ZEROS OF ENTIRE FUNCTIONS REPRESENTED BY FOURIER TRANSFORMS

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Abstract

The distributions of zeros of entire functions represented by finite or infinite Fourier transforms are investigated. In particular, this thesis explores those properties of kernels that ensure that their Fourier transforms will have only real zeros. With the aid of the Mellin transform and the theory of multiplier sequences, several classical results of G. Pólya are extended. In addition, special attention is given to the finite Fourier transforms of kernels of the form $e^{-t^\alpha}$, $\alpha > 0$. A related open problem posed by G. Csordas and C.-C. Yang is partially solved.
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Introduction

In this thesis we present results pertaining to the location of the zeros of entire functions which can be represented as Fourier transforms. Specifically, we examine those properties of kernels that ensure that their finite or infinite Fourier transforms have only real zeros. It has been stated several times in the literature (see, for example, [7], [15], [34], or [41]):

*Today, there are no known explicit necessary and sufficient conditions that a kernel must satisfy in order for its Fourier transform to have only real zeros.*

Following the Introduction, in Chapter 1, we recall preliminary results related specifically to Fourier transforms, including the Paley-Wiener Theorem, the Fourier Inversion Theorem, and the Riemann-Lebesgue Lemma. We also introduce the requisite material related to the Laguerre-Pólya class as well as that of linear operators, including multiplier sequences and differential operators.

Chapters 2 and 4 are partly guided by the excellent 108 page survey of D. Dimitrov and P. Rusev published in 2011, *Zeros of entire Fourier transforms* [18] and also include a result of R. Duffin and A. Schaeffer [20]. The results in these chapters are a variety of necessary or sufficient conditions for a kernel to have a Fourier transform with only real zeros. In Chapter 3, we establish a partial answer to a problem posed by G. Csordas and C.-C. Yang in their 2005 paper *Finite Fourier transforms and the zeros of the Riemann \( \xi \)-function*, [16, Problem 1.6]. Problem 1.6 asks when the finite Fourier transform of \( e^{-t^2n}, n = 1, 2, 3, \ldots \), has only real zeros. Our final chapters, Chapters 5 and 6, are based on the 1927 paper by G. Pólya, *Über trigonometrische Integrale mit nur reellen Nullstellen* [34]. In his paper, Pólya studies the connections between certain linear operators and the location of zeros of entire functions.

The methods used to establish the new results that follow include a powerful theorem of Laguerre (Theorem 1.44), representation of kernels via their Mellin transform (Theorem 6.4), as well as Pólya’s *universal factors* (Theorem 5.10). We conclude the paper with four open questions related closely to these results.
1 Preliminary Results and Definitions

1.1 The Fourier Transform

As noted in the introduction, we begin with preliminary definitions and results, some of which will be used with little or no further comment in the remainder of the paper.

Definition 1.1. The Fourier transform of a function of the real variable $t$, $K(t)$, called a kernel, is given by the integral

$$f(z) = \int_{-r}^{r} K(t) e^{izt} \, dt,$$

where $0 < r \leq \infty$ and $z \in \mathbb{C}$, provided the integral exists. When $0 < r < \infty$, $f(z)$ is a finite Fourier transform of $K(t)$ and, when $r = \infty$, $f(z)$ is the infinite Fourier transform of $K(t)$.

Remark 1.2. Beyond existence of the transform, there are certain properties that a kernel will be frequently assumed to possess. We will occasionally say that a kernel has the usual properties which is understood to mean

1. the kernel is a real-valued function,
2. even, and
3. continuous over the region of integration.

In particular, when a kernel is even and real-valued, we will occasionally and without mention, recast the Fourier transform in another familiar form,

$$f(z) = \int_{-r}^{r} K(t) e^{izt} \, dt = 2 \int_{0}^{r} K(t) \cos zt \, dt,$$

which makes it clear that the Fourier transform itself, $f(z)$, is an even function which is real-valued for real $z$.

If the interval of integration in (1) is finite, $0 < r < \infty$, and $K(t)$ is continuous on the interval (including at the endpoints), then the transform defines an entire function. This result is a simple application of Leibniz's integral rule, a version of which we now state (cf. [1, p. A-11]).

