \int_{\|y-x\|>\delta} p\left(t-s,x,y\right)\, dy \\
&\le (t-s)\varepsilon^{-2}\left(2\pi\right)^{-d/2}\int_{\|\zeta\|>(t-s)^{-\frac{1}{2}}\varepsilon} \|\zeta\|^2 e^{-\frac{1}{2}\|\zeta\|^2}\, d\zeta \text{ by (5)} \\
&\le (t-s)\varepsilon^{-2}\left(2\pi\right)^{-d/2}\int_{\mathbb{R}^d} \|\zeta\|^2 e^{-\frac{1}{2}\|\zeta\|^2}\, d\zeta.
\end{aligned}$$

Recall (6) that

$$\int_{\mathbb{R}^d} \|\zeta\|^2 e^{-\frac{1}{2}\|\zeta\|^2}\, d\zeta < \infty.$$

Hence, (8) is shown to be true. ■

It follows that $\lim_{s\to t, s\epsilon S}\mu\left[\|X_t - X_s\| > \varepsilon\right] = 0$. Then, X_t is a limit in probability of X_s. Y_t is also such a limit. So, $Y_t = X_t$, a.s.. Then, the distribution of Y_t satisfies definition 2.

3 Properties of Brownian motion

Proposition 11 *If* $(\Sigma, P_x, \{X_t : t \in [0,\infty)\})$ *is Brownian motion in* $\mathbb{R}^d$ *starting at* x*, then*

$$P_x(X_0 = x) = P_x(\{\omega \in \Omega : X_0(\omega) = \omega(0) = x\}) = 1.$$

Proof. By definition of Ω, $t \mapsto X_t(\omega) = \omega(t)$ is continuous for $t \geq 0$. Thus $\lim_{t\to 0} X_t(\omega) = X_0(\omega) = \omega(0)$ exists. If $\omega(0) \neq x$, then there is some $t_0 > 0$ such that

$$t \in [0, t_0] \Rightarrow \|\omega(t) - \omega(0)\| < \tfrac{1}{2}\|x - \omega(0)\|,$$

and hence for $t \in [0, t_0]$,

$$\|x - \omega(0)\| \leq \|x - \omega(t)\| + \|\omega(t) - \omega(0)\| < \|x - \omega(t)\| + \tfrac{1}{2}\|\omega(t) - \omega(0)\|.$$

Thus, for some $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and some $t_0 > 0$, we have

$$\|x - \omega(t)\| > \tfrac{1}{2}\|x - \omega(0)\| \geq n^{-1} \text{ for all } t \in [0, t_0].$$

Let

$$B_{m,n} := \left\{\omega \in \Omega : \|X_{m^{-1}}(\omega) - x\| \geq n^{-1}\right\}.$$

We then have (where s.t. means "such that")

$$X_0(\omega) \neq x \Rightarrow \exists (n, M) \in \mathbb{N} \times \mathbb{N} \text{ s.t. } (m \geq M \Rightarrow \omega \in B_{m,n}).$$

Conversely, if $\exists (n, M) \in \mathbb{N} \times \mathbb{N}$ s.t. $(m \geq M \Rightarrow \omega \in B_{m,n})$, then

$$\lim_{m\to\infty} \|X_{m^{-1}}(\omega) - x\| \geq n^{-1} \text{ and so } X_0(\omega) \neq x.$$

Thus,

$$X_0(\omega) \neq x \Leftrightarrow \omega \in \cup_{n\in\mathbb{N}} \cup_{M\in\mathbb{N}} (\cap_{m\geq M} B_{m,n})$$

If we can show that $\lim_{m\to\infty} P_x(B_{m,n}) = 0$ for any fixed $n \in \mathbb{N}$, then

$$0 \leq P_x(\cap_{m\geq M} B_{m,n}) \leq P_x(B_{m,n}) \text{ for all } m \geq M$$

$$\begin{aligned}
&\Rightarrow \quad P_x(\cap_{m\geq M} B_{m,n}) = 0 \\
&\Rightarrow \quad P_x(\cup_{M\in\mathbb{N}} (\cap_{m\geq M} B_{m,n})) = 0 \\
&\Rightarrow \quad P_x(\cup_{n\in\mathbb{N}} \cup_{M\in\mathbb{N}} (\cap_{m\geq M} B_{m,n})) = 0 \\
&\Rightarrow \quad P_x(\{\omega \in \Omega : X_0(\omega) = \omega(0) \neq x\}) = 0 \\
&\Rightarrow \quad P_x(\{\omega \in \Omega : X_0(\omega) = \omega(0) = x\}) = 1.
\end{aligned}$$

Thus, it remains to prove $\lim_{m\to\infty} P_x(B_{m,n}) \to 0$. Note that $B_{m,n} = A_{m^{-1},n^{-1}}$, where

$$A_{t,\varepsilon} := \{\omega \in \Omega : \|X_t(\omega) - x\| \geq \varepsilon\}.$$

By definition,

$$(dP_x)_{X_t} = p(t,x,y)\ dy = (2\pi t)^{-d/2}\, e^{-\frac{1}{2}\|y-x\|^2/t} dy.$$

Hence, if C_d is the d-volume of the unit sphere in $\mathbb{R}^d$, then

$$\begin{aligned}
P_x(A_{t,\varepsilon}) &= \int_{\{y\in\mathbb{R}^d:\|y-x\|\geq\varepsilon\}} (2\pi t)^{-d/2}\, e^{-\frac{1}{2}\|y-x\|^2/t} dy \\
&= (2\pi t)^{-d/2} \int_{\{z\in\mathbb{R}^d:\|z\|\geq\varepsilon\}} e^{-\frac{1}{2}\|z\|^2/t}\, dz \quad \text{with } z = y-x \\
&\leq (2\pi t)^{-d/2}\, C_d \int_{\varepsilon}^{\infty} e^{-\frac{1}{2}r^2/t} r^{d-1}\, dr \\
&= (2\pi t)^{-d/2}\, C_d \int_{\varepsilon/\sqrt{t}}^{\infty} e^{-\frac{1}{2}s^2} t^{(d-1)/2} s^{d-1}\, t^{1/2} ds \quad \text{with } s = r/\sqrt{t} \\
&= (2\pi t)^{-d/2}\, t^{d/2} C_d \int_{\varepsilon/\sqrt{t}}^{\infty} e^{-\frac{1}{2}s^2} s^{d-1}\, ds \\
&= (2\pi)^{-d/2}\, C_d \int_{\varepsilon/\sqrt{t}}^{\infty} e^{-\frac{1}{2}s^2} s^{d-1}\, ds \to 0 \text{ as } t \to 0^+.
\end{aligned}$$

Thus $\lim_{m\to\infty} P_x(B_{m,n}) = \lim_{m\to\infty} P_x\left(A_{m^{-1},n^{-1}}\right) \to 0$, as required. ■

Proposition 12 *Let $(\Sigma, P_x, \{X_t : t \in [0,\infty)\})$ be Brownian motion in $\mathbb{R}^d$ starting at x. For $n \geq 1$ and $t_1 < t_2 < \cdots < t_n$, we have that*

$$X_{t_1}, X_{t_2} - X_{t_1},\ X_{t_3} - X_{t_2}, \ldots,\ X_{t_n} - X_{t_{n-1}}$$

are independent.

