A LOOK AT SOME FAMILIES OF GEOMETRIES AND THEIR LATTICES OF FLATS

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ABSTRACT. We look at a family of geometries of a particularly simple type. We then consider the lattices of flats of these geometries. Since the geometries satisfy the exchange property, their flat lattices will be geometric. We consider the question of whether the Whitney numbers of these lattices are logarithmically concave.

1. CLOSURE OPERATORS, THE EXCHANGE PROPERTY AND EXAMPLES

In this paper we discuss the unimodality of the Whitney numbers of several types of geometric lattices. In the process, we construct a family of geometries whose lattice of flats are tractable, and show that some of these are logarithmically concave, hence unimodal.

Let us first give some definitions needed for the context of this paper. A closure operator $\sigma$ on a given set $X$ is an increasing, idempotent function sending $\mathcal{P}(X)$ to $\mathcal{P}(X)$, such that for all $S \subseteq X$, $S \subseteq \sigma(S)$. The sets in the range of this function are referred to as closed sets. Any set $S \subseteq X$ that has the property that $\sigma(S) \neq \sigma(S \setminus \{s\})$ for all $s \in S$ is called an independent set. For all closed sets $Y$, any set $A \subseteq Y$ such that $\sigma(A) = Y$ is referred to as a generating set of $Y$. A basis of $Y$ is any independent set that generates $Y$.

For our purposes, we will be interested in algebraic closure operators that have the exchange property. An algebraic closure operator $\sigma$ on a set $X$ is a closure operator such that for all $B \subseteq X$, $\sigma(B) = \bigcup\{\sigma(F) | F$ is a finite subset of $B\}$. The pair $X, \sigma$ are said to satisfy the exchange property if for all $Y \subseteq X$, and $a, x \in X$, if $a \in \sigma(Y \cup \{x\})$ and $a \notin \sigma(Y)$, then $x \in \sigma(Y \cup \{a\})$. Equivalently $X, \sigma$ satisfy the exchange property if for all $Y \subseteq X$ closed, and $a, x \in X$, if $a \in \sigma(Y \cup \{x\})$ and $a \notin Y$, then $x \in \sigma(Y \cup \{a\})$. The exchange property was introduced by Mac Lane[4]. The importance of the exchange property is shown in the following lemmas.

**Lemma 1.1.** Let $X, \sigma$ satisfy the exchange property. If $A$ is an independent set and $x \notin \sigma(A)$, then $A \cup \{x\}$ is an independent set.

**Proof.** We are given that $x \notin \sigma(A)$. Suppose that $a \in \sigma(A)$ such that $a \in \sigma((A \setminus \{a\}) \cup \{x\})$, then by the exchange property $x \in \sigma((A \setminus \{a\}) \cup \{a\}) = \sigma(A)$, which is a contradiction. Thus $A \cup \{x\}$ is an independent set.

**Lemma 1.2.** Let $\sigma$ be an algebraic closure operator such that $X, \sigma$ satisfy the exchange property. If $A$ is a finite independent set, and $B$ is a generating set of $X$, then $|A| \leq |B|$.

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Proof. Let $A = \{a_1, \ldots, a_n\}$, and $B = B_0$ (wlog $n > 0$). If $a_1 \in B_0$ then let $B_1 = B_0$. If $a_1 \not\in B_0$ then there exists a finite set $C \subseteq B_0$ such that $a_1 \in \sigma(C)$, since $B_0$ generates $X$, and $\sigma$ is algebraic. We may assume that $C$ is chosen to be minimal with respect to set containment. Since $a_1 \not\in C$, $a_1 \notin \sigma(C)$, and $a_1 \notin \sigma(A \setminus \{a_1\})$ there exists $c_1 \in C \setminus A$. Let $D = C \setminus \{c_1\}$. Then $a_1 \notin \sigma(D)$ and $a_1 \notin \sigma(D \cup \{c_1\})$. Whence $c_1 \in \sigma(D \cup \{a_1\}) \subseteq \sigma((B_0 \setminus \{c_1\}) \cup \{a_1\})$. Let $B_1 = ((B_0 \setminus \{c_1\}) \cup \{a_1\})$ and note that $B = ((B_0 \setminus \{c_1\}) \cup \{c_1\}) \subseteq \sigma(B_1)$. In either case we have $A$ is an independent set, and $a_1 \in B_1$ with $|B_1| = |B|$, and $B_1$ generates $X$. Continuing this process recursively (starting with $A, B_1$) we obtain $B_k$ such that $\{a_1, \ldots, a_k\} \subseteq B_k$, $|B_k| = |B|$, and $B_k$ generates $X$. It follows that $A \subseteq B_n$ where $|A| \leq |B_n| = |B|$. \hfill \square

**Theorem 1.3.** Let $\sigma$ be an algebraic closure operator such that $X, \sigma$ satisfy the exchange property. If $X$ has a finite basis, then every basis has the same cardinality.

The above theorem is directly proved from the results of the previous two lemmas.

For a given set $X$ and algebraic closure operator that satisfy the exchange property, with a finite basis, the size of a basis is invariant. Thus we can define the *dimension* of $X$ to be the cardinality of any of its bases.

### 1.1. Examples.

**Example 1.4.** Consider the boolean algebra of all subsets of $X$ with the closure operatate $\sigma(S) = S$, for all subsets $S$ of $X$. Thus every set is an independent set, and every set is its own generating set, and hence basis. The dimension of a set is simply its cardinality. Here the exchange property works trivially. That is, if $a \in S \cup \{x\}$ and $a \notin S$, then $a = x$, and hence $x \in S \cup \{a\}$.