Theorem 1.3 (Leibniz’s Integral Rule). Let $(a, b), -\infty \leq a < b \leq \infty$, be a real interval and $S$ be an open subset of the complex plane. Suppose $\varphi(t, z) : (a, b) \times S \to \mathbb{C}$, $\varphi(t, z)$ and $\frac{\partial \varphi}{\partial z}(t, z)$ are continuous functions for all $t \in (a, b)$ and $z \in S$, and $\varphi(t, z)$ is absolutely integrable (with respect to $t$) over $[a, b]$ for all $z \in S$. If there exists a piecewise continuous function $g(t)$ such that, for all $z$ and $t$,

$$\left| \frac{\partial \varphi}{\partial z}(t, z) \right| \leq g(t)$$

and $g(t)$ is integrable, then

$$\frac{d}{dz} \int_{a}^{b} \varphi(t, z) \, dt = \int_{a}^{b} \frac{\partial}{\partial z} \varphi(t, z) \, dt$$

and, thus, $f(z)$ is analytic on $S$.  

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We now investigate properties of kernels of Fourier transforms which pertain to the transforms’ growth and decay. The first result of this sort we present is the Riemann-Lebesgue Lemma (cf. [6, p. 22]). This classic result shows that a Fourier transform will tend toward zero along horizontal lines in the complex plane.

**Theorem 1.4 (Riemann-Lebesgue Lemma).** Suppose that $K(t) \in C(-r, r)$ and that $K(t) e^{\pm|t|}$ is absolutely integrable over the interval $(-r, r)$, $0 < r \leq \infty$, for some $y_0 \geq 0$. Then, for $x$, $y$ real and $|y| \leq y_0$,

$$\lim_{|x| \to \infty} \int_{-r}^{r} K(t) e^{ixy} dt = 0.$$  

**Proof.** We prove this result only for $x \to \infty$ as the proof for $x \to -\infty$ is similar.

Let $r > \eta > 0$, $y_0 \geq 0$, and $x > x_0 > 0$ where $x_0$ is large enough that $\pi/x_0 < \eta$. Let $|y| \leq y_0$ be a real number and

$$f_\eta(x) = \int_{-r+\eta}^{r-\eta} K(t) e^{ixy} dt,$$

so that

$$-f_\eta(x) = e^{-\pi} \int_{-r+\eta}^{r-\eta} K(t) e^{ixy} dt.\]

Then, with the substitution $u = t - \pi/x$,

$$-f_\eta(x) = \left( \int_{-r+\eta}^{r-\eta-\pi/x} + \int_{-r+\eta-\pi/x}^{r-\eta} \right) K(u + \frac{\pi}{x}) e^{i\pi x y (u + \pi/x)} du. \]  \tag{2}$$

Replacing the variable $u$ with $t$ in (2) and subtracting $-f_\eta(x)$ from $f_\eta(x)$ yields

$$2|f_\eta(x)| = \left| \int_{r-\eta}^{r-\eta-\pi/x} K(t) e^{ixy} dt + \int_{r-\eta}^{r-\eta-\pi/x} K(t - e^{-\pi y/x} K(t + \frac{\pi}{x}) e^{ixy} dt - e^{-\pi y/x} \int_{r-\eta}^{r-\eta-\pi/x} K(t + \frac{\pi}{x}) e^{ixy} dt \right|$$

$$\leq \int_{r-\eta}^{r-\eta-\pi/x} |K(t)| e^{-\pi y} dt + \int_{r-\eta}^{r-\eta-\pi/x} \left| K(t) - e^{-\pi y/x} K(t + \frac{\pi}{x}) \right| e^{-\pi y} dt + e^{-\pi y/x} \int_{r-\eta}^{r-\eta-\pi/x} \left| K(t + \frac{\pi}{x}) \right| e^{-\pi y} dt. \] \tag{3}$$

