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ABSTRACT

We study some properties of the American derivative security price for which the payoff is the length of the current run of heads. We discuss the distribution, expected value, and variance of the longest run of heads in a coin flip sequence. Furthermore, we study the optimal stopping condition and prove that for $N$ independent tosses of a fair coin, the price is on the order of $\log N$. 
# TABLE OF CONTENTS

Acknowledgments .................................................. ii

Abstract .......................................................... iii

1. American Derivative Securities ............................... 1

   1.1. Introduction .............................................. 1

      1.1.1. Options ............................................. 1

      1.1.2. American option ................................. 1

      1.1.3. The Binomial Model ......................... 2

      1.1.4. Binomial Asset Price Process .............. 2

   1.2. Martingales ............................................. 2

      1.2.1. Stopping Time ................................... 3

   1.3. General American Derivatives ....................... 4

2. The Exact Distribution of the Longest Run for a Fair Coin .......................... 7

   2.1. Properties of the Distribution of the Longest Run ..................... 10

      2.1.1. Expectation and variance of the longest head run ........... 10

3. American option pricing and optimal stopping for success runs .................. 12

References .......................................................... 17
CHAPTER 1

American Derivative Securities

1.1 Introduction

1.1.1 Options

An option gives the holder the right to trade (buy or sell) a specified quantity of an underlying asset at a fixed price (also called the strike price) at any time on or before a given date or expiration date. Since it is a right and not an obligation the owner of the option can choose not to exercise the option and allow the option to expire.

In any option contract, there are two parties involved. An investor that buys an option (that is, an option holder) and an investor that sells an option (that is, an option writer). The options holder is said to take a long position while the options writer is said to take a short position.

1.1.2 American option

An American option can be exercised at any time up to and including the expiry date of the option. The early exercise feature of these options complicates the evaluation process as the standard Black-Scholes continuous time model cannot be used. The most common model for valuating American Options is the binomial model.

Let us define the intrinsic value of an option. Since an American option can be exercised at any time prior to its expiration, it can never be worth less than the pay off associated with the immediate exercise and we call this the intrinsic value of
an option.

In the following sections, we will go over some definitions and theorems from Shreve’s book([13], Chapter 4).

1.1.3 The Binomial Model

The binomial model presents a way to describe the random asset price dynamics. Let us take two possible asset prices per time-step, by increasing the number of time-steps in the limit we will eventually arrive at the correct price of the option and find an alternative way to represent the value of the option namely the risk-neutral expectation formula.

1.1.4 Binomial Asset Price Process

The binomial model starts out with two state market model. If \( S_0 \) is the spot price of a risky asset at time \( t = 0 \) after some time period \( T \), it can only assume two distinct values: \( uS_0 \) and \( dS_0 \), where \( u \) and \( d \) are real numbers such that \( u > d \). Moreover we will assume the existence of a risk-less asset with a constant yield \( r \).

Thus, we can say that an investment of \( S_0 \) dollars at time \( t = 0 \) yields \( e^{rT}S_0 \) time \( t = T \). The no arbitrage argument is also valid here, we must require that

\[
S_0d < S_0u.
\]

1.2 Martingales

**Definition 1.1.** Consider the binomial asset-pricing model as in section 1.1. Let \( M_0, M_1, ..., M_N \) be a sequence of random variables, with each \( M_n \) depending only on the first \( n \) coin tosses. Such a sequence of random variables is called an adapted stochastic process.
(1) If
\[ M_n = \mathbb{E}_n[M_{n+1}], \quad n = 0, 1, \ldots, N - 1 \]
we say this process is a martingale.

(2) If
\[ M_n \leq \mathbb{E}_n[M_{n+1}], \quad n = 0, 1, \ldots, N - 1 \]
we say the process is a submartingale.

(3) If
\[ M_n \geq \mathbb{E}_n[M_{n+1}], \quad n = 0, 1, \ldots, N - 1 \]
we say the process is a supermartingale.