Proof. We know from Proposition 5 that $X_t - X_s$ has density $p(t-s,\cdot)$ for $t > s$. Thus, $X_{t_k} - X_{t_{k-1}}$ has density $p(t_k - t_{k-1}, \cdot)$. Now X_{t_1} and $X_{t_2} - X_{t_1}$ are independent if

$$\begin{aligned}
&P_x\left((X_{t_1}, X_{t_2} - X_{t_1}) \leq (D_1, D_2)\right) \\
=\ &\int_{-\infty}^{D_1}\int_{-\infty}^{D_2} p(t_1, y_1)\, p(t_2 - t_1, y_2)\ dy_1 dy_2.
\end{aligned}$$

Here D_1 and $D_2 \in \mathbb{R}^d$ and "$\leq$" operates componentwise. We in fact have

$$\begin{aligned}
& P_x\left(\left(X_{t_1}, X_{t_2}-X_{t_1}\right) \leq \left(D_1, D_2\right)\right) \\
= & \int_{-\infty<y_1 \leq D_1, -\infty<y_2-y_1 \leq D_2} p\left(t_1, x, y_1\right) p\left(t_2-t_1, y_1, y_2\right) \; dy_1 dy_2 \\
= & \cdots \text{(making the change of variable } z=y_2-y_1\text{)} \\
= & \int_{-\infty}^{D_2} \int_{-\infty}^{D_1} p\left(t_1, x, y_1\right) p\left(t_2-t_1, y_1, y_1+z\right) \; dy_1 dz \\
= & \int_{-\infty}^{D_2} \int_{-\infty}^{D_1} p\left(t_1, x, y_1\right) p\left(t_2-t_1, z\right) \; dy_1 dz \\
= & \int_{-\infty}^{D_2} \int_{-\infty}^{D_1} p\left(t_1, x, y_1\right) p\left(t_2-t_1, y_2\right) \; dy_1 dy_2 .
\end{aligned}$$

In general,

$$\begin{aligned}
& P_x\left(\left(X_{t_1}, X_{t_2}-X_{t_1}, \ldots, X_{t_n}-X_{t_{n-1}}\right) \leq \left(D_1, D_2, \ldots, D_n\right)\right) \\
= & \int_{-\infty<y_1 \leq D_1, \ldots, -\infty<y_n-y_{n-1} \leq D_{n-1}} p\left(t_1, x, y_1\right) p\left(t_2-t_1, y_1, y_2\right) \cdots \\
& \cdots p\left(t_n-t_{n-1}, y_{n-1}, y_n\right) \, dy_1 \cdots dy_n \\
= & \left(\text{with } z_2=y_2-y_1, \ldots, z_n=y_n-y_{n-1}\right)= \\
= & \int_{-\infty}^{D_n} \cdots \int_{-\infty}^{D_1} p\left(t_1, y_1\right) p\left(t_2-t_1, z_1\right) \cdots p\left(t_n-t_{n-1}, z_n\right) \; dz_1 \cdots dz_n .
\end{aligned}$$

■

The following implies that to construct Brownian motion in $\mathbb{R}^d$, it suffices to construct it on $\mathbb{R}^1$.

Proposition 13 *Suppose that $(\Sigma, P_x, \{X_t : t \in [0,\infty)\})$ is Brownian motion in $\mathbb{R}^d$ starting at x, and $(\Sigma', P'_{x'}, \{X'_t : t \in [0,\infty)\})$ is Brownian motion in $\mathbb{R}^{d'}$ starting at x'. Then $(\Sigma \times \Sigma', P_x \times P'_{x'}, \{(X_t, X'_t) : t \in [0,\infty)\})$ is Brownian motion in $\mathbb{R}^{d+d'}$ starting at (x,x').*

Proof. We need to check that for each finite sequence $0 < t_1 < t_2 < \cdots < t_n$, the joint distribution of $\left(X_{t_1}, X'_{t_1}\right), \left(X_{t_2}, X'_{t_2}\right), \ldots, \left(X_{t_n}, X'_{t_n}\right)$ is given by

$$\begin{aligned}
& d\mu_{\left(\left(X_{t_1}, X'_{t_1}\right), \left(X_{t_2}, X'_{t_2}\right), \ldots, \left(X_{t_n}, X'_{t_n}\right)\right)}\left(\left(x_1, x'_1\right), \ldots, \left(x_n, x'_n\right)\right) \\
= & p^{d+d'}\left(t_1, \left(x, x'\right), \left(x_1, x'_1\right)\right) p^{d+d'}\left(t_2-t_1, \left(x_1, x'_1\right), \left(x_2, x'_2\right)\right) \\
& \cdots p^{d+d'}\left(t_n-t_{n-1}, \left(x_{n-1}, x'_{n-1}\right), \left(x_n, x'_n\right)\right) \, dx_1 dx'_1 \cdots dx_n dx'_n
\end{aligned}$$

on $\left(\mathbb{R}^{d+d'}\right)^n$, where

$$p^{d+d'}(t, (x,x'), (y,y')) := (2\pi t)^{-\left(d+d'\right)/2} \exp\left(-\tfrac{1}{2}\left(\left\|(y,y')-(x,x')\right\|^2\right)/t\right).$$

Using $\|(y,y')-(x,x')\|^2 = \|y-x\|^2 + \|y'-x'\|^2$, we obtain

$$\begin{aligned} p^{d+d'}(t,(x,x'),(y,y')) &= (2\pi t)^{-d/2} \exp\left(-\tfrac{1}{2}\left(\|y-x\|^2\right)/t\right)(2\pi t)^{-d'/2}\exp\left(-\tfrac{1}{2}\left(\|y'-x'\|^2\right)/t\right) \\ &= p^d(t,x,y)p^{d'}(t,x',y'). \end{aligned}$$

Since $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ and $\left(X'_{t_1}, X'_{t_2}, \dots, X'_{t_n}\right)$ are independent,

$$\begin{aligned} & d\mu_{\left((X_{t_1},X'_{t_1}),(X_{t_2},X'_{t_2}),\dots,(X_{t_n},X'_{t_n})\right)}\left((x_1,x'_1),\dots,(x_n,x'_n)\right) \\ = {} & d\mu_{(X_{t_1},X_{t_2},\dots,X_{t_n})}(x_1,\dots,x_n)\, d\mu_{(X'_{t_1},X'_{t_2},\dots,X'_{t_n})}(x'_1,\dots,x'_n) \\ = {} & p^d(t_1,x,x_1)\,p^d(t_2-t_1,x_1,x_2)\cdots p^d(t_n-t_{n-1},x_{n-1},x_n)\,dx_1\cdots dx_n \\ & p^{d'}(t_1,x',x'_1)\,p^{d'}(t_2-t_1,x'_1,x'_2)\cdots p^{d'}\left(t_n-t_{n-1},x'_{n-1},x'_n\right)\,dx'_1\cdots dx'_n \\ = {} & p^d(t_1,x,x_1)p^{d'}(t,x',x'_1)p^d(t_2-t_1,x_1,x_2)p^{d'}(t_2-t_1,x'_1,x'_2) \\ & p^d(t_n-t_{n-1},x_{n-1},x_n)\,p^{d'}\left(t_n-t_{n-1},x'_{n-1},x'_n\right)dx_1dx'_1\cdots dx_ndx'_n \\ = {} & p^{d+d'}(t_1,(x,x'),(x_1,x'_1))\,p^{d+d'}(t_2-t_1,(x_1,x'_1),(x_2,x'_2)) \\ & \cdots p^{d+d'}\left(t_n-t_{n-1},\left(x_{n-1},x'_{n-1}\right),(x_n,x'_n)\right)\,dx_1dx'_1\cdots dx_ndx'_n, \end{aligned}$$

as required. ■

4 Holder Continuity

Brownian motion is interesting in that it is continuous yet non-differentiable. This section explores the Holder continuity of Brownian motion.

By Proposition 13, we only need to consider one-dimensional Brownian motion starting at 0, which will be referred to as standard Brownian motion.