**Example 1.5.** For a vector space we use $Span(Y)$ as the closure operator, which takes subsets of the vector space to their linear span, and the closed sets are the subspaces. An independent set is one whose span changes when any vector is removed from the set, i.e., linearly independent sets. A generating set is any set of vectors whose span is the whole vector space. A basis for the vector space is any set of linearly independent vectors that generates the whole space. Vector spaces also have the exchange property.

Let $V$ be a vector space over a field $F$ and $X$ some subspace of $V$. Suppose $\vec{u}, \vec{v} \in V$ such that $\vec{u} \in Span(X \cup \vec{v})$ and $\vec{u} \notin X$ Then $\vec{u} = a\vec{v} + b\vec{w}$ for some $a, b \in F$ with $a \neq 0$. Thus $\vec{v} = \frac{1}{a}(\vec{u} - b\vec{w})$. Hence $\vec{v} \in Span(X \cup \{\vec{u}\})$, proving the exchange property holds.

**Example 1.6.** An affine space is similar to a vector space, but with an extra requirement on its closed sets. That is in a vector space we have,

$$Span(W) = \{a_1\vec{w}_1 + \cdots + a_k\vec{w}_k | k \in \omega, a_1, \ldots, a_k \in F, \vec{w}_1, \ldots, \vec{w}_k \in W\},$$

whereas in an affine space we require,

$$\sigma(W) = \{a_1\vec{w}_1 + \cdots + a_k\vec{w}_k | k \in \omega, a_1, \ldots, a_k \in F, \vec{w}_1, \ldots, \vec{w}_k \in W, a_1 + \cdots + a_k = 1\}.$$ 

Thus affine spaces are extremely similar to vector spaces. An independent set is one whose span changes when any vector is removed from the set, i.e., linearly independent sets. A generating set is any set of vectors whose span is the whole
affine space. A basis for the affine space is any set of linearly independent vectors that generates the whole space. Affine spaces also have the exchange property.

Let $V$ be an affine space over a field $F$ and $X$ some subspace of $V$. Suppose $\vec{u}, \vec{v} \in V$ such that $\vec{u} \in \sigma(X \cup \vec{v})$ and $\vec{u} \notin X$. Then $\vec{u} = a\vec{v} + b\vec{w}$ for some $a, b \in F$ with $a \neq 0$, and $a + b = 1$. Thus $\vec{v} = \frac{1}{a} \vec{u} - \frac{b}{a} \vec{w}$, and $\frac{1-b}{a} = 1$, since $a = 1 - b$. Hence $\vec{v} \in \sigma(X \cup \{\vec{u}\})$, proving the exchange property holds.

**Example 1.7.** We use the classic [7] definition of a projective geometry $\mathcal{A}_P = \langle P, L \rangle$, where $P$ is a set of points to define the closure operator such that $L$ is a collection of subsets (lines) of $P$ satisfying the following:

1. For any two distinct points in $P$, there exist a unique line, $l \in L$ containing those points.
2. If $l_1, l_2, l_3$ form a triangle and the line $l_4$ that crosses both $l_1$ and $l_2$ (not at their point of intersection), then there exists a point where $l_4$ also intersects $l_3$.
3. For every line $l \in L$ there are at least three distinct points $p_1, p_2, p_3 \in P$, with $p_1, p_2, p_3 \in l$.

A set $F \subseteq P$ is a flat of the geometry if whenever $p_1, p_2$ are distinct points in $F$, then the line $l(p_1, p_2) \subseteq F$. Note the intersection of a collection of flats is again a flat. Hence the closed sets of the geometry are its flats. Let $\phi(Y)$ denote the flat generated by the set $Y$.

Let us show that projective geometries satisfy the exchange property. Let $S$ be a flat, and let $x$ be some point not in $S$. Let $T = \{a \in P|a \in l(x, s), \text{ for some } s \in S\}$.

We claim that $T$ is closed and $T = \phi(S \cup \{x\})$. Thus we need to show that for any distinct $a, b \in T$, $l(a, b) \subseteq T$.

Let $a, b \in P$ such that for some $s_1, s_2 \in S$, $a \in l(x, s_1)$, $b \in l(x, s_2)$. Thus we have $\triangle x s_1 s_2$ and line $l(a, b)$ that intersects two of its sides. It follows by (2) that there is some point $s_3 \in l(s_1, s_2)$ such that $s_3 \in l(a, b)$. We note that since $S$ is a flat, $s_3 \in S$. Now suppose that $c$ is some point on the line $l(a, b)$. With out loss of generality we may assume $c$ is a new point. Next consider the line $l(x, c)$, and triangle $\triangle a s_1 s_3$. We note that $l(x, c)$ crosses the lines $l(a, s_1)$, and $l(a, s_3)$. Thus
again by (2) there exist a point \( s_4 \in l(s_1, s_3) \) such that \( s_4 \in l(x, c) \). Again we note that since \( S \) is a flat, \( s_4 \in S \). Hence \( c \in T \), and \( T \) is closed. Thus projective geometries satisfy the exchange property. This argument is illustrated in Figure 1.