1.2.1 Stopping Time

The time at which an American security derivative should be exercised is random; it depends of the price movement of the underlying asset.

Definition 1.2. In an N-period binomial model, a stopping time is a random variable \( \tau \) that takes values 0, 1, \ldots, N or \( \infty \) and satisfies the condition that if
\[ \tau(\omega_1, \omega_2, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_N) = n \]
then
\[ \tau(\omega_1, \omega_2, \ldots, \omega_n, \omega'_{n+1}, \ldots, \omega'_N) = n \]
for all \( \omega'_{n+1}, \ldots, \omega'_N \).

Theorem 1.3 (optimal sampling-Part 1). A martingale stopped at a stopping time is a martingale. A supermartingale(or submartingale) stopped at a stopping time is a supermartingale(or submartingale, respectively).
Theorem 1.4 (optimal sampling-Part 2). Let $X_n, n = 0, 1, \ldots, N$ be a submartingale, and let $\tau$ be a stopping time. Then $\mathbb{E}X_{n\wedge \tau} \leq \mathbb{E}X_n$. If $X_n$ is supermartingale, then $\mathbb{E}X_n \leq \mathbb{E}X_{n\wedge \tau}$; if $X_n$ is a martingale, then $\mathbb{E}X_{n\wedge \tau} = \mathbb{E}X_n$.

1.3 General American Derivatives

In this section, we introduce American derivative securities whose intrinsic value is allowed to be path-dependent. We will study the optimal exercise time.

Definition 1.5. For each $n = 0, 1, \ldots, N$, let $G_n$ be a random variable depending on the first $n$ coin tosses. An American derivative security with intrinsic value process $G_n$ is a contract that can be exercised at any time prior to and including time $N$ or not exercised at all, and, if exercised at time $n$, pays $G_n$. We define the price process $V_n$ for this contract by American risk-neutral formula

$$ V_n = \max_{\tau \in \mathcal{S}_n} \mathbb{E}_n\left[ \prod_{\tau \leq N} (1/(1 + r)^{\tau-n}) G_\tau \right], n = 0, 1, \ldots, N. \quad (1.1) $$

Theorem 1.6 (13, Theorem 4.4.2). The American derivative security price process has the following properties:

1. $\max\{G_n, 0\} \leq V_n$ for all $n$;

2. The discounted process $[1/(1 + r)^n] V_n$ is a supermartingale;

3. If $Y_n$ is another process satisfying $\max\{G_n, 0\} \leq Y_n$ for all $n$ and for which $[1/(1 + r)^n] Y_n$ is a supermartingale, then $V_n \leq Y_n$ for all $n$.

Now we generalize the American pricing algorithm for the path-dependent securities.
**Theorem 1.7** (13, Theorem 4.4.3). We have the following American pricing algorithm for the path-dependent derivative security price process:

(1) \( V_N(\omega_1 \ldots \omega_N) = \max\{G_N(\omega_1 \ldots \omega_N), 0\} \).

(2) \( V_n(\omega_1 \ldots \omega_n) = \max\{G_n(\omega_1 \ldots \omega_n), (1/(1+r))[pV_{n+1}(\omega_1 \ldots \omega_nH) + qV_{n+1}(\omega_1 \ldots \omega_nH)]\} \)

for \( n = N - 1, \ldots, 0 \).

The following theorem justifies Definition 1.5 for American derivative security prices, since it shows that the short position can be hedged using these prices.