By taking $x$ sufficiently large, the first and last integrals in (3) may be made arbitrarily small (see [45, p. 88]). In addition, since

$$\int_{r-\eta}^{r-\eta-\pi/x} \left| K(t) - e^{-\pi y/x} K(t + \frac{\pi}{x}) \right| e^{-\pi y} dt \leq \int_{r-\eta}^{r-\eta-\pi/x} |K(t)| e^{-\pi y} dt + \int_{r-\eta}^{r-\eta-\pi/x} \left| e^{-\pi y/x} K(t + \frac{\pi}{x}) \right| e^{-\pi y} dt \leq \int_{r-\eta}^{r-\eta} |K(t)| e^{-\pi y} dt + e^{-\pi y/x} \int_{r-\eta}^{r-\eta} \left| K(t + \frac{\pi}{x}) \right| e^{-\pi y} dt,$$
by the dominated convergence theorem,
\[
\lim_{x \to \infty} \int_{-r+\eta}^{r-\eta-\pi/x} |K(t) - e^{-\pi y/x} K(t + \frac{\pi}{x})| e^{-yt} dt \leq \int_{-\eta}^{\eta} \lim_{x \to \infty} |K(t) - e^{-\pi y/x} K(t + \frac{\pi}{x})| e^{-yt} dt = 0.
\]
That is, for \( r > \eta > 0 \), \( \lim_{x \to \infty} \int_{-r+\eta}^{r-\eta-\pi/x} |K(t) - e^{-\pi y/x} K(t + \frac{\pi}{x})| e^{-yt} dt = 0 \).

A related property of the Fourier transform is that the process may be ‘undone.’ To illustrate this, we produce the Fourier Inversion Theorem (cf. [22, p. 217]). This result shows that if a kernel is sufficiently nice on the real axis, then it may be recovered from its Fourier transform.

In order to produce this theorem, we need a lemma of more measure theoretic nature. We omit the somewhat lengthy proof of this lemma, but it too may be found in [22, p. 208].

**Lemma 1.5.** Let \( g(t) \) be an even and real-valued function which is absolutely integrable and continuous on the real axis. Suppose \( \int_{-\infty}^{\infty} g(t) \, dt = c \), \( c \in \mathbb{C} \). Let \( f(t) \) be continuous and suppose, for real \( t \), the convolution of \( f \) and \( g \),
\[
f * g(t) = \int_{-\infty}^{\infty} f(t - s)g(s) \, ds,
\]
converges. Then
\[
\lim_{\epsilon \to 0} \left[ f * \left( \frac{g(x/\epsilon)}{\epsilon} \right) \right](t) = cf(t),
\]
and the convergence is uniform on compact intervals.

**Theorem 1.6** (Fourier Inversion Theorem). If \( K(t) \) is absolutely integrable and continuous on the real axis, then, for real \( t \),
\[
K(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\epsilon z^2/2} e^{-izt} dz,
\]
(4)
where \( f(z) \) is the Fourier transform of \( K(t) \). If \( f(z) \) is absolutely integrable, then \( K(t) \) is continuous for real \( t \) and
\[
K(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z) e^{itz} dz.
\]
Proof. Let \( \epsilon > 0 \). By hypothesis, \( f(z) \) is the Fourier transform of the continuous and absolutely integrable function \( K(t) \). Then the Riemann-Lebesgue Lemma implies \( e^{-\epsilon(z)^2/2} f(z) \) is absolutely integrable along the real axis and, thus,

\[
\int_{-\infty}^{\infty} e^{-\epsilon(z)^2/2} f(z) e^{-isz} \, dz = \int_{-\infty}^{\infty} e^{-\epsilon(z)^2/2} \int_{-\infty}^{\infty} K(t) e^{izt} \, dt \, e^{-isz} \, dz \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\epsilon(z)^2/2} K(t) e^{iz(t-s)} \, dt \, dz.
\]

As \( K(t) \) is absolutely integrable, \( e^{-\epsilon(z)^2/2} K(t) \) is also absolutely integrable. Then, by Fubini’s Theorem, we may interchange the order of integration in (5). This yields

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\epsilon(z)^2/2} K(t) e^{iz(t-s)} \, dt \, dz = \int_{-\infty}^{\infty} K(t) \int_{-\infty}^{\infty} e^{-\epsilon(z)^2/2} e^{iz(t-s)} \, dz \, dt \\
= \frac{\sqrt{2\pi}}{\epsilon} \int_{-\infty}^{\infty} K(t) e^{-\frac{(t-s)^2}{2\epsilon^2}} \, dt.
\]

Then by Lemma 1.5, the assumption that \( K(t) \) is continuous, and using the substitution \( u = s - t \),

\[
\lim_{\epsilon \to 0} \frac{\sqrt{2\pi}}{\epsilon} \int_{-\infty}^{\infty} K(t) e^{-\frac{(t-s)^2}{2}} \, dt = \lim_{\epsilon \to 0} \frac{\sqrt{2\pi}}{\epsilon} \int_{-\infty}^{\infty} K(s-u) e^{-\frac{(u)^2}{2}} \, du = 2\pi K(s).
\]