Lemma 14 *([KS], Problem 2.9.3, p. 104)* ***Strong Law of Large Numbers*** *When $X = \{X_t; 0 \le t < \infty\}$ is a standard Brownian motion*

$$\lim_{t\to\infty} \frac{X_t}{t} = 0, a.s.$$

Proof. First note that $\forall \tau > 0$, $E(X_\tau) = 0$ and $Var(X_\tau) = \tau$. Then $\forall k > 0$,

$$\begin{aligned} k\Pr[\sup_{\sigma\le t\le\tau} \frac{X_t}{t} \ge \frac{k}{t}] &= k\Pr[\sup_{\sigma\le t\le\tau} X_t \ge k] \\ &\le E(X_\tau^+) \;\; \text{by [KS] Theorem 1.3.8, p. 13} \\ &= \frac{1}{\sqrt{2\pi\tau}}\int_0^\infty x\exp\left[\frac{-x^2}{2\tau}\right]dx \\ &= \sqrt{\frac{\tau}{2\pi}} \end{aligned}$$

Dividing by k, we get

$$\Pr[\sup_{\sigma \le t \le \tau} \frac{X_t}{t} \ge \frac{k}{t}] = \frac{1}{k}\sqrt{\frac{\tau}{2\pi}}$$

Choose an arbitrary $c > 0$. Let $k^* = c\sigma$. Then, for $\sigma \le t$, $\frac{k^*}{t} = \frac{c\sigma}{t} \le c$. So,

$$\begin{aligned}\Pr[\sup_{\sigma \le t \le \tau} \frac{X_t}{t} &\ge c] \le \Pr[\sup_{\sigma \le t \le \tau} \frac{X_t}{t} \ge \frac{k^*}{t}] \\ &= \frac{1}{k^*}\sqrt{\frac{\tau}{2\pi}} \\ &= \frac{1}{c\sigma}\sqrt{\frac{\tau}{2\pi}}\end{aligned}$$

It follows that $\forall c > 0$,

$$\Pr[\sup_{2^n \le t \le 2^{n+1}} \frac{X_t}{t} \ge c] \le \frac{1}{c\sqrt{\pi}} \frac{1}{\sqrt{2}^n}$$

Now, let E_n be the event that, for some $t \in [2^n, 2^{n+1}]$, $\left|\frac{X_t}{t}\right| \ge c$. Then, $\Pr(E_n \;\; i.o.) = 0$ by the Borel-Cantelli Lemma. Then, for a fixed c there exists a t^* such that $\forall t > t^*$, $\left|\frac{X_t}{t}\right| < c$. So, $\lim_{t\to\infty} \frac{X_t}{t} = 0$ a.s. ∎

Lemma 15 *([KS], Lemma 9.4) When $X = \{X_t; 0 \le t < \infty\}$ is a standard Brownian motion, so are the processes obtained from the following "equivalence transformations":*

(i) Scaling: $X = \{X_t; 0 \le t < \infty\}$ defined for $c > 0$ by

$$X_t = \frac{1}{\sqrt{c}} X_{ct}, 0 \le t < \infty$$

(ii) Time-Inversion: $Y = \{Y_t; 0 \le t < \infty\}$ defined by

$$Y_t = \left\{ \begin{array}{ll} tX_{1/t} & 0 < t < \infty \\ 0 & t = 0 \end{array} \right\}$$

(iii) Time-reversal: $Z = \{Z_t; 0 \le t < \infty\}$ defined for $T > 0$ by

$$Z_t = X_T - X_{T-t}, 0 \le t \le T$$

(iv) Symmetry: $-X = \{-X_t; 0 \le t < \infty\}$

Proof. The proof of (ii) will be shown in detail. The others are similar.

It must be shown that Y_t is a zero mean Gaussian process with $Cov(Y_s, Y_t) = s \wedge t$.([KS] Remark 2.9.2, p. 103).

Recall the following fact

$$\forall a, b \in \mathbb{R},\ X \sim N(\mu, \sigma^2) \implies aX + b \sim N(a\mu + b, a^2\sigma^2)$$

Then, for $t \neq 0$, $Y_t \sim N(0, t)$. Continuity (in t) at the origin is guaranteed by Lemma 14 above.

Further,

$$\begin{aligned} Cov(Y_s, Y_t) &= E(Y_s Y_t) - E(Y_s)E(Y_t) \\ &= E(Y_s Y_t) \\ &= stE(X_{1/s} X_{1/t}) \\ &= st[E(X_{1/s} X_{1/t}) - E(X_{1/s})E(X_{1/t})] \\ &= stCov(X_{1/s}, X_{1/t}) \\ &= st(\frac{1}{s} \wedge \frac{1}{t}) \\ &= s \wedge t. \end{aligned}$$

■

Remark 16 *It is easy to show that*

$$\int_0^\infty \rho^m e^{-\frac{1}{2}\rho^2} d\rho = 2^{(m-1)/2}\Gamma(\tfrac{m+1}{2}),\ \textit{for } m > -1.$$

Substitute $x = \frac{1}{2}\rho^2$ *in the integral for the gamma function to get (for* $t > 0$*)*

$$\Gamma(t) := \int_0^\infty e^{-x}x^{t-1}dx = \int_0^\infty e^{-\frac{1}{2}\rho^2}\left(\tfrac{1}{2}\rho^2\right)^{t-1}\rho\, d\rho = \tfrac{1}{2^{t-1}}\int_0^\infty e^{-\frac{1}{2}\rho^2}\rho^{2t-1}d\rho.$$

Let $m = 2t - 1$, *or* $t = (m+1)/2$, *and note that* $2^{t-1} = 2^{(m+1)/2-1} = 2^{(m-1)/2}$.

Proposition 17 *For almost every* $\omega \in \Omega$, *the Brownian path* $X(\omega)$ *is Holder continuous with exponent* $\gamma < \frac{1}{2}$.

This result will be shown, by the following two lemmas.

Lemma 18 *([KS], Problem 2.10, p. 55) If* $B_t - B_s$, $0 \leq s < t$, *is normally distributed with mean zero and variance* $t - s$, *then for each positive integer* n, *there is a positive constant* C_n *for which*

$$E\left[|B_t - B_s|^{2n}\right] = C_n\,|t - s|^n$$

Proof.

$$
\begin{aligned}
& E\left[|B_t - B_s|^{2n}\right] \\
=\ & \int_{-\infty}^{\infty} \left|x^{2n}\right| \frac{1}{\sqrt{2\pi h}} e^{-\frac{1}{2}x^2/h}\, dx \quad \text{where } h = t - s \\
=\ & \frac{2}{\sqrt{2\pi h}} \int_0^{\infty} x^{2n} e^{-\frac{1}{2}x^2/h}\, dx
\end{aligned}
$$

Using Remark 6, we get (setting $\rho = xh^{-\frac{1}{2}}$)

$$
\begin{aligned}
2^{(m-1)/2}\Gamma(\frac{m+1}{2}) &= \int_0^{\infty} \rho^m e^{-\frac{1}{2}\rho^2} d\rho = \int_0^{\infty} \left(xh^{-\frac{1}{2}}\right)^m e^{-\frac{1}{2}x^2/h} h^{-\frac{1}{2}} dx \\
&= \int_0^{\infty} x^m h^{-\frac{m+1}{2}} e^{-\frac{1}{2}x^2/h}\, dx = h^{-\frac{m+1}{2}} \int_0^{\infty} x^m e^{-\frac{1}{2}x^2/h}\, dx.
\end{aligned}
$$

Thus,

$$
\int_0^{\infty} x^m e^{-\frac{1}{2}x^2/h}\, dx = 2^{(m-1)/2}\Gamma(\frac{m+1}{2}) h^{\frac{m+1}{2}}, \text{ and so}
$$

$$
\frac{2}{\sqrt{2\pi h}} \int_0^{\infty} x^{2n} e^{-\frac{1}{2}x^2/h}\, dx = \frac{2}{\sqrt{2\pi}} 2^{(2n-1)/2}\Gamma(\frac{2n+1}{2}) h^n = \frac{2^n}{\sqrt{\pi}}\Gamma\left(n + \frac{1}{2}\right) h^n.
$$

■

Lemma 19 *([KS], Theorem 2.8, p. 53) Suppose that a process* $X = \{X_t; 0 \leq t \leq T\}$ *on a probability space* $(\Omega, \mathcal{B}, P)$ *satisfies the condition*

$$
E[|X_t - X_s|^{\alpha}] \leq C\,|t - s|^{1+\beta}, \quad 0 \leq s, t \leq T \tag{9}
$$

for some positive constants $\alpha, \beta,$ *and* C*. Then there exists a continuous modification* $X^* = \{X_t^*; 0 \leq t \leq T\}$ *of* X*, which is locally Holder continuous with exponent* γ *for every* $\gamma \in (0, \beta/\alpha)$*, i.e.*

$$
\Pr\left[\omega : \sup_{\substack{0 < t-s < h(\omega) \\ s,t \in [0,T]}} \frac{|X_t^*(\omega) - X_s^*(\omega)|}{|t - s|^{\gamma}} \leq \delta\right] = 1, \tag{10}
$$

where $h(\omega)$ *is an a.s. positive random variable and* $\delta > 0$ *is an appropriate constant.*

Proof. For notational simplicity, take $T = 1$.