**Theorem 1.8** (13, Theorem 4.4.4). Consider an \( N \)-period binomial asset-pricing model with \( 0 < d < 1 + r < u \) and with

\[
\tilde{p} = \frac{(1 + r - d)}{(u - d)}
\]

and

\[
\tilde{q} = \frac{(u - 1 - r)}{(u - d)}
\]

For each \( n, n = 0, 1, \ldots, N \), let \( G_n \) be a random variable depending on the first \( n \) coin tosses. With \( V_n, n = 0, 1, \ldots, N \), given by Definition 1.5, we define

\[
\Delta_n(\omega_1 \ldots \omega_n) = \frac{V_{n+1}(\omega_1 \ldots \omega_nH) - V_{n+1}(\omega_1 \ldots \omega_nT)}{S_{n+1}(\omega_1 \ldots \omega_nH) - S_{n+1}(\omega_1 \ldots \omega_nT)}
\]

\[
C_n(\omega_1 \ldots \omega_n) = V_n(\omega_1 \ldots \omega_n) - \frac{1}{1 + r}[\tilde{p}V_{n+1}(\omega_1 \ldots \omega_nH) + \tilde{q}V_{n+1}(\omega_1 \ldots \omega_nT)]
\]

where \( n \) ranges between 0 and \( N - 1 \). We have \( C_n \geq 0 \) for all \( n \). If we set \( X_0 = V_0 \) and define recursively forward in time the portfolio values \( X_1, \ldots, X_N \) by

\[
X_{n+1} = \Delta_nS_{n+1} + (1 + r)(X_n - C_n - \Delta_nS_n)
\]
then we have

\[ X_n(\omega_1\ldots\omega_n) = V_n(\omega_1\ldots\omega_n) \]

for all \( n \) and \( \omega_1\ldots\omega_n \). In particular, \( X_n \geq G_n \) for all \( n \).
In this chapter we will study Boyd’s result following the exposition in Schilling’s paper [12] about the exact distribution of the longest run for a fair coin, its variance and some of its properties.

Consider \( n \) independent tosses for a fair coin, and let \( R_n \) represent the length of the longest run of heads. To study the stochastic behavior of the longest run of heads, we can study the distribution function

\[
F_n(x) = P(R_n \leq x). \tag{2.1}
\]

Let \( A_n \) be the number of sequences of length \( n \) in which the longest run of heads does not exceed \( x \). Clearly, \( F_n(x) = 2^{-n} A_n(x) \). We can calculate \( A_n(x) \) with a simple recursive formula by partitioning the set of favorable outcomes according to the number of heads, if any, that occur before the first tail. Consider the two following cases:

1. \( n \leq 3 \).

Then, \( A_n(3) = 2^n \) since any outcome is a favorable one.

2. \( n > 3 \).

Each favorable sequence begins with either T, HT, HHT or HHHT and is followed by a string with no more than three consecutive heads. Thus,

\[
A_n(3) = A_{n-1}(3) + A_{n-2}(3) + A_{n-3}(3) + A_{n-4}(3), \quad \text{for } n > 3. \tag{2.2}
\]
\[
\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 A_n(3) & 1 & 2 & 4 & 8 & 15 & 29 & 56 & 108 \\
\end{array}
\]

\[
\begin{array}{cccc}
 N = 4 & 0 & 1 & 2 \\
P(R_N = k) & 1/16 & 7/16 & 5/16 & 2/16 & 1/16 \\
\end{array}
\]

Figure 2.1: Probability of the longest head run in 4 tosses of a fair coin

Using recursion, the value of \( A_n(3) \) can be computed:

In the general case we obtain

\[
A_n(x) = \begin{cases} 
\sum_{0 \leq j \leq x} A_{n-1-j}(x) & \text{for } n > x \\
2^n & \text{for } n \leq x 
\end{cases}
\]

Note that for \( n = 1, 2, 3, \ldots \) the number of \( A_n(1) \) of sequences of length \( n \) that contain no consecutive heads is the \((n+2)\)nd Fibonacci number.

Now we examine some numerical examples. If a fair coin is flipped four times:

The expected length of the longest head run is 1.687.

Next, let us check the case when we flip a fair coin for 5 times or 6 times and calculate the expected length of the longest head run.