We have established (4). Under the hypothesis that \( f(z) \) is absolutely integrable and the observation that, for real \( z \),

\[
|f(z)| \geq |f(z) e^{-itz-(\epsilon z)^2/2}|,
\]

we may use the dominated convergence theorem and interchange the integral and limit in (4),

\[
K(t) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon(z)^2/2} f(z) e^{-itz} \, dz \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} e^{-\epsilon(z)^2/2} f(z) e^{-itz} \, dz \\
= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(z) e^{-itz} \, dz,
\]

as desired. In this case, the continuity of \( K(t) \) also follows from the dominated convergence theorem.

Remark 1.7. In the case when \( f(z) \) is not absolutely integrable, we may replace the requirement that \( K(t) \) is continuous in the Fourier Inversion Theorem (Theorem 1.6), by the hypothesis that \( K(t) = [K(t^+) + K(t^-)]/2 \). This extension is not necessary for the results in this paper.

We now continue our investigation of properties of kernels which pertain to the convergence and rate of growth of their finite and infinite Fourier transforms. For the finite Fourier transform, the Paley-Wiener Theorem (cf. [2, p. 103]) estimates the growth of the transform. We prove this theorem after introducing the required terminology and definitions.
**Definition 1.8.** Let \( f(z) \) be an entire function. For \( R > 0 \), let

\[
M_f(R) := \max\{|f(z)| : |z| = R\},
\]

the maximum modulus of \( f(z) \) on a circle of radius \( R \) about the origin. Define

\[
\rho := \limsup_{R \to \infty} \frac{\ln \ln M_f(R)}{\ln R}.
\]

Then \( 0 \leq \rho \leq \infty \) and \( \rho \) is called the *order* of the entire function \( f(z) \).

**Definition 1.9.** Let \( f(z) \) be an entire function of order \( 0 < \rho < \infty \). For \( R > 0 \), let \( M_f(R) \) be as in Definition 1.8. Define

\[
\sigma := \limsup_{R \to \infty} \frac{\ln M_f(R)}{R^\rho}.
\]

Then \( 0 \leq \sigma \leq \infty \) and \( \sigma \) is called the *type* of the entire function \( f(z) \).

The following proposition follows directly from Definitions 1.8 and 1.9.

**Proposition 1.10.** If \( f(z) \) is an entire function of finite order \( \rho_0 > 0 \) and finite type \( \sigma_0 \), then for all \( \rho > \rho_0 \) and \( \sigma > 0 \) there exists \( R_0 > 0 \) such that, for \( R > R_0 \),

\[
M_f(R) \leq e^{\sigma R^\rho},
\]

but not for any \( \rho < \rho_0 \). Similarly, for all \( \sigma > \sigma_0 \) there exists \( R_1 > 0 \) such that, for \( R > R_1 \),

\[
M_f(R) \leq e^{\sigma R^\rho_0},
\]

but not for any \( \sigma < \sigma_0 \).

With these definitions, we are ready to state Hadamard’s factorization theorem (see [29, p. 289]).

**Theorem 1.11** (Hadamard’s Factorization Theorem). If \( f(z) \) is an entire function of finite order \( \rho \), then

\[
f(z) = e^{g(z)}z^m \prod_{k=1}^{\omega} \left(1 - \frac{z}{z_k}\right) \exp \left\{ \sum_{n=1}^{N} \frac{z^n}{n^{z_k}} \right\}, \quad 0 \leq \omega \leq \infty,
\]

where \( g(z) \) is a polynomial of degree less than or equal to \( \rho \), \( m \in \mathbb{N}_0 \), and \( 0 \leq N \leq \lfloor \rho \rfloor \) (with the convention that the exponential factors in the product are not present when \( N = 0 \)).

The integer \( q := \max\{\deg(g), N\} \) from (6) is called the *genus* of \( f(z) \) and it may be shown that either \( q = \lfloor \rho \rfloor \) or \( q = \lfloor \rho \rfloor - 1 \) (see [29, pp. 288, 294]).

**Remark 1.12.** It is clear from Hadamard’s Factorization Theorem that if a function has only finitely many zeros and is of order zero then it must be a polynomial.