**Example 1.8.** Let \( A_G \) be the graph algebra of the graph \( G = \langle V, E \rangle \), where the closure operator \( \gamma \) of \( A_G \) acts on the sets of edges of \( G \). A set \( Y \subseteq E \) is closed if and only if whenever \( Y \) contains all the edges of a circuit except one, then it contains the entire circuit. These sets are again called flats. The independent sets are exactly the forests of \( G \), and a generating set of \( Y \) is any minimal forest that spans \( Y \). If \( S \) is a flat of \( A_G \), and \( x, y \in E \) are such that \( x \not\in S \) and \( x \in \sigma(S \cup \{y\}) \), then there exist edges \( e_1, \ldots, e_n \in S \) such that \( e_1, \ldots, e_n, y, x \) make a cycle, not necessarily in that order. Thus \( y \in \sigma(S \cup \{x\}) \). Hence graph algebras satisfy the exchange property.

2. Skeleton Algebras

Now we want to consider geometries \( G \) where for lines with three or more points there are no cycles.

**Definition 2.1.** A skeleton, \( \Gamma \), on a set \( X \) is a collection of subsets, \( \gamma_i \ (i \in I) \), called bones, such that:

1. \( \bigcup_{i \in I} \gamma_i = X \);
2. for all \( \gamma \in \Gamma \), \( |\gamma| \geq 3 \);
3. for \( i \neq j \), \( |\gamma_i \cap \gamma_j| \leq 1 \);
4. if \( i_1, \ldots, i_k \) are distinct, and \( k \geq 2 \), then \( \gamma_{i_1} \cap \gamma_{i_2} = \{a_1\}, \gamma_{i_2} \cap \gamma_{i_3} = \{a_2\}, \ldots, \gamma_{i_k} \cap \gamma_{i_1} = \{a_k\} \) if and only if \( a_1 = a_2 = \cdots = a_k \).

If \( \{x\} = \gamma_i \cap \gamma_j \) for any \( i, j \leq n \), we call \( x \) a joint.

We want to define a geometry \( G \) on \( X \) such that the bones in \( \Gamma \) are exactly the lines of \( G \) with at least three points. Thus we define a subset \( S \) of \( X \) to be a flat (closed) as follows.

**Definition 2.2.** Let \( \Gamma \) be a skeleton on \( X \). For all \( S \subseteq X \), \( S \) is a flat if and only if for all \( \gamma \in \Gamma \), \( |S \cap \gamma| \geq 2 \) implies \( \gamma \subseteq S \).

Note that the intersection of any collection of flats is again a flat. Thus the flats form a closure system which is the set of closed subsets of an algebraic closure operator \( \sigma \) on \( X \). As usual a closed set can also be constructed recursively. For all \( Y \subseteq X \), we define:

\[
S_0 = Y \\
S_{k+1} = S_k \cup \bigcup \{\gamma_i : |S_k \cap \gamma_i| \geq 2\} \\
\sigma(Y) = \bigcup_{k \in \omega} S_k
\]

If \( X \) is finite there is some minimum \( r \in \mathbb{Z}^+ \) such that \( S_r = S_j \) for all \( j \geq r \). Hence \( \sigma(Y) = S_r \).

It follows that an independent set of a skeleton is any set of points where for every bone, there are no more than two points from the bone in the set. For any two points \( a, x \in X \), if there exists bones \( \gamma_1, \ldots, \gamma_k \in \Gamma \) with \( a \in \gamma_1 \), \( x \in \gamma_k \) and for each \( 1 < i \leq k \), \( |\gamma_{i-1} \cap \gamma_i| = 1 \) then we say there is a bone-path from \( a \) to \( x \) of length \( k \). This idea is illustrated in Figure 3.
Lemma 2.3. Let $\Gamma$ be a skeleton on a set $X$, and $\sigma$ be the above closure operator. Let $Y$ be a flat with $a \notin Y$. Then $a \in \sigma(Y \cup \{x\})$ if and only if there exists a bone-path with points $x = a_0, a_1, \ldots, a_k = a \in X$, $y_1, \ldots, y_k \in Y$, and bones $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that $\{a_{i-1}, a_i, y_i\} \subseteq \gamma_i$. Furthermore the bone-path from $x$ to $a$ is unique.

Proof. Suppose that there exists such a bone-path from $a$ to $x$. Then $a_0, y_1 \in \sigma(Y \cup \{x\})$, and $a_0, y_1 \in \gamma_1$. By Definition 2.2, $\gamma_1 \subseteq \sigma(Y \cup \{x\})$, whence $a_1 \in \sigma(Y \cup \{x\})$. Continuing this process recursively (starting with $a_1, y_2, \gamma_2$) we obtain $a_{k-1} \in \sigma(Y \cup \{x\})$. Thus $a_{k-1}, y_k \in \sigma(Y \cup \{x\})$, and $a_{k-1}, y_k \in \gamma_k$. Again by Definition 2.2, $\gamma_k \subseteq \sigma(Y \cup \{x\})$, whence $a_k = a \in \sigma(Y \cup \{x\})$.