The expected length of the longest head run is 1.936.
Figure 2.2: Probability of the longest head run in 4 tosses of a fair coin

\[
P(k) = R_N
\]

<table>
<thead>
<tr>
<th>(N = 5)</th>
<th>0</th>
<th>1/32</th>
<th>12/32</th>
<th>11/32</th>
<th>5/32</th>
<th>2/32</th>
<th>1/32</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(R_N = k))</td>
<td>1/32</td>
<td>12/32</td>
<td>11/32</td>
<td>5/32</td>
<td>2/32</td>
<td>1/32</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.3: Probability of the longest head run in 5 tosses of a fair coin

\[
P(k) = R_N
\]

<table>
<thead>
<tr>
<th>(N = 6)</th>
<th>0</th>
<th>1/64</th>
<th>20/64</th>
<th>23/64</th>
<th>12/64</th>
<th>5/64</th>
<th>2/64</th>
<th>1/64</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(R_N = k))</td>
<td>1/64</td>
<td>20/64</td>
<td>23/64</td>
<td>12/64</td>
<td>5/64</td>
<td>2/64</td>
<td>1/64</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.4: Probability of the longest head run in 5 tosses of a fair coin

\[
P(k) = R_N
\]

<table>
<thead>
<tr>
<th>(N = 6)</th>
<th>0</th>
<th>1/64</th>
<th>20/64</th>
<th>23/64</th>
<th>12/64</th>
<th>5/64</th>
<th>2/64</th>
<th>1/64</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(R_N = k))</td>
<td>1/64</td>
<td>20/64</td>
<td>23/64</td>
<td>12/64</td>
<td>5/64</td>
<td>2/64</td>
<td>1/64</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.5: Probability of the longest head run in 6 tosses of a fair coin
2.1 Properties of the Distribution of the Longest Run

2.1.1 Expectation and variance of the longest head run

In 1972 D. W. Boyd calculated $E(R_N)$ and $\text{Var}(R_N)$. Let $F(n, k)$ be the probability that in a sequence of $n$ Bernoulli trials, the longest run of losses has length at most $k$. He assumed throughout the success probability $p$ satisfies $0 \leq p \leq 1$. The distribution function $F(n, k)$ is the probability that there is no losing run of length greater than $k$ in a sequence of $n$ trails. He set up a recurrence relation to compute $F(n, k)$ and also the expected maximum run of losses in $n$ trials($M(n)$). Then, He shows that the variance of the distribution $F(n, k)$ is bounded as $n \to \infty$. By properties of Poisson distribution of long runs he is able to show $F(n, k) \to \Phi(t)$ if $npq^{k_n+1} \to pq^{t+1}$ as $n \to \infty$. The Poisson distribution of long runs explains that if the number of $k$ of trials and the length $r$ of runs both tends to infinity so that $kqp^r \to \lambda$, then
the probability of having exactly \( n \) runs of length \( r \) tends to \( e^{-\frac{\lambda}{n!}} \). In other words \( np_n = \lambda \) means that if \( X \) is binomial\((n, p_n)\) then the distribution of \( X \) converges to the Poisson\((\lambda)\) distribution as \( n \) goes to infinity. The methods that he uses in the proof are Rouche’s theorem, Poisson’s summation formula and generating functions. Please see [3] for details of the proof.

In our situation we can identify heads with success and tail with failure. The following asymptotic-based formula for the expectation and variance of the longest head run can be derived:

\[
ER_n = \log_1 / p(nq) + \gamma / \ln(1) - 1/2 + r_1(n) + \epsilon_1(n), \tag{2.4}
\]

\[
Var R_n = \pi^2 / 6 \ln^2(1/p) + 1/12 + r_2(n) + \epsilon_2(n), \tag{2.5}
\]

where \( \gamma = 0.577 \) is Euler’s constant, \( r_1(n) \) and \( r_2(n) \) are very small periodic functions of \( \log_1 / p(n) \), and \( \epsilon_1(n) \) and \( \epsilon_2(n) \) tend to zero as \( n \rightarrow \infty \) (See [6]). These result were first obtained through the use of generating functions by Boyd [3] for the case \( p = 1/2 \).