$$
\begin{aligned}
\Pr\left[|X_t - X_s| \geq \varepsilon\right] &\leq \frac{E\left[|X_t - X_s|^{\alpha}\right]}{\varepsilon^{\alpha}} \text{ by Chebyshev's Inequality} \\
&\leq \frac{C\,|t - s|^{1+\beta}}{\varepsilon^{\alpha}} \quad \text{by (9).}
\end{aligned} \tag{11}
$$

Then, $X_s \to X_t$ in probability. In (11), let $t = \frac{k}{2^n}$, $s = \frac{k-1}{2^n}$, and $\varepsilon = 2^{-\gamma n}$. We have

$$\Pr\left[\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \geq 2^{-\gamma n}\right] \leq C(2^{-\gamma n})^{-\alpha}(2^{-n})^{1+\beta} \leq C \cdot 2^{-n(1+\beta-\alpha\gamma)}$$

and consequently,

$$\begin{aligned}
& \Pr\left[\max_{1\leq k\leq 2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \geq 2^{-\gamma n}\right] \\
\leq\ & \Pr\left[\bigcup_{k=1}^{2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \geq 2^{-\gamma n}\right] \\
\leq\ & \sum_{k=1}^{2^n}\Pr\left[\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \geq 2^{-\gamma n}\right] \\
\leq\ & 2^n \Pr\left[\left|X_{1/2^n}\right| \geq 2^{-\gamma n}\right] \\
\leq\ & 2^n C \cdot 2^{-n(1+\beta-\alpha\gamma)} \\
=\ & C \cdot 2^{-n(\beta-\alpha\gamma)}
\end{aligned}$$

Note that $\sum_{n=1}^{\infty}\Pr\left[\max_{1\leq k\leq 2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \geq 2^{-\gamma n}\right] < \infty$, so by the Borel-Cantelli lemma, $\max_{1\leq k\leq 2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| \geq 2^{-\gamma n}$ occurs for at most finitely many n. Then there is a set Ω^*, with $\Pr(\Omega^*) = 1$, such that for each $\omega \in \Omega^*$

$$\max_{1\leq k\leq 2^n}\left|X_{k/2^n} - X_{(k-1)/2^n}\right| < 2^{-\gamma n}, \quad \forall n \geq n^*(\omega), \tag{12}$$

where $n^*(\omega)$ is a positive integer-valued random variable.

For each integer $n \geq 1$, consider the partition $D_n = \{(k/2^n) : k = 0, 1, 2, ..., 2^n\}$ of $[0,1]$, and let $D = \cup_{n=1}^{\infty} D_n$ be the set of dyadic rationals in [0,1]. Fix $\omega \in \Omega^*$, $n \geq n^*(\omega)$. It will be shown for all $m \geq n$, we have

$$|X_t(\omega) - X_s(\omega)| \leq 2\sum_{j=n+1}^{m} 2^{-\gamma j} \quad \forall s, t \in D_m,\ 0 < t - s < 2^{-n}. \tag{13}$$

Consider $m = n+1$. Then, we can only have $t = \frac{k}{2^m}$, $s = \frac{k-1}{2^m}$, and $2\sum_{j=n+1}^{m} 2^{-\gamma j} = 2^{-\gamma m+1} < 2^{-\gamma m}$, so (13) follows from (12). Next, suppose that (13) is valid for $m = n+1, ..., M-1$. We want to show that (13) is valid for M. Consider $s < t$, $s, t \in D_M$. Let $t^{(1)} = \max\{u \in D_{M-1} : u \leq t\}$, $s^{(1)} = \min\{u \in D_{M-1} : u \geq s\}$. Note that $s \leq s^{(1)} \leq t^{(1)} \leq t$, $s^{(1)} - s \leq 2^{-M}$, $t - t^{(1)} \leq 2^{-M}$. Then, from (12), $|X_{s^{(1)}} - X_s| < 2^{-\gamma M}$ and $|X_t - X_{t^{(1)}}| < 2^{-\gamma M}$. From (13) with $m = M-1$,

$$|X_{t^{(1)}}(\omega) - X_{s^{(1)}}(\omega)| \leq 2\sum_{j=n+1}^{M-1} 2^{-\gamma j}\ .$$

Then,

$$
\begin{aligned}
& |X_t(\omega) - X_s(\omega)| \\
\leq \; & |X_t(\omega) - X_{t^{(1)}}(\omega)| + |X_{t^{(1)}}(\omega) - X_{s^{(1)}}(\omega)| + |X_{s^{(1)}}(\omega) - X_s(\omega)| \\
\leq \; & 2^{-\gamma M} + 2 \sum_{j=n+1}^{M-1} 2^{-\gamma j} + 2^{-\gamma M} \\
\leq \; & 2 \sum_{j=n+1}^{M} 2^{-\gamma j},
\end{aligned}
$$

so we have (13) for M.

Now, we can show that $\{X_t(\omega) : t \in D\}$ is uniformly continuous in t for every $\omega \in \Omega$. Let $h(\omega) := 2^{-n^*(\omega)}$. For any numbers $s, t \in D$ with $0 < t - s < 2^{-n^*(\omega)} = h(\omega)$, we select $n \geq n^*(\omega)$ such that $2^{-(n+1)} \leq t - s \leq 2^{-n}$. Then,

$$
\begin{aligned}
& |X_t(\omega) - X_s(\omega)| \\
\leq \; & 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j} \quad \text{by (13)} \\
\leq \; & 2 \cdot 2^{-(n+1)\gamma} \sum_{j=0}^{\infty} 2^{-\gamma j} \\
\leq \; & 2 \cdot 2^{-(n+1)\gamma} \frac{1}{1 - 2^{\gamma}} \\
\leq \; & 2 \, |t - s|^{\gamma} \, \frac{1}{1 - 2^{\gamma}} \qquad \text{since } 2^{-(n+1)} \leq t - s \\
\leq \; & \delta \, |t - s|^{\gamma}, \;\; \text{where } \delta = \frac{2}{1 - 2^{\gamma}}.
\end{aligned}
$$

Thus, we have uniform continuity on D.

Now, define X^* for $t \in [0, 1]$ as

$$
X_t^*(\omega) := \begin{Bmatrix} 0 & \text{for } \omega \notin \Omega^* \\ X_t(\omega) & \text{for } \omega \in \Omega \end{Bmatrix}.
$$

For $\omega \in \Omega^*$, $t \in [0,1] \cap D^C$, choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$. Uniform continuity and the Cauchy criterion imply that $\{X_{s_n}\}_{n=1}^{\infty}$ has a limit which depends on t but not on the particular sequence $\{s_n\}$ chosen. We let $X_t^*(\omega) = \lim_{s_n \to t} X_{s_n}(\omega)$. The resulting process X^* is thereby continuous. Further, X^* satisfies

$$
|X_t^*(\omega) - X_s^*(\omega)| \leq \delta \, |t - s|^{\gamma}, \;\; 0 < t - s < h(\omega)
$$

So (10) is established.

To see that X^* is a modification of X, observe that $X_t^* = X_t$ a.s. for $t \in D$. For $t \in [0,1] \cap D^C$ and $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$, we have $X_{s_n} \to X_t$ in probability and $X_{s_n} \to X_t^*$ a.s., so $X_t^* = X_t$ a.s. ∎

So by the last two lemmas, we have that for almost every $\omega \in \Omega$, the Brownian sample path $X(\omega)$ is Holder continuous with exponent $\gamma < \frac{1}{2}$. We shall see that these are the only γ-values for which the Brownian path is Holder continuous.