Now let $T$ be the union of $Y$ and the set of all $t \in X$ such that there is a bone-path from $x$ to $t$ containing points $y_i \in Y$, as in the statement of the lemma. We claim that $T$ is a flat with $T = \sigma(Y \cup \{x\})$ and each bone-path from $x$ to a fixed point $t$ is unique. It suffices to show that for all $\gamma \in \Gamma$ if $|\gamma \cap T| \geq 2$, then $\gamma \subseteq T$. Suppose $\gamma \in \Gamma$ and $\{a, b\} \subseteq T \cap \gamma$, with $a \neq b$. If $a, b \in Y$ then $\gamma \subseteq Y$ since $Y$ is a flat. Assume $a, b \notin Y$, and let $x = a_0 = b_0, a = a_k, b = b_n$, and $a_1, \ldots, a_{k-1}, b_1, \ldots, b_{n-1} \subseteq X$. Let $y_1, \ldots, y_k, z_1, \ldots, z_n \in Y$. Let $\gamma_1, \ldots, \gamma_k, \xi_1, \ldots, \xi_n \in \Gamma$ be such that $\{a_{i-1}, a_i, y_i\} \subseteq \gamma_i$ and $\{b_{n-1}, b_m, z_m\} \subseteq \xi_m$, for all $1 \leq i \leq k$, $1 \leq m \leq n$. Note that if $a_i \in Y$ for any $i \leq k$ then $a_{k-1} \in Y$, whence by induction $a_0 = a \in Y$, which is a contradiction. Thus $a_i \notin Y$ for all $i \leq k$. Without loss of generality we may assume that $k$ and $n$ are chosen to be minimal with respect to bone-path length. It follows that $\gamma \cap \gamma_k = \{a\}$, $\gamma_k \cap \gamma_{k-1} = \{a_{k-1}\}, \ldots, \gamma_2 \cap \gamma_1 = \{a_1\}$, $\gamma_1 \cap \xi_1 = \{x\}$, $\xi_1 \cap \xi_2 = \{a_1\}, \ldots, \xi_{n-1} \cap \xi_n = \{b_n\}$, and $\xi_n \cap \gamma = \{b\}$. By Definition 2.1 (4) we have $a = a_{k-1} = \cdots = a_3 = x = b_1 = b_{k-1} = b$. This contradicts $a \neq b$. It must then be the case that $n + k < 2$, whence $a, b, x \in \gamma \subseteq Y$. This argument is illustrated by Figure 3. 

\[\square\]
Lemma 2.3 shows that skeletons satisfy the exchange property. The exchange property for skeletons then implies that either \( x, a \) share a bone, or there exists a unique bone path from \( x \) to \( a \) as described in the Lemma.

3. Background

We begin with some definitions from Grätzer [3].

**Definition 3.1.** A geometric lattice \( L \) is a lattice that satisfies the following:

1. \( L \) is semimodular.
2. \( L \) is algebraic.
3. The compact elements of \( L \) are exactly the finite joins of the atoms of \( L \).

Equivalently a lattice is geometric if it satisfies the following:

1. \( L \) is semimodular.
2. \( L \) is complete.
3. \( L \) is atomistic.
4. All atoms are compact.

We now pick apart the meanings behind this definition. Both definitions require that \( L \) be semimodular. A lattice is semimodular if \( b \succ a \wedge b \) implies that \( a \vee b \geq a \).

**Definition 3.2.** A lattice \( L \) is complete if for any subset \( H \) of \( L \), \( \bigvee H \) and \( \bigwedge H \) exist.

**Definition 3.3.** An element \( x \) of \( L \) is compact if for any subset \( H \) of \( L \), \( x \leq \bigvee H \), then \( x \leq \bigvee H' \), for some finite \( H' \subseteq H \).

**Definition 3.4.** A lattice is algebraic if it is complete, and every element is the join of a set, possibly infinite, of compact elements.

**Definition 3.5.** A lattice is atomistic if every element is a join of atoms.

**Lemma 3.6.** Let \( \Gamma \) be an algebraic closure operator on a set \( X \). Then \( C_\Gamma \), the closed sets induced by \( \Gamma \), is an algebraic lattice whose compact elements are \( \{ \Gamma(F) : F \text{ is a finite subset of } X \} \).

For proof of this lemma, see Nation [5].

We now look at an important theorem of Mac Lane [4] which will give importance to our previous examples.

**Theorem 3.7.** A lattice \( L \) is geometric if and only if \( L \) is isomorphic to the lattice of closed sets of an algebraic closure operator with the exchange property.

**Proof.** Following the proof of Nation [5], we let \( L \) be a geometric lattice. We then define a closure operator \( \Gamma \) on \( A \), the set of all atoms of \( L \), via:

\[
\Gamma(X) = \{ a \in A | a \leq \bigvee X \}.
\]

Since \( L \) is complete \( \Gamma \) is well defined, and it is clear that \( X \subseteq \Gamma(X) \), and that for all \( X \subseteq Y \subseteq A \), we have \( \Gamma(X) \subseteq \Gamma(Y) \). Let us then consider \( \Gamma(\Gamma(X)) \). Let \( a \in \Gamma(\Gamma(X)) \), then \( a \leq \bigvee \Gamma(X) \). Since \( a \) is an atom of a geometric lattice, \( a \) is compact, and so there exist \( x_1, \ldots, x_n \in \Gamma(X) \) such that \( a \leq x_1 \vee \cdots \vee x_n \). For every \( 1 \leq i \leq n \), \( x_i \leq \bigvee X \), hence \( a \leq x_1 \vee \cdots \vee x_n \leq \bigvee X \). Thus \( a \in \Gamma(X) \).
Therefore $\Gamma$ is an algebraic closure operator. Next let $a, b, t \in \mathcal{L}$, with $a, b$ atoms. If $a \leq b \lor t$ and $a \not\leq t$, then $a \lor t \leq b \lor t$, $t \not< a \lor t$ and $t < b \lor t$, whence $a \lor t = b \lor t$. This proves that $\Gamma$ satisfies the exchange property. Again let $\mathcal{C}_\Gamma$ denote the closed sets of $\Gamma$. Since $\mathcal{L}$ is atomistic, the function $\phi: \mathcal{L} \to \mathcal{C}_\Gamma$, given by $\phi(x) = \{a \in A | a \leq x\}$ is an isomorphism.