We now prove the Paley-Wiener Theorem. Note that, with Remark 1.7 in mind, the following Paley-Wiener result can be made slightly stronger, as R. Boas does in his book, [2, p. 103]. Again, such an extension is not required for the results in this paper.
Theorem 1.13 (Paley-Wiener Theorem). If $K(t)$ is continuous on the interval $[-r, r]$, $0 < r < \infty$, $f(z) = \int_{-r}^{r} K(t) e^{izt} dt$, then $f(z)$ is entire, and of order one and type less than or equal to $r$.

(In partial converse:) If $f(z)$ is an entire function of order one and type $r$ and is absolutely integrable on the real axis then there exists a continuous function $K(t)$ such that $f(z) = \int_{-r}^{r} K(t) e^{izt} dt$.

Proof. Since $K(t)$ is continuous, as a direct consequence of Leibniz’s Integral Rule (Theorem 1.3), $f(z)$ is an entire function.

Also by $K(t)$ continuous, there exists $M > 0$ such that $|K(t)| \leq M$ for $t \in [-r, r]$. If $z = x + iy$ with $x$ real and $y \geq 1$, then

$$
\left| \int_{-r}^{r} K(t) e^{izt} dt \right| \leq \int_{-r}^{r} |K(t)| e^{b|yt|} dt \\
\leq M \int_{-r}^{r} e^{b|yt|} dt \\
\leq M e^{b|r|},
$$

which, by Proposition 1.10, implies that this integral has order less than or equal to one and type less than or equal to $r$.

Conversely, suppose $f(z)$ is an entire function, absolutely integrable on the real axis, and of order one and type $0 < r < \infty$. By the Fourier Inversion Theorem (Theorem 1.6), there exists a continuous function $K(t)$ such that $f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} dt$.

We aim to show that $K(t) = 0$ for $|t| > r$ as this will imply $f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} dt = \int_{-r}^{r} K(t) e^{izt} dt$.

This argument requires two parts, we prove only that $K(t) = 0$ for $t < -r$ as the proof for $t > r$ is similar.

Let $t < -r$ and consider the contour integral

$$
\oint_{\partial C} f(z) e^{-izt} dz = 0,
$$

where $C = \{z \in \mathbb{C} : -R < \Re z < R, 0 < \Im z < S\}$, with $R > 0$ and $S > 1$ and has the counterclockwise orientation. In particular, as the integrand is an entire function, by hypothesis and Cauchy’s Theorem, the contour integral along the upper three sides of the rectangle $\partial C$ taken in the
opposite direction, which we will denote by $\gamma$, is equal to the integral along the base,
\[ \int_{-R}^{R} f(z) e^{-izt} \, dz. \]

We now estimate the contour integral along $\gamma$,
\[
\left| \oint_{\gamma} f(z) e^{-izt} \, dz \right| = \left| i \int_{0}^{S} f(R + iy) e^{-i(R + iy)t} \, dy - \int_{-R}^{R} f(x + iS) e^{-i(x + iS)t} \, dx 
- i \int_{0}^{S} f(-R + iy) e^{-i(-R + iy)t} \, dy \right|
\leq \int_{0}^{S} |f(R + iy)| e^{yt} \, dy + e^{Rt} \int_{-R}^{R} |f(x + iS)| \, dx
+ \int_{0}^{S} |f(-R + iy)| e^{yt} \, dy.
\]

Let $\epsilon > 0$. By (7), there exists $M > 0$ such that for the second integral in (8),
\[ e^{Rt} \int_{-R}^{R} |f(x + iS)| \, dx \leq 2RM e^{(t + r)R}. \]

Then, as $t < -r$, when $R$ is sufficiently large,
\[ 2RM e^{(t + r)R} < \epsilon/3. \]  

(9)

The first and third integrals in (8) may be estimated in a similar manner; we look only at the first. Letting $T > 1$,
\[
\int_{0}^{S} |f(R + iy)| e^{yt} \, dy = \left( \int_{0}^{T} + \int_{T}^{S} \right) |f(R + iy)| e^{yt} \, dy
\leq \int_{0}^{T} |f(R + iy)| e^{yt} \, dy + \frac{M}{t + r} [e^{(t + r)S} - e^{(t + r)T}].
\]