Lemma 20 *([PM], Lemma 2.5) Let $Z \sim N(0,1)$. Then $\forall x \geq 0$,*

$$\frac{x}{x^2+1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \leq \Pr(Z > x) \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Proof. First the right inequality will be shown. Recall that

$$\Pr(Z > x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}e^{-u^2/2}du$$

In this integral $u \geq x$, so we have

$$\begin{aligned}
\Pr(Z &> x) \leq \int_x^\infty \frac{u}{x}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du \\
&= \frac{1}{x}\frac{1}{\sqrt{2\pi}}\int_\infty^x e^{-u^2/2}(-u)du \\
&= \frac{1}{x}\frac{1}{\sqrt{2\pi}}\left[e^{-u^2/2}\right]_\infty^x \\
&= \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}
\end{aligned}$$

To prove the left inequality, let $f(x) = xe^{-x^2/2} - (x^2+1)\int_x^\infty e^{-u^2/2}du$. Note that

$$\begin{aligned}
&(i) \quad f(0) < 0 \text{ and} \\
&(ii) \quad \lim_{x\to\infty} f(x) = 0.
\end{aligned}$$

Further,

$$\begin{aligned}
f'(x) &= e^{-x^2/2} + xe^{-x^2/2}(-x) - 2x\int_x^\infty e^{-u^2/2}du - (x^2+1)(-e^{x^2/2}) \\
&= (1 - x^2 + x^2 + 1)e^{-x^2/2} - 2x\int_x^\infty e^{-u^2/2}du \\
&= 2e^{-x^2/2} - 2x\int_x^\infty e^{-u^2/2}du \\
&= -2x(\int_x^\infty e^{-u^2/2}du - \frac{1}{x}e^{-x^2/2})
\end{aligned}$$

By the right inequality, we have $\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-u^2/2}du \leq \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ which gives us that $f'(x) \geq 0$, $\forall x \geq 0$. Hence, by (i) and (ii), we get that $f(x) \leq 0$, $\forall x \geq 0$.

Then

$$\begin{aligned} xe^{-x^2/2} &\leq (x^2+1)\int_x^\infty e^{-u^2/2}du \\ \frac{x}{(x^2+1)}e^{-x^2/2} &\leq \int_x^\infty e^{-u^2/2}du \\ \frac{x}{(x^2+1)}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} &\leq \Pr(Z>x). \end{aligned}$$

■

Theorem 21 *([PM], Theorem 1.11, p. 25) For almost every* $\omega \in \Omega$*, the Brownian sample path* $X(\omega)$*, for every constant* $c<\sqrt{2}$ *and every* $\varepsilon > 0$*, there exist* $0 < h < \varepsilon$ *and* $0 \leq t \leq 1-h$ *with*

$$|X_{t+h}(\omega) - X_t(\omega)| \geq c\sqrt{h\ln(h)}$$

Proof. Let $c < \sqrt{2}$ and define, for integers $k, n \geq 0$,

$$A_{k,n} := \{\omega \in \Omega : |X_{(k+1)/e^n}(\omega) - X_{k/e^n}(\omega)| > c\sqrt{n}e^{-n/2}\}$$

$$\begin{aligned} \Pr(A_{k,n}) &= \Pr(X_{e^{-n}} > c\sqrt{n}e^{-n/2}) \\ &= \Pr(Z > c\sqrt{n}) \\ &\geq \frac{c\sqrt{n}}{c^2n+1}e^{-c^2n/2} \quad \text{by Lemma 20} \end{aligned}$$

Since $c < \sqrt{2}$, $e^n \Pr(A_{k,n}) \to 0$ as $n \to \infty$. Then, using $1 - x \leq e^{-x}$, for all x,

$$\begin{aligned} \Pr\left(\bigcap_{k=0}^{\lfloor e^n - 1\rfloor} A_{k,n}^C\right) &= (1 - \Pr(A_{k,n}))^{e^n} \\ &\leq \exp(-e^n \Pr(A_{0,n})) \\ &\to 0 \end{aligned}$$

Consider that for any $\varepsilon > 0$, there exist $h \in (0, \varepsilon)$ and $n \in \mathbb{N}$ such that $h = e^{-n}$. Then, it follows that for any $\varepsilon > 0$,

$$\Pr\{\omega \in \Omega : \forall h \in (0,\varepsilon),\ \forall t \in [0, 1-h],\ \ |X_{t+h}(\omega) - X_t(\omega)| < c\sqrt{h\ln(h)}\} = 0$$

■

Theorem 22 *For almost every $\omega \in \Omega$, the Brownian path $X(\omega)$ is nowhere Holder continuous with exponent $\gamma > \frac{1}{2}$.*

Proof. For $\gamma > \frac{1}{2}$ and sufficiently small h, $1 > he^{h^{2\gamma-1}}$. Then, $\frac{1}{h} > e^{h^{2\gamma-1}}$. It follows that $\sqrt{h \ln(\frac{1}{h})} > h^{\gamma}$. Then, apply the previous theorem. ■

Theorem 23 *([PM], Exercise 4.4) For almost every $\omega \in \Omega$, the Brownian path $X(\omega)$ is nowhere Holder continuous with exponent $\gamma = \frac{1}{2}$.*

Proof. Let $j > 0$. Let $H = \{\omega \in \Omega : |X_{t+h} - X_t| \leq jh^{1/2},\ \forall t > 0, \forall h > 0\}$. In other words, H is the set of Brownian paths that are Holder continuous with exponent $\frac{1}{2}$ and coefficient j. It must be shown that $\Pr(H) = 0$ (regardless of j) .

Let $n^* \in \mathbb{N}\ n^* > j^2$. Let c be such that $1 < c < \sqrt{2\ln(2)}$. Let $A_{k,n} := \{\omega \in \Omega : |X_{(k+1)2^{-n}} - X_{k2^{-n}}| > c\sqrt{n}2^{-n/2}\}$. Since $j < c\sqrt{n}$, $H \subseteq \cap_{n=n^*}^{\infty} \cap_{k=1}^{2^n} A_{k,n}^C$. Then it suffices to show that $\Pr(\cap_{n=n^*}^{\infty} \cap_{k=1}^{2^n} A_{k,n}^C) = 0$.

$$\Pr(A_{k,n}) = \Pr(X_1 > c\sqrt{n}) \geq \frac{c\sqrt{n}}{c^2 n + 1} e^{-c^2 n/2} \quad \text{by Lemma 20} \tag{14}$$

Since $c < \sqrt{2\ln(2)}$, we have $\frac{c^2}{2} < \ln(2)$, $\frac{-c^2}{2} > \ln(\frac{1}{2})$, $e^{-c^2/2} > \frac{1}{2}$ and $2e^{-c^2/2} > 1$. Thus,

$$\begin{aligned} \lim_{n\to\infty} 2^n e^{-c^2 n/2} \frac{c\sqrt{n}}{c^2 n + 1} &= \infty \text{ and} \\ \lim_{n\to\infty} 2^n \Pr(A_{k,n}) &= \infty \quad \text{by (14).} \end{aligned}$$

Then,

$$\begin{aligned} \Pr(\cap_{k=1}^{2^n} A_{k,n}^C) &= [1 - \Pr(A_{k,n})]^{2^n} \\ &\leq e^{-2^n Pr(A_{k,n})} \text{ since } 1 - x \leq e^{-x} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

■

5 Non-differentiability

The goal of this section is to show that the Brownian sample path is nowhere differentiable.

Theorem 24 *([KS], Theorem 9.9) For almost every $\omega \in \Omega$, the sample path $X(\omega)$ is monotone in no interval.*

Proof. Suppose $X(\omega)$ is monotone on some interval I. There exist two rationals in I, q_1 and q_2 (Assume $q_1 < q_2$.) Then, $X(\omega)$ is monotone on $[q_1, q_2]$. By Lemma 15 part (i), there exists a Brownian motion with a sample path that is monontone on $[0,1]$. Hence, the theorem will be proven once it has been shown that on $[0,1]$, $X(\omega)$ is monotone for almost no ω. By Lemma 15 part (iv), it suffices to show that on $[0,1]$, $X(\omega)$ is nondecreasing for almost no ω.

$\forall\omega,\ \forall s,t,\ \ s > t,\ \Pr(X_s(\omega) - X_t(\omega) \geq 0) = \frac{1}{2}$

Let $A_n := \{\omega \in \Omega : \forall i \in \{0,1,2,...,n-1\},\ \ X_{(i+1)/n}(\omega) - X_{i/n}(\omega) \geq 0\}$
$\Pr(A_n) = \Pi_{i=0}^{n-1} \Pr(X_{(i+1)/n}(\omega) - X_{i/n}(\omega) \geq 0) = 2^{-n}$
Let $A := \{\omega \in \Omega : X(\omega)$ is nondecreasing on $[0,1]\}$
Then, $A = \cap_{n=1}^{\infty} A_n$. $\Pr(A) \leq \lim\limits_{n\to\infty} \Pr(A_n) = 0$ ■

Definition 25 *([KS], Definition 2.9.16) For a continuous function $f : [0,\infty) \to \mathbb{R}$, we denote*

the upper right Dini derivative by $D^+ f(t) = \overline{\lim\limits_{h\to 0^+}} \frac{f(t+h)-f(t)}{h}$
the upper left Dini derivative by $D^- f(t) = \overline{\lim\limits_{h\to 0^-}} \frac{f(t+h)-f(t)}{h}$
the lower right Dini derivative by $D_+ f(t) = \underline{\lim\limits_{h\to 0^+}} \frac{f(t+h)-f(t)}{h}$
the lower left Dini derivative by $D_- f(t) = \underline{\lim\limits_{h\to 0^-}} \frac{f(t+h)-f(t)}{h}$

The function f is said to be differentiable at $t > 0$ if the four Dini derivatives are finite numbers and equal. At $t = 0$, differentiability only requires the two right Dini derivatives agree.