Now assume that $\Gamma$ is an algebraic closure operator satisfying the exchange property. We claim that $\mathcal{C}_\Gamma$ is a geometric lattice. By Lemma 3.6, $\mathcal{C}_\Gamma$ is an algebraic lattice. Since $\Gamma$ satisfies the exchange property for any singleton set, $\Gamma(\{x\})$, the closure is either the least element $\Gamma(\emptyset)$, or an atom of $\mathcal{C}_\Gamma$. Furthermore if $x \in \Gamma(y)$ then $y \in \Gamma(x)$, thus $\Gamma(x) = \Gamma(y)$. Hence for any closed set $B$ we have $B = \bigvee_{b \in B} \Gamma(b)$. Let $B$ and $C$ be closed sets with $B \supseteq B \cap C$. Then $B = \Gamma(\{x\} \cup (B \cap C))$ for any $x \in B \setminus (B \cap C)$. Hence $B \lor C = \Gamma(\{x\} \cup (B \cap C)) = \Gamma(\{x\} \cup (B \cap C)) = \Gamma(\{x\} \cup C)$. Suppose $C \not\leq D \leq (B \lor C) = \Gamma(B \lor C)$ and let $y \in (D \setminus C)$. Fix $x \in B \setminus (B \lor C)$. Then $y \in \Gamma(C \cup \{x\})$ and $y \not\in C$. Thus by the exchange property $x \in \Gamma(C \cup \{y\})$, and $B \leq \Gamma(C \cup \{y\}) \leq D$. Hence $C \not< C \lor B$, and $\mathcal{C}_\Gamma$ is semimodular.$\square$

**Definition 3.8.** Given a finite semimodular lattice $\mathcal{L}$ we let $W_j$ equal the number of elements of height $j$ of $\mathcal{L}$, and we say that $W_j$ is the $j^{th}$ Whitney number of the second kind.

G. C. Rota [1] conjectured that for a finite geometric lattice, Whitney numbers of the second kind are unimodal, that is, $W_j \geq \min\{W_i, W_k\}$ whenever $i \leq j \leq k$. He later conjectured that the Whitney numbers of the second kind of a finite geometric lattice are logarithmically concave: $W_j^2 \geq W_{j+1}W_{j-1}$ for all $j$. If this second conjecture is true, then so is the first due to this next lemma.

**Lemma 3.9.** Logarithmic concavity implies unimodality.

**Proof.** We follow the proof of Stonesifer [6]. Suppose $\mathcal{L}$ is not unimodal. Then there exists $i < j < k$ such that $W_j < \min\{W_i, W_k\}$. Let $W = \min\{W_m : i < m < k\}$, and let $n = \max\{m : W_m = W, i < m < k\}$. Then $W_{n+1} > W_n \leq W_{n-1}$, and $\frac{W_j^2}{W_{n+1}W_{n-1}} < 1$. Hence $\mathcal{L}$ is not logarithmically concave.$\square$

One last conjecture states that for a geometric lattice of rank $n$, the Whitney numbers satisfy $W_j \leq W_{n-j}$, whenever $j \leq n/2$. The idea behind this conjecture is supported by the findings of Dowling [8] which show that the top of the lattice is “heavier” than the bottom. Furthermore, Dukes [2] shows that if log-concavity fails at some level $k$, we can truncate the lattice and take the free extension and the lattice will still fail. These two results show that our focus should be in the upper half of our lattices.

4. **Examples of log concavity**

**Example 4.1.** Since the finite boolean algebra $B_n$ is isomorphic to the lattice of subsets of an $n$ element set, we have the $k^{th}$ Whitney number for all $k, W_k = \binom{n}{k}$.

Thus we must check that $\frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} \geq 1$. Simple calculation gives us

$$\frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} = \frac{kn - k^2 + n + 1}{kn + k^2} \geq 1.$$ 

Thus finite boolean algebras are log-concave.
Example 4.2. The $j^{th}$ Whitney number of a vector space of dimension $n$ over a field of order $q$ has the following formula

$$W_j = \frac{\prod_{i=0}^{j-1}(q^n - q^i)}{\prod_{i=0}^{j-1}(q^j - q^i)}.$$ 

It has been shown these lattices are log-concave.

Example 4.3. The $j^{th}$ Whitney number for an affine geometry of dimension $n$ over a field of order $q$ has the following formula

$$W_j = \frac{q^n \prod_{i=0}^{j-2}(q^n - q^i)}{q^{j-1} \prod_{i=0}^{j-2}(q^{j-1} - q^i)}.$$ 

It can be shown that these lattices satisfy log-concavity. Furthermore, this formula also gives the $j^{th}$ Whitney number for an affine geometry of order $q$ and geometric dimension $n - 1$.

5. **The Flat Lattices of Skeletons and Their Rank Functions**

Let $\sigma$ be the algebraic closure operator on a set $X$ with skeleton $\Gamma$. Let $L$ be the set of flats of $\sigma$. The set of flats ordered by set inclusion forms the flat lattice $\mathcal{L}$ of $\Gamma$.