Note that the leading term of \( E(R_n) \) is consistent with the heuristic argument given above for \( R_n \). For \( p = 1/2 \) we get the simple approximation

\[
ER_n = \log_2(n/2) + \gamma / \ln 2 - 1/2 = \log_2 n + O(1). \tag{2.6}
\]
In 1975 Ross [11] studied optimal stopping for success runs and a way to recursively compute the price of this option.

**Theorem 3.1. (Chebyshev’s Inequality).** Let $X$ be a random variable. For all $t > 0$

$$P(|X - E(X)| \geq t) \leq \frac{Var(X)}{t^2}.$$ 

**Definition 3.2.** The Run option is the option where the payoff is the current run of heads in a coin tossing sequence.

**Theorem 3.3.** Let $V^A$ be the price of American option. For each $n = 0, 1, \ldots, N,$ let $G_n(\omega) = \max\{r : \omega_{n-r+1} \ldots \omega_n, \text{all heads}\}$

and let $\tau_t$ be defined by

$$\tau_t(\omega_1 \ldots \omega_N) = \min\{s : G_s = [E(R_N) - t]\}$$

Then

(1) If $t_N = t$ maximizes $f(t) = E(G_{\tau_t})$, then

$$t_N \in \Theta(\sqrt[3]{\ln N})$$

(2) There is a sequence of $\varepsilon_N$, $\lim_{N \to +\infty} \varepsilon_N = 0$ and there exists $c_2$ and $c$,

$$E(G_{\tau_{t_N}}) \geq (\log_2 N - c_2 - c\sqrt[3]{\ln N})(1 - \varepsilon_N).$$
Proof. (1) We define the price process $V_n^A$ for this contract by the American risk-neutral formula

$$V_n^A = \max_{\tau \in S_n} \mathbb{E}_n[\mathbb{I}_{\tau \leq N} G_{\tau}], \quad \text{for } n = 0, 1, \ldots, N.$$ 

So

$$V_n^A \geq \mathbb{E}_n[\mathbb{I}_{t_N \leq N} G_{\tau}] .$$

Now we maximize $\mathbb{E}(G_{\tau_t})$. Now we maximize $\mathbb{E}(G_{\tau_t})$.

$$\mathbb{E}(G_{\tau_t}) = \mathbb{E}(G_{\tau_t}|G_{\tau_t} > 0) \mathbb{P}(G_{\tau_t} > 0)$$

$$= ([E(R_N)] - t)(P\{R_N \geq [E(R_N)] - t\})$$

$$\geq ([E(R_N)] - t)(P\{R_N \geq E(R_N) - t + 1\})$$

$$\geq ([E(R_N)] - t)(P\{|R_N - \mathbb{E}(R_N)| \leq (t - 1)\})$$

By Chebyshev’s inequality this is

$$\geq ([E(R_N)] - t) \left( 1 - \frac{Var(R_N)}{(t - 1)^2} \right)$$
\[ \geq ((E(R_N) - 1 - t) \left(1 - \frac{Var(R_N)}{(t - 1)^2}\right) \]

By (2.5)

\[ Var(R_n) = \frac{\pi^2}{6} \ln^2(1/p) + 1/12 + r_2(n) + \varepsilon_2(n), \]

\(r_2(n)\) is a very small periodic function of \(\log_{1/p} n\), and \(\varepsilon_2(n)\) tend to zero as \(n \to \infty\). For the case \(p = 1/2\) we get the simple approximation

\[ VarR_N = \frac{\pi^2}{6} \ln^2(2) + 1/12 + r_2(N) + \varepsilon_2(N) = 3.424 \leq 4. \quad (3.1) \]