Theorem 26 *([KS], Theorem 2.9.18) For almost every $\omega \in \Omega$, the Brownian sample path $X(\omega)$ is nowhere differentiable.*

Proof. By Lemma 15 part (i), It is enough to consider the interval $[0,1]$. Let

$$A_{j,k} := \bigcup_{t\in[0,1]} \bigcap_{h\in[0,1/k]} \{\omega \in \Omega : |X_{t+h}(\omega) - X_t(\omega)| \leq jh\}$$

(The intuitive idea is that with the graph of a Brownian path from $A_{j,k}$, when we draw a secant line through any 2 points whose $t-$values are less than $\frac{1}{k}$ apart, then the slope of the secant line is between $-j$ and j.) Note that

$$\{\omega \in \Omega : -\infty < D_+ X_t(\omega) \leq D^+ X_t(\omega) < \infty \ \text{ for some } t \in [0,1]\} = \cup_{k=1}^{\infty} \cup_{j=1}^{\infty} A_{j,k}$$

Then, for each fixed j,k, we want to produce an event C such that $A_{j,k} \subseteq C$ and $\Pr(C) = 0$. Fix $\omega \in A_{j,k}$. Let $t \in [0,1]$ such that $|X_{t+h}(\omega) - X_t(\omega)| \leq jh$ for every $h, 0 \le h \le \frac{1}{k}$. Let $n \le 4k$. Let i be such that $1 \le i \le n$ and $\frac{i-1}{n} \le t \le \frac{i}{n}$. Consider the intervals $[\frac{i}{n}, \frac{i+1}{n}], [\frac{i+1}{n}, \frac{i+2}{n}]$,and $[\frac{i+2}{n}, \frac{i+3}{n}]$.

$$\begin{aligned} |X_{(i+1)/n}(\omega) - X_{i/n}(\omega)| &\le |X_{(i+1)/n}(\omega) - X_t(\omega)| + |X_{i/n}(\omega) - X_t(\omega)| \\ &\le \frac{2j}{n} + \frac{j}{n} = \frac{3j}{n}, \end{aligned}$$

$$\begin{aligned} |X_{(i+2)/n}(\omega) - X_{(i+1)/n}(\omega)| &\le |X_{(i+2)/n}(\omega) - X_t(\omega)| + |X_{(i+1)/n}(\omega) - X_t(\omega)| \\ &\le \frac{3j}{n} + \frac{2j}{n} = \frac{5j}{n}, \end{aligned}$$

$$\begin{aligned} |X_{(i+3)/n}(\omega) - X_{(i+2)/n}(\omega)| &\le |X_{(i+3)/n}(\omega) - X_t(\omega)| + |X_{(i+2)/n}(\omega) - X_t(\omega)| \\ &\le \frac{4j}{n} + \frac{3j}{n} = \frac{7j}{n}. \end{aligned}$$

Let

$$C_i^{(n)} := \cap_{v=1}^3 \{\omega \in \Omega : |X_{(i+v)/n}(\omega) - X_{(i+v-1)/n}(\omega)| \le \frac{2v+1}{n} j\}. \qquad (15)$$

Observe that $A_{j,k} \subseteq \cup_{i=1}^n C_i^{(n)}$ holds for every $n \ge 4k$. Let

$$Z_v = \sqrt{n}(X_{(i+v)/n} - X_{(i+v-1)/n}) \text{ for } v = 1, 2, 3$$

Then, each Z_v is an independent, standard normal random variable. Recall that

$$\Pr(|Z_v| \le \varepsilon) \le \varepsilon \qquad (16)$$

So by (15) and (16),

$$\Pr(C_i^{(n)}) \le \Pi_{v=1}^3 \sqrt{n} \frac{2v+1}{n} j = \frac{105 j^3}{n^{3/2}}, \quad \text{for} \quad i = 1, 2, 3, ..., n.$$

Then, $\Pr(\cup_{i=1}^n C_i^{(n)}) \le n \frac{105j^3}{n^{3/2}} = \frac{105j^3}{\sqrt{n}}$. Let $C := \cap_{n=4k}^{\infty} \cup_{i=1}^n C_i^{(n)}$. Then, $\Pr(C) \le \inf_{n \ge 4k} \Pr(\cup_{i=1}^n C_i^{(n)}) = 0$. ■

6 Alternate Construction of Brownian Motion (One-dimensional)

Consider this alternate definition of Brownian motion.

Definition 27 ***One-dimensional Brownian motion process*** *consists of a probability space* $(\Omega, \mathcal{B}, P)$ *and a family* $\{X_t, t \geq 0\}$ *of random variables defined on it satisfying*

(i) $X_0(\omega) = 0$ *(a.s.)*
(ii) *If* $0 = t_0 < t_1 < .... < t_n$, *then the random variables* $X_{t_{i+1}} - X_{t_i}, i = 0, 1, ..., n-1,$ *are independent.*
(iii) *For each* $s, t \geq 0$, $X_{t+s} - X_t$ *is normally distributed with mean* 0 *and variance* s.
(iv) *For almost all* $\omega \in \Omega$, *the function* $X_t = X_t(\omega)$ *is everywhere continuous in* t.

This definition is equivalent to Definition 2. To see this, suppose that we have a family of random variables $\{X_t\}$ that satisfies Definition 2. Then Definition 27 is satisfied by Propositions 11, 12, and 5, and the argument regarding equation (7). Conversely, suppose that we have a family of random variables that satisfies Definition 27. Then, Definition 2 is satisfied using (iii) and (ii) above and (1).

Theorem 28 *A Brownian motion process on* $[0, \infty)$ *exists*

Proof. Let $D_n = \{k \ / \ 2^n : 0 \leq k \leq 2^n\}$. Let $D = \cup_{n=0}^{\infty} D_n$. We will first construct a family of random variables $\{X_t\}$ on $[0,1] \cap D$ and then extend it to $[0, \infty)$. It will be shown that this family of random variables satisfies Definition 27. Let $\{Z_d\}_{d \in D}$ be a collection of independent $N(0,1)$ random variables. Let $X_0 = 0$, $X_1 = Z_1$. Then for $d \in D_n \setminus D_{n-1}$ define

$$X_d = \frac{X_{d^-} + X_{d^+}}{2} + \frac{Z_d}{2^{(n+1)/2}}, \tag{17}$$

where $d^- = d - 2^{-n}$ and $d^+ = d + 2^{-n}$. Note that the vectors $\{X_d : d \in D_n\}$ and $\{Z_d : d \in D \backslash D_n\}$ are independent.

Claim 1: For $r < s < t$ in D_n, $X_t - X_s \sim N(0, t-s)$ and $X_t - X_s$ is independent of $X_s - X_r$.

It will be shown by induction that Claim 1 holds for $\forall d \in D$. Assume $n = 1$. Then we have

$$\begin{aligned} X_{\frac{1}{2}} - X_0 &= X_{\frac{1}{2}} \\ &= \frac{1}{2}(X_0 + X_1) + \frac{1}{2} Z_{\frac{1}{2}} \text{ by (17)} \\ &= \frac{1}{2} X_1 + \frac{1}{2} Z_{\frac{1}{2}} \\ &= \frac{1}{2}(X_1 + Z_{\frac{1}{2}}). \end{aligned} \tag{18}$$

Further,

$$\begin{aligned}
X_1 - X_{\frac{1}{2}} &= X_1 - \left[\frac{1}{2}(X_1) - \frac{1}{2}Z_{\frac{1}{2}}\right] \\
&= \frac{1}{2}X_1 - \frac{1}{2}Z_{\frac{1}{2}} \\
&= \frac{1}{2}(X_1 - Z_{\frac{1}{2}}). \qquad (19)
\end{aligned}$$

Since X_1 and $Z_{\frac{1}{2}}$ are independent, each with distribution $N(0,1)$, their sum and their difference are independent, each with distribution $N(0,2)$ (by Corollary 37 in the Appendix). By (18) and (19), $X_{\frac{1}{2}} - X_0$ and $X_1 - X_{\frac{1}{2}}$ are independent with distribution $N(0,\frac{1}{2})$. So, Claim 1 is true for $n = 1$.