**Definition 5.1.** Let $\rho : \mathcal{L} \to \omega$ be given by, for all $Q \in \mathcal{L}$,

$$\rho(Q) = \min\{|A| : \sigma(A) = Q\}.$$ 

**Theorem 5.2.** $\rho$ is the rank function for $\mathcal{L}$.

**Proof.** Let $P, P_\ast \in \mathcal{L}$ such that $P \succ P_\ast$ as lattice elements, and therefore $P_\ast \subseteq P \subseteq X$, as sets. Let $A \subseteq P_\ast$ such that $\sigma(A) = P_\ast$ and $A$ is minimal. For all $a \in A$, $\sigma(A \setminus \{a\}) \subseteq \sigma(A) = P_\ast$. Hence $A$ is an independent set. Let $x \in P \setminus P_\ast$. Consider the set $A \cup \{x\}$. By Lemma 1.1, $A \cup \{x\}$ is also an independent set. Thus we have $P_\ast = \sigma(A) \subseteq \sigma(A \cup \{x\})$. Furthermore $\sigma(A \cup \{x\}) \subseteq P$ since $A \subseteq P$ and $x \in P$. However $\sigma(A \cup \{x\}) \subseteq \mathcal{L}$ and $P \succ P_\ast$. Therefore $\sigma(A \cup \{x\}) = P$. Hence $A \cup \{x\}$ is a basis of $P$. By Theorem 1.3, $\rho(P) = |A \cup \{x\}| = \rho(P_\ast) + 1$.

### 5.1. Recursive Relations for the Whitney Numbers of Zig-zags and Stars.

#### 5.1.1. Zig-zags.

We will be looking at the Whitney numbers of two specific families of skeletons, the first of which we refer to as *zig-zags* and can be described in the following manner.

Let $X_n = \{x_1, x_2, \ldots, x_{\sum_{i=1}^{n-1}(k_i)+1}\}$ (where $k > 2, n \geq 1$), and let $\Gamma_n = \{\gamma_1, \ldots, \gamma_n\}$ be the skeleton on $X_n$ such that:

$$\gamma_j = \{x_{\sum_{i=1}^{j-1}(k_i)+1}, x_{\sum_{i=1}^{j-1}(k_i)+2}, \ldots, x_{\sum_{i=1}^{j-1}(k_i)+2^j}, \ldots, x_{\sum_{i=1}^{j-1}(k_i)+n}\},$$

so that $|\gamma_j| = k_j + 1$, for all $j \leq n$.

We note that in this family the sequence $\{k_1, \ldots, k_n\}$ is a sequence of positive integers greater than or equal to two, but that is the only requirement we put on any $k_i$. Furthermore, $|\gamma_n| = n$, with $X_n = X_{n-1} \cup \gamma_n$, for all $n$. Let $\sigma_n$ be the associated closure operator. Whence for each $Y \subseteq X_n$, $\sigma_n(Y) = \sigma_{n+1}(Y)$. We also note that for each $1 \leq m \leq n$, $\gamma_m \cap \gamma_{m+1} = \{x_{\sum_{i=1}^{m}(k_i)+1}\}$, these are the only joints...
of the skeleton. We set \( x_{\sum_{i=1}^{m}(k_i)+1} = e_m \) for ease of reference later. Figure 4 shows the first five general zig-zags (including the trivial skeleton). Figure 5 shows the \( n+1 \)st general zig-zag.

Let \( L_n \) be the flats of \( \sigma_n \), and \( \mathcal{L}_n = \langle L_n, \subseteq \rangle \) the lattice of flats, and \( W_j^n \) be the \( j \)th Whitney number of \( \mathcal{L}_n \).

**Theorem 5.3.** The Whitney numbers for the family of zig-zags can be generated by the recursive relation:

\[
W_j^{n+1} = W_j^n + k_{n+1}W_{j-1}^n - (k_{n+1} - 1)W_{j-2}^{n-1}.
\]

The proof of this theorem comes completely from the following two lemmas.

**Lemma 5.4.** The Whitney numbers for the family of zig-zags can be generated by the recursive relation:

\[
W_j^{n+1} = W_j^n + k_{n+1}W_{j-1}^n - (k_{n+1} - 1)U_{j-1}^n.
\]

Where \( U_{j-1}^n \) is the the number of flats of \( X_n \), with rank \( j-1 \), that contain the element \( e_n \).

**Proof.** We start by remarking that for all \( Y \in \mathcal{L}_n \) with \( \rho_n(Y) = j \), \( Y \in \mathcal{L}_{n+1} \), and \( \rho_{n+1}(Y) = j \). There are \( W_j^n \) such elements of \( \mathcal{L}_{n+1} \).

Next we note that \( |X_{n+1} \setminus X_n| = k_{n+1} = |\gamma_{n+1} \setminus \{e_n\}|. \) For each point \( x_i \in \gamma_{n+1} \setminus \{e_n\} \) and each flat \( Z \in \mathcal{L}_n \) such that \( \rho_n(Z) = j-1 \), we have \( \rho_n(\sigma_n(Z \cup x_i)) = j \). Thus there are \( k_{n+1}W_{j-1}^n \) of these sets in \( \mathcal{L}_{n+1} \).

Since \( \{e_n\} \in \mathcal{L}_n \) there are some of the previously counted sets \( Z \) that contain \( e_n \). If \( e_n, x_i \in Z \) for some \( x_i \in \gamma_{n+1} \), then by Definition 2.2 \( \gamma_{n+1} \subseteq Z \). Thus we have counted each of these sets \( k_{n+1} \) times. Hence we have over-counted by a total of \( (k_{n+1} - 1)U_{j-1}^n \) times. Lastly, we note that \( \mathcal{L}_{n+1} \) is atomistic and the atoms of the lattice are exactly the singleton subsets of \( X_{n+1} \). It follows we have counted all the possible sets of rank \( j \). 

\( \square \)
Lemma 5.5. For the family of zig-zags, $U^n_j = W^{n-1}_{j-1}$.