Then, for $T$ and $S$ sufficiently large, since $t < -r$,
\[ \frac{M}{t + r} [e^{(t + r)S} - e^{(t + r)T}] < \epsilon/6. \]  

(10)

By the Riemann-Lebesgue Lemma (Theorem 1.4), for $y \in [0, T]$, $\lim_{R \to \infty} f(R + iy) = 0$. Hence, there exists $R_0 > 0$ such that for $R > R_0$,
\[ \int_{0}^{T} |f(R + iy)| e^{yt} \, dy < \epsilon/6. \]  

(11)

As already noted, this holds, mutatis mutandis, for the third integral in (8). Thus, by (8), (9), (10), and (11),
\[
\left| \oint_{\gamma} f(z) e^{-izt} \, dz \right| \leq \int_{0}^{S} |f(R + iy)| e^{yt} \, dy + e^{Rt} \int_{-R}^{R} |f(x + iS)| \, dx
+ \int_{0}^{S} |f(-R + iy)| e^{yt} \, dy
\leq (\epsilon/6 + \epsilon/6) + \epsilon/3 + (\epsilon/6 + \epsilon/6).
\]
Note that the proof that $K(t) = 0$ for $t > r$ proceeds in the same manner except that the contour selected is, instead of $\partial C$, the rectangle in the lower half-plane. Since $K(t) = 0$ for $|t| > r$,

$$f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} \, dt = \int_{-r}^{r} K(t) e^{izt} \, dt.$$

\[\square\]

**Remark 1.14.** As previously mentioned, Theorem 1.13 is a slightly weakened version of the Paley-Wiener Theorem. In what was just proved, we used the hypothesis that $f(z)$ is *absolutely* integrable on $\mathbb{R}$ to employ the version of the Fourier Inversion Theorem (Theorem 1.6), proved above. The full Paley-Wiener result omits this hypothesis and is of a slightly more measure theoretic nature.

We provide a refinement of the Paley-Wiener Theorem (cf. [2, p. 104]).

**Corollary 1.15.** If $K(t)$ is continuous on the interval $[-r, r]$, $0 < r < \infty$,

$$f(z) = \int_{-r}^{r} K(t) e^{izt} \, dt,$$

and

$$f(z) \neq \int_{-r+\epsilon}^{r-\epsilon} K(t) e^{izt} \, dt,$$

for all $\epsilon > 0$, $f(z)$ is an entire function of order one and type $r$.

**Proof.** Due to the Paley-Wiener Theorem (Theorem 1.13), it is sufficient to prove that $f(z)$ is of type no less than $r$.

Suppose $K(t)$ is continuous and

$$f(z) = \int_{-r}^{r} K(t) e^{izt} \, dt$$

has order one and type $s < r$. Then, by the Fourier Inversion Theorem and the Paley-Wiener Theorem,

$$f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} \, dt.$$

That is, if $f(z)$ has type less than $r$, then $K(t)$ is supported on a closed interval properly contained in $[-r, r]$. This completes the proof.

In order to facilitate the treatment of the *infinite* Fourier transform, we introduce the “big O notation.”

**Notation 1.16.** Let $K(t), h(t) : \mathbb{R} \to \mathbb{C}$. If there exist positive constants $A$ and $t_0$ such that

$$|K(t)| < A|h(t)|$$

when $|t| > t_0$, then we write

$$K(t) = O(h(t)), \text{ as } t \to \pm \infty$$
Theorem 1.17. If \( K(t) \in C(\mathbb{R}) \) and if for every positive \( \sigma \) there exists \( \beta > 0 \) such that
\[
K(t) = \mathcal{O}\left(e^{-(\sigma+\beta)|t|}\right), \quad \text{as } t \to \pm \infty,
\]
then the infinite Fourier transform of \( K(t) \),
\[
f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} \, dt,
\]
defines an entire function.