Next, assume that Claim 1 holds for $n-1$. First we will show that Claim holds for three consecutive values in D_n. Let $d \in D_n/D_{n-1}$. Consider $r = d^-$, $s = d$, $t = d^+$. Note that

$$\begin{aligned}
X_{d^+} - X_d &= X_{d^+} - \left[\frac{1}{2}\left(X_{d^+} + X_{d^-}\right) + \frac{1}{2^{(n+1)/2}}Z_d\right] \\
&= \frac{1}{2}\left(X_{d^+} - X_{d^-}\right) - \frac{1}{2^{(n+1)/2}}Z_d.
\end{aligned}$$

Further,

$$\begin{aligned}
X_d - X_{d^-} &= \left[\frac{1}{2}\left(X_{d^+} + X_{d^-}\right) + \frac{1}{2^{(n+1)/2}}Z_d\right] - X_{d^-} \\
&= \frac{1}{2}\left(X_{d^+} - X_{d^-}\right) + \frac{1}{2^{(n+1)/2}}Z_d,
\end{aligned}$$

$d^-, d^+ \in D_{n-1}$ and $d^+ - d^- = \frac{1}{2^{n-1}}$. Then, by induction, we have that $X_{d^+} - X_{d^-} \sim N(0, \frac{1}{2^{n-1}})$. Then, $\frac{1}{2}\left(X_{d^+} - X_{d^-}\right) \sim N(0, \frac{1}{2^{n+1}})$. $d \notin D_{n-1}$, so Z_d is independent of X_{d^+} and X_{d^-}. Hence, by Corollary 37, Claim 1 holds for d^-, d, and d^+. A similar argument shows that Claim 1 holds for d, d^+, and $d^+ + \frac{1}{2^n}$.

Next consider arbitrary $r < s < t \in D_n$. To show that $X_t - X_s$ has the correct distribution, we need only consider that

$$X_t - X_s = X_t - X_{t-\frac{1}{2^n}} + X_{t-\frac{1}{2^n}} - X_{t-\frac{2}{2^n}} + \cdots + X_{s+\frac{2}{2^n}} - X_{s+\frac{1}{2^n}} + X_{s+\frac{1}{2^n}} - X_s$$

This will get us that $X_t - X_s \sim N(0, t-s)$, if the increments over non-adjacent intervals of length 2^{-n} are shown to be independent. This pairwise independence of the increments must be shown. Consider two such intervals in D_n. There exists $c \in D_j$, $j < n$ such that the two intervals are contained in $[c - \frac{1}{2^j}, c]$ and $[c, c + \frac{1}{2^j}]$. By induction the increments over these intervals of length 2^{-j} are independent. The increments over the intervals of length 2^{-n} are then constructed (possibly through several induction steps) from the independent increments $X_c - X_{c-\frac{1}{2^j}}$ and $X_{c+\frac{1}{2^j}} - X_c$ using a disjoint set of variables $\{Z_d : d \in D_n \backslash D_j\}$. Hence the increments over non-adjacent intervals are shown to be independent. Claim 1 is true for all n.

Next, we will define X_t for $t \notin D$. Let

$$F_0(t) = \left\{ \begin{array}{c} Z_1, \text{ for } t = 1 \\ 0, \text{ for } t = 0 \\ \text{linear, in between} \end{array} \right| .$$

Then, for $n = \{1, 2, 3, ...\}$, let

$$F_n(t) = \left\{ \begin{array}{c} 2^{-(n-1)/2} Z_t, \text{ for } t \in D_n \backslash D_{n-1} \\ 0, \text{ for } t \in D_{n-1} \\ \text{linear, between consecutive points in } D_n \end{array} \right|$$

Notice that $\forall n$, $\forall t \in [0, 1]$, $F_i(t)$ is continuous and it follows that $\sum_{i=0}^{n} F_i(t)$ is continuous. Also, $\forall n$, $\forall d \in D_n$,

$$X_d = \sum_{i=0}^{n} F_i(d) = \sum_{i=0}^{\infty} F_i(d).$$

This can be shown by induction. Suppose that it holds for $n - 1$. Let $d \in D_n \backslash D_{n-1}$. Then, for $0 \leq i \leq n - 1$,

$$\begin{aligned} F_i(d) &= \frac{1}{2}\left[F_i(d^-) + F_i(d^+)\right] \\ \sum_{i=0}^{n-1} F_i(d) &= \sum_{i=0}^{n-1} \frac{1}{2}\left[F_i(d^-) + F_i(d^+)\right] \\ &= \frac{1}{2}\left[X_{d^-} + X_{d^+}\right], \text{ and} \end{aligned}$$

$$\begin{aligned} \sum_{i=0}^{n} F_i(d) &= \sum_{i=0}^{n-1} F_i(d) + F_i(d) \\ &= \frac{1}{2}\left[X_{d^-} + X_{d^+}\right] + 2^{-(n-1)/2} Z_d \\ &= X_d. \end{aligned}$$

It remains to show that $\forall t > s \in [0, 1]$, $X_t - X_s \sim N(0, t - s)$. For n large enough, $\Pr(|Z_d| \geq c\sqrt{n}) \leq \exp(-\frac{c^2 n}{2})$ for $c > 0$. (This is Lemma 20) Fix $c > \sqrt{2\ln(2)}$. Let E_n be the event that there exists $d \in D_n$ with $|Z_d| \geq c\sqrt{n}$.

$$\begin{aligned} \Pr(E_n) &\leq \sum_{d=0}^{2^n} \Pr(|Z_d| \geq c\sqrt{n}) \\ \sum_{n=0}^{\infty} \Pr(E_n) &\leq \sum_{n=0}^{\infty} (2^n + 1) \exp(-\frac{c^2 n}{2}) \end{aligned}$$

Since $c > \sqrt{2\ln(2)}$, we have $\frac{c^2}{2} > \ln(2)$, $\frac{-c^2}{2} < \ln(\frac{1}{2})$, $e^{-c^2/2} < \frac{1}{2}$ and $2e^{-c^2/2} < 1$. Then, $\Sigma_{n=0}^{\infty} \Pr(E_n) < \infty$. By the Borel-Cantelli Lemma, there exists a finite N such that $\forall n \geq N$, $\forall d \in D_n$, $|Z_d| < c\sqrt{n}$. It follows that $\forall n \geq N$, $\|F_n\|_\infty < c\sqrt{n}2^{-n/2}$. This upper bound gives us that $\sum_{i=0}^{\infty} F_i(t)$ is uniformly convergent on $[0,1]$. Then, define for $t \in [0,1] \cap D^c$, $X_t = \sum_{i=0}^{\infty} F_i(t)$.

Let t_1,t_2,t_3,... and s_1,s_2,s_3,... be sequences in D converging to t and s respectively. Then, $X_t - X_s = \lim_{k\to\infty} (X_{t_k} - X_{s_k})$ is a limit of normal random variables. Hence, it is normal with mean 0 and variance $\lim_{k\to\infty} (t_k - s_k)$. Thus, we have Brownian motion on $[0,1]$. To extend to $[0,\infty)$, for $n \geq 0$, let $\{X_t^{(n)}\}_n$ be independent Brownian motions on $[0,1]$. Then, $X_t = X_{t-\lfloor t\rfloor}^{\lfloor t\rfloor} + \sum\limits_{0\leq i\leq\lfloor t\rfloor} X_1^{(i)}$

■

7 Appendix

Definition 29 *The **joint distribution measure** of two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ is the distribution measure $\mu_{(X,Y)}$ of $(X,Y) : \Omega \to \mathbb{R}^2$. We say that X and Y are **independent** if $\mu_{(X,Y)}$ is the product measure $\mu_X \times \mu_Y$, determined by*

$$(\mu_X \times \mu_Y)(A,B) = \mu_X(A)\mu_Y(B),$$

for all measurable $A \subset X$ and $B \subset Y$. Similarly one defines the joint distribution measure of any finite number of random variables, as well the notion of their independence.