Proof. Note $U^n_j$ is the number of flats of $X^n$ of rank $j$ that contain the element $e_n$. For all bones of $\Gamma_n$, $e_n$ is only in $\gamma_n$. Thus we can look back at $L_{n-1}$ to find the size of $U^n_j$. An argument similar to Lemma 5.4 we note there are $W^{n-1}_j$ many sets $Y \in L_{n-1}$ with $\rho_n(Y) = j - 1$, and $\rho_n(\sigma_n(Y \cup \{e_n\})) = j$. We have counted each of these sets only once because, if $x \neq e_n$ and $x \in \gamma_n$ and $x \in Y$, then $x = e_{n-1}$ since $\gamma_n \cap X_{n-1} = \{e_{n-1}\}$.

Using this recursive formula we have managed to test for homogeneous zig-zags, with $n$ bones, and $k+1$ points per bone:

- $k = 2, 1 \leq n \leq 740$;
- $k = 3, 1 \leq n \leq 580$;
- $k = 4, 1 \leq n \leq 480$;
- $k = 5, 1 \leq n \leq 430$.

We’ve also tested with a different number of points per bone up to:

- $n \leq 7, 1 \leq k_1, k_2, k_3, k_4, k_5, k_6, k_7 \leq 16$.

So far everything we have tested is log-concave and our empirical evidence seems to suggest that $\limsup_{n \to \infty} \frac{W^n_j}{W^{n+1}_{j-1}} = 1$.

5.1.2. Stars. The second family of skeletons we consider we refer to as stars and can be described as follows:

Let $X_n = \{x_1, x_2, \ldots, x_{k \sum_{i=1}^n (k_i)+1}\}$ where $k \geq 2, n \geq 1$. Let $\Xi_n = \{\xi_1, \ldots, \xi_n\}$ be the skeleton on $X_n$ such that:

$$\xi_j = \{x_1, x_{\sum_{i=1}^{j-1} (k_i)+2}, x_{\sum_{i=1}^{j-1} (k_i)+3}, \ldots, x_{\sum_{i=1}^{j} (k_i)}, x_{\sum_{i=1}^{j} (k_i)+1}\},$$

thus $|\xi_j| = k_j + 1$, for all $j \leq n$. For ease of reference we let $e_j = x_{\sum_{i=1}^{j} (k_i)+1}$.

![Figure 6](image-url)

Note that as with the zig-zag family, for the star family the sequence $\{k_1, \ldots, k_n\}$ is a sequence of positive integers greater than or equal to two, but that is the only requirement we put on any $k_i$. Furthermore, $|\Xi_n| = n$, and for each $1 \leq i \leq n$, $|\xi_i| = k_i + 1$. Again $X_n = X_{n-1} \cup \xi_n$, for all $n$. Thus for each $Y \subseteq X_n$, $\sigma_n(Y) = \sigma_{n+1}(Y)$. 


However for stars, if \( a \neq b \), then \( \xi_a \cap \xi_b = \{ x_1 \} \). Hence there is only one joint. Figure 6 illustrates the \( n \)th star.

To avoid confusion let \( \lambda_n \) be the closure operators of the star family, and \( H_n \) be the set of flats of \( \lambda_n \). Then \( H_n = \langle H_n, \subseteq \rangle \) denotes the lattice of flats. Furthermore, let \( V^n_j \) be the \( j \)th Whitney number of \( H_n \).

**Theorem 5.6.** The \( j \)th Whitney number for the family of star lattices is:

\[
V^n_j = \sum_{1 \leq \alpha_1 < \cdots < \alpha_j \leq n} (k_{\alpha_1} \cdot \cdots \cdot k_{\alpha_j}) + \binom{n}{j-1},
\]

where \( k_{\alpha_i} + 1 = |\xi_{\alpha_i}| \), for some \( \xi_{\alpha_i} \in \Xi \).

**Proof.** First note that we can partition the flats of rank \( j \) in two, those containing the joint \( x_1 \) and those not. Suppose \( A \in H \) such that \( x_1 \not\in A \) and \( \rho_n(A) = j \). For any two \( x_d, x_{\bar{d}} \in A \), with \( d \neq \bar{d} \); \( x_d, x_{\bar{d}} \) must be in different bones. Thus we have \( j \) different points each in its own bone. However, once a bone is picked, say \( \xi_{\alpha_i} \), there are \( k_{\alpha_i} \) points to choose from on that bone. Since this is true for each bone and we have \( j \) different points at each time we have a total of \( \sum_{1 \leq \alpha_1 < \cdots < \alpha_j \leq n} (k_{\alpha_1} \cdot \cdots \cdot k_{\alpha_j}) \) sets not containing the joint with rank \( j \). Finally suppose \( B \in H \) and \( x_1 \in B \). For any \( x_n \neq x_1 \), the bone \( \xi \) containing \( x_1 \) will be contained in \( B \). Thus a flat like \( B \) can be formed by choosing \( j - 1 \) points each from a unique bone. Note that we don’t care about the size of a bone, since we only need a single point (not \( x_1 \)) on the bone for the whole bone to be a subset of the flat. 

\[ \square \]

**Remark 5.7.** If we fix the number of nodes in each bone, that is, if \( k_i = k \) for each \( 1 \leq i \leq n \) for some fixed \( k \in \mathbb{Z}^+ \) it follows from the argument in Theorem 5.6 that \( V^n_j = k^j \binom{n}{j} + \binom{n}{j-1} \).