Also from (2.6)

\[ \mathbb{E}(R_n) = \log_{1/p}(nq) + \gamma/\ln(1/p) - 1/2 + r_1(n) + \varepsilon_1(n), \]

\(r_1(n)\) is a very small periodic function of \(\log_{1/p} n\), and \(\varepsilon_1(n)\) tend to zero as \(n \to \infty\). For the case \(p = 1/2\) we get the simple approximation

\[ \mathbb{E}(R_N) = \log_2(N/2) + \gamma/\ln 2 - 1/2 = \log_2 N + O(1). \quad (3.2) \]

Now we find the \(t_N\) such that \(\mathbb{E}(G_{t_N})\) is maximized. Let \(\mathbb{E}(R_N) = a\) and \(t_N = t\) then we get \((a - t - 1)(1 - 4/(t - 1)^2)\). This third degree polynomial has negative discriminant. Therefore it has one real root that can be calculated by Mathematica

\[ t = \left(\frac{3}{2}\right)^{2/3} \sqrt[3]{9 \ln^2(2) \ln(N) + \sqrt{3} \sqrt{27 \ln^4(2) \ln^2(N) + 4 \ln^6(2)}} \]

\[ \ln(2) \]

14
\[
\frac{2^{\frac{2}{3}} \ln(2)}{\sqrt[3]{9 \ln^2(2) \ln(N)} + \sqrt{3} \sqrt[3]{27 \ln^4(2) \ln^2(N)} + 4 \ln^6(2)}
\]

By the second derivative test, \(d^2/dt^2[(a - t)(1 - 4/t^2)] = -3t^2 - 4 \leq 0\) we can see that \(t\) maximizes this polynomial.

As \(n \to \infty\),

\[
\frac{2^{\frac{2}{3}} \ln(2)}{\sqrt[3]{9 \ln^2(2) \ln(n)} + \sqrt{3} \sqrt[3]{27 \ln^4(2) \ln^2(N)} + 4 \ln^6(2)}
\]

goes to zero. Hence,

\[
\frac{t}{(4/\ln 2)^{1/3}(\sqrt[3]{\ln N})}
\]

goes to 1. Therefore, \(t_N \in \Theta(\sqrt[3]{\ln N})\).

(2) In the previous part we showed that

\[
\mathbb{E}(G_{t_N}) \geq ((E(R_N) - 1 - t) \left(1 - \frac{\text{Var}(R_N)}{(t - 1)^2}\right))
\]

For sufficiently large \(N\),

\[
\text{Var}R_n = \pi^2/6 \ln^2(2) + 1/12 + r_2(n) + \varepsilon_2(n) \leq 4.
\]

Then,

We also proved that for sufficiently large \(N\), \(t = t_N \in \Theta(\sqrt[3]{\ln N})\).

Therefore

\[
\mathbb{E}(G_{\tau_N}) \geq (\log_2 N - c_2 - c\sqrt[3]{\ln N})(1 - \varepsilon_N).
\]
Corollary 3.4. Let $V^A$ be the price of American option. Then $V^A \sim \log_2 N$.

Proof. In the previous theorem we proved that $V^A$ is bounded below by the payoff strategy that waits for $[E(R_N)] - t_N$ heads and then exercises. On the other hand, $V^A$ is bounded above $\mathbb{E}(R_N)$. Therefore

$$\mathbb{E}(R_N) - t_N \leq V^A \leq \mathbb{E}(R_N).$$

Therefore,

$$\log_2 N - c_2 - \frac{3\ln N}{\sqrt{N}} \leq V^A \leq \log_2 N + O(1)$$

Dividing by $\log_2 N$ we get

$$1 - o(1) \leq \frac{V^A}{\log_2 N} \leq 1 + o(1)$$

Therefore,

$$V^A \sim \log_2 N.$$
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URL = ”http://www.math.ubc.ca/ boyd/bern.runs/bernoulli.html”.