Definition 30 *The **characteristic function** $\phi_X : \Omega \to \mathbb{C}$ of a real-valued random variable X on a probability measure space (Ω, μ), is given by*

$$\phi_X(\lambda) := E\left(e^{i\lambda X}\right) = \int_\Omega e^{i\lambda X}\, d\mu = \int_{-\infty}^{\infty} e^{i\lambda x}\, d\mu_X,$$

where $\mu_X(A) := \mu(X^{-1}(A))$ defines the distribution measure of X. If μ_X has a density function $p_X(x)$, i.e., $d\mu_X = p_X(x)dx$, then

$$\phi_X(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x}\, d\mu_X = \int_{-\infty}^{\infty} e^{i\lambda x}\, p_X(x)\, dx = (2\pi)^{\frac{1}{2}}\, \widehat{p_X}(-\lambda),$$

where we recall (see [BC], p. 423) that the Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}$ of $f \in L^1(\mathbb{R})$ is given by

$$\widehat{f}(\lambda) := (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx.$$

If $p_X(x)$ and $p_Y(y)$ are the probability densities of *independent* random variables X and Y, then $p_X(x)\,p_Y(y)$ is the joint density function of X and Y, and

$$\begin{aligned}\mu_{(X,Y)}(X+Y\le b) &= \int\int_{-\infty<x+y\le b} p_X(x)\,p_Y(y)\;dx\,dy = (\text{where } z = x+y)\\ &= \int_{-\infty}^{b}\int_{-\infty}^{\infty} p_X(x)\,p_Y(z-x)\;dx\,dz.\end{aligned}$$

Thus, the **convolution**

$$(p_X * p_Y)(z) := \int_{-\infty}^{\infty} p_X(x)\,p_Y(z-x)\;dx = \int_{-\infty}^{\infty} p_X(z-y)\,p_Y(y)\;dx$$

is the density for $X+Y$; i.e.,

$$d\mu_{X+Y} = \left(\int_{-\infty}^{\infty} p_X(x)\,p_Y(z-x)\;dx\right)dz. \tag{20}$$

The Convolution Theorem (see [BC], p. 436) says that for suitably decaying functions f and g (Actually $f,g\in L^1(\mathbb{R})$ is enough – see [DM], p. 42)

$$(f*g)^{\wedge}(\xi) = \sqrt{2\pi}\,\widehat{f}(\xi)\,\widehat{g}(\xi). \tag{21}$$

In terms of characteristic functions,

$$\phi_{X+Y}(\lambda) = \phi_X(\lambda)\phi_Y(\lambda).$$

This can also be shown more directly:

$$\begin{aligned}\phi_{X+Y}(\lambda) &= E\left(e^{i\lambda(X+Y)}\right) = E\left(e^{i\lambda X}e^{i\lambda Y}\right) \overset{(X,Y\text{ indep.})}{=} E\left(e^{i\lambda X}\right)E\left(e^{i\lambda Y}\right)\\ &= \phi_X(\lambda)\phi_Y(\lambda).\end{aligned}$$

In the case where $X:\Omega\to\mathbb{R}^d$, we can define a characteristic function $\phi_X:\mathbb{R}^d\to\mathbb{C}$ by

$$\phi_X(\lambda) = E\left(e^{i\lambda\cdot X}\right),\ \text{where } \lambda\cdot X = \lambda_1X_1+\cdots+\lambda_dX_d.$$

In higher dimensions, for $f\in L^1\left(\mathbb{R}^d\right)$, we have

$$\widehat{f}(\lambda) = \widehat{f}(\lambda_1,\ldots,\lambda_d) := (2\pi)^{-\frac{d}{2}}\int_{-\infty}^{\infty} e^{-i\lambda\cdot x} f(x_1,\ldots,x_n)\,dx_1\ldots dx_n.$$

If $d\mu_X = p_X(x)\,dx = p_X(x_1,\ldots,x_d)\,dx_1\cdots dx_d$, then

$$\phi_X(\lambda) = (2\pi)^{d/2}\,\widehat{p_X}(-\lambda_1,\ldots,-\lambda_d) = \int_{\mathbb{R}^d} e^{i\lambda\cdot x}p_X(x)\,dx.$$

Theorem 31 ***The Borel-Cantelli Theorem*** *For arbitrary events* $\{E_n\}$, $\sum_n \Pr(E_n) < \infty \Rightarrow \Pr(E_n \text{ i.o.}) = 0$, *where i.o. stands for "infinitely often". For independent events* $\{E_n\}$, $\sum_n \Pr(E_n) = \infty \Rightarrow \Pr(E_n \text{ i.o.}) = 1$.

For a proof, see [Ch] p. 69-72.

Definition 32 *A mapping* $f : X \to Y$ *between metric spaces* X *and* Y *is said to be* ***Holder continuous of order*** $\gamma > 0$, *if there exists a* $j > 0$ *such that* $\forall s, t \in X$,

$$d_Y(f(s), f(t)) < j \cdot [d_X(s,t)]^\gamma$$

Theorem 33 ***Chebyshev Inequality*** *([Ch] p. 46) If* φ *is a strictly positive and increasing functionon* $(0, \infty)$, $\varphi(u) = \varphi(-u)$ *and* X *is a random variable such that* $E[\varphi(X)] < \infty$, *then for* $u > 0$,

$$\Pr(|X| \geq u) \leq \frac{E[\varphi(X)]}{\varphi(u)}$$

We apply this theorem with $\varphi(u) = |u|^\alpha$ in the proof of Theorem 19.

Definition 34 *([PM] Def. II.3.2, p. 255) A random variable* $X = (X_1, ..., X_d)^T$ *with values in* $\mathbb{R}^d$ *has the* ***d-dimensional standard Gaussian distribution*** *if its d coordinates are standard normally distributed and independent.*

Definition 35 *([PM] Def. II.3.3, p.255) A random variable* Y *with values in* $\mathbb{R}^n$ *is called* ***Gaussian*** *if there exists d-dimensional standard Gaussian* X, *an* $n \times d$ *matrix* A, *and an n-dimensional vector* b *such that* $Y^T = AX + b$. *The covariance matrix of the column vector* Y *is given by* $Cov(Y) = E\left[(Y - E(Y))(Y - E(Y))^T\right] = AA^T$ *where the expectations are defined componentwise.*

Lemma 36 *([PM] Lemma II.3.4) If* A *is an orthogonal* $d \times d$ *matrix (i.e.* $AA^T = I_d$*) and* X *is a d-dimensional standard Gaussian vector, then* AX *is also a d-dimensional standard Gaussian vector.*

Proof.

$$\begin{aligned} f(x_1, ...x_d) &= \prod_{i=1}^{d} e^{-x_i^2/2} \\ &= \frac{1}{(2\pi)^{d/2}} e^{-\|x\|^2/2} \end{aligned}$$

Then, the density of AX is $f(A^{-1}x) \cdot \|\det(A^{-1})\|$ (This is the substitution rule for multiple variables.) Orthogonal matrices have a determinant of 1 and they preserve the Euclidean norm, so AX has the same density as X. ■

Corollary 37 *([PM] Cor. II.3.5, p. 255) Let X_1 and X_2 be independent and normally distributed with expectation 0 and variance $\sigma^2 > 0$. Then, $X_1 + X_2$ and $X_1 - X_2$ are independent and normally distributed with expectation* 0 *and variance* $2\sigma^2$.

Proof. Note that the vector $Y = \left(\frac{X_1}{\sigma}, \frac{X_2}{\sigma}\right)^T$ is standard Gaussian.

$$\text{Let } A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Then,

$$AY = \begin{pmatrix} \frac{X_1+X_2}{\sqrt{2}\sigma} \\ \frac{X_1-X_2}{\sqrt{2}\sigma} \end{pmatrix}$$

which must have independent standard normal coordinates by Lemma 36 ■

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