**Theorem 5.8.** The family of stars for a fixed \( k \) are logarithmically concave.

**Proof.** We let \( |\xi_i| = k+1 \) for all \( i \) and some \( k > 1 \). Thus by Remark 5.7, \( V^n_j = k^j \binom{n}{j} + \binom{n}{j-1} \).

\[
\frac{(V^n_j)^2}{V^n_{j+1}V^n_{j-1}} = \frac{\binom{k^j \binom{n}{j} + \binom{n}{j-1}}{2}}{\binom{k^{j+1}(\binom{n}{j+1}) + \binom{n}{j}}{2} \cdot \binom{k^{j-1}(\binom{n}{j-1}) + \binom{n}{j-2}}{2}}
\]

\[
= \frac{n^2(k^j(n-(j-1))+j)^2}{j^2(n-(j-1))^2} \cdot \frac{(j+1)!(n-j)1}{n!(k^{j+1}(n-j)+j+1)} \cdot \frac{(j-1)!(n-(j-2))}{n!(k^{j-1}(n-(j-2))+j-1)}
\]

\[
= \frac{(k^j(n-(j-1))+j)^2(j+1)(n-(j-1))}{(k^{j+1}(n-j)+j+1)(k^{j-1}(n-(j-2))+j-1)(n-(j-1))}
\]

Subtracting the bottom from the top gives us:

\[
(V^n_j)^2 - V^n_{j+1}V^n_{j-1} = (k^j(n-j+1)+j^2(j+1)(n-j-2)) - [(k^{j+1}(n-j)+j+1)(k^{j-1}(n-j+2)+j-1)j(n-j+1)]
\]

Note we need to show the above is positive. For ease of reading we let \( m = n-j+1 \), and denote the above total by \( \Delta \). Hence,
\[\Delta \]
\[= (k^j m + j)^2 (j + 1)(m + 1) - [(k^{j+1}(m - 1) + (j + 1))(k^{j-1}(m + 1) + (j - 1))jm] \]
\[= (k^{2j}m^2 + 2k^j mj + j^2)(mj + m + j + 1) \]
\[- mj[(k^{2j}(m^2 - 1) + k^{j+1}(m - 1)(j - 1) + k^{j-1}(m + 1)(j + 1) + (j^2 - 1)] \]

Expanding, we get:

\[\Delta = k^{2j} m^3 j + k^{2j} m^3 + k^{2j} m^2 j + k^{2j} m^2 + 2k^j m^2 j^2 + 2k^j m^2 j + 2k^j mj^2 + 2k^j mj + m^3 j + m^2 j + j^2 - k^{2j} m^3 j + k^{2j} mj \]
\[- k^{j+1} m^2 j^2 + k^{j+1} m^2 j + k^{j+1} mj^2 - k^{j+1} mj \]
\[- k^{j-1} m^2 j^2 - k^{j-1} m^2 j - k^{j-1} mj^2 - k^{j-1} mj - m^3 j + mj \]

Simplifying, we break \(\Delta\) into pieces.

\[\alpha_1 = k^{2j} m^3 + k^{2j} m^2 j + k^{2j} m^2 + k^{2j} mj \]
\[\alpha_2 = 2k^j mj(m + 1)(j + 1) - k^{j+1} mj(m - 1)(j - 1) - k^{j-1} mj(m + 1)(j + 1) \]
\[\alpha_3 = mj^2 + j^3 + j^2 + mj \]
\[\Delta = \alpha_1 + \alpha_2 + \alpha_3 \]

Note that \(\alpha_3\) is positive since \(n, j, k \geq 1\). Also,
\[\alpha_1 = k^{2j} m^3 + k^{2j} m^2 j + k^{2j} m^2 + k^{2j} mj \]
\[= k^{2j} m(m^2 + mj + m + j) \]
\[= k^{2j} m(m + 1)(m + j) \]

and
\[\alpha_2 \geq 2k^j mj(m + 1)(j + 1) - k^{j+1} mj(m + 1)(j + 1) - k^{j-1} mj(m + 1)(j + 1) \]
\[= -mj(k^{j+1} - 2k^j + k^{j-1})(m + 1)(j + 1) \]
\[= -k^{j-1} mj(k - 1)^2(m + 1)(j + 1) \]

whence
\[\Delta \geq \alpha_1 + \alpha_2 \]
\[= k^{2j} m(m + 1)(m + j) - k^{j-1} mj(k - 1)^2(m + 1)(j + 1) \]
\[= k^{j-1} m[k^{j+1}(m + 1)(m + j) - j(k - 1)^2(m + 1)(j + 1)] \]
\[\geq k^{j-1} m[k^{j+1}(m + 1)(j + 1) - j(k - 1)^2(m + 1)(j + 1)] \]
\[= k^{j-1} m(m + 1)(j + 1)[k^{j+1} - j(k - 1)^2] \geq 0 \]

since \(k, j \geq 1, k^{j+1} - j(k - 1)^2 \geq 0\).

\[\square\]

**Theorem 5.9.** For a fixed \(k > 1\), the family of stars satisfies the top-heavy conjecture, \(V_j \leq V_{n-j}\) for all \(j \leq n/2\).
Proof. Again we let $|\xi_i| = k + 1$ for all $i$ and some $k > 1$. Thus by Remark 5.7,

$$V_j^n = \binom{k}{j} + \binom{n}{j}.$$ 

We make the following observations that hold true for any $j \leq n/2$:

$$\binom{n}{j} = \binom{n}{n-j},$$

$$k^j \leq k^{n-j},$$

$$k^j \binom{n}{j} \leq k^{n-j} \binom{n}{n-j},$$

$$\binom{n}{j-1} \leq \binom{n}{j+1}$$

whence the inequality holds.

$\square$

REFERENCES