Proof. Let \( z = x + iy \) and \( S \) be a compact subset of the complex plane which is contained in a disk of radius \( r \) centered at the origin. Then there exists positive constants \( t_0, \beta, \) and \( A \) such that, for \( z \in S \),
\[
\left| \frac{\partial}{\partial z} K(t) e^{izt} \right| = |tK(t) e^{izt}| \leq g(t) = \begin{cases} |tK(t)| e^{r|t|}, & t \in [-t_0, t_0] \\ Ae^{-\beta|t|}, & t \in (-\infty, -t_0) \cup (t_0, \infty) \end{cases}.
\]
Observing that \( g(t) \) is absolutely integrable over the real line, by Leibniz’s Integral Rule (Theorem 1.3),
\[
\frac{\partial}{\partial z} f(z) = \int_{-\infty}^{\infty} \frac{\partial}{\partial z} K(t) e^{izt} \, dt, \quad z \in S.
\]
This shows that \( f(z) \) is analytic on every open subset of \( S \). In other words, \( f(z) \) is an entire function. \( \square \)

Our last introductory results pertaining to the Fourier transform of a kernel, \( K(t) \), shows that, with slightly stronger assumptions on the decay of the \( K(t) \), we can bound the growth of its infinite Fourier transform (see G. Pólya [34]).

Theorem 1.18. If \( K(t) \in C(\mathbb{R}) \) and if there exists \( \alpha > 0 \) such that
\[
K(t) = \mathcal{O}\left(e^{-|t|^{2+\alpha}}\right), \quad \text{as } t \to \pm \infty,
\]
then the infinite Fourier transform of \( K(t) \),
\[
f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} \, dt,
\]
is an entire function of order \( \rho \), \( \rho \leq \frac{2+\alpha}{1+\alpha} < 2 \).

Proof. By Theorem 1.17 \( f(z) \) is an entire function. We now proceed to estimate the growth of \( f(z) \).

Consider,
\[
f(z) = \int_{0}^{\infty} K(t) e^{-izt} \, dt + \int_{-\infty}^{0} K(t) e^{-izt} \, dt. \tag{12}
\]
By symmetry, the two integrals will have the same order. Since the sum of two entire functions of order \( \rho \) has order less than or equal to \( \rho \) (see [27, p. 3]), it suffices to estimate the growth of only the first integral in (12).
By hypothesis, there exist positive numbers $A$, $\alpha$, and $t_0$ such that $|K(t)| \leq Ae^{-|t|^{2+\alpha}}$ when $|t| > t_0$. Then

$$
\left| \int_0^\infty K(t) e^{itz} dt \right| \leq \left| \int_0^{t_0} K(t) e^{itz} dt \right| + A \left| \int_0^\infty e^{-|t|^{2+\alpha}} e^{itz} dt \right|. 
$$

(13)

By the Paley-Wiener Theorem (Theorem 1.13), the first integral on the right in (13) is of order one, thus, we need only to bound the second integral. Thus,

$$
\left| \int_0^\infty e^{-|t|^{2+\alpha}} e^{itz} dt \right| \leq \int_0^\infty e^{|z|t-t^{2+\alpha}} dt
$$

$$
= \int_0^{2|z|^{1/(\alpha+1)}} e^{|z|t-t^{2+\alpha}} dt + \int_0^\infty e^{|z|t-t^{2+\alpha}} dt
$$

$$
< \int_0^{2|z|^{1/(\alpha+1)}} e^{|z|t} dt + \int_0^\infty e^{-t/2+\alpha} dt
$$

$$
< |z|^{-1} \exp\left\{2|z|^{(2+\alpha)/(1+\alpha)}\right\} + \int_0^\infty e^{-t/2+\alpha} dt
$$

$$
< |z|^{-1} \exp\left\{2|z|^{(2+\alpha)/(1+\alpha)}\right\} + \sqrt{\pi}.
$$

In the above argument we used the facts

$$
|z|t - t^{2+\alpha} < |z|t, \text{ when } 0 < t < 2|z|^{1/(\alpha+1)},
$$

and

$$
|z|t - t^{2+\alpha} < (t/2)^{1+\alpha}t - t^{2+\alpha} < -(t/2)^{2+\alpha}, \text{ when } 2|z|^{1/(\alpha+1)} < t.
$$

We have established that the order of $h(z) = \int_0^\infty e^{-|t|^{2+\alpha}} e^{itz} dt$ and thus of $f(z)$ is less than or equal to $\frac{2+\alpha}{1+\alpha} < 2$. \qed

**Proposition 1.19.** If $K(t) \in C(\mathbb{R})$ is such that

$$
K(t) = O\left(e^{-at^2}\right), \text{ as } t \to \pm\infty
$$

for some $a > 0$, then

$$
f(z) = \int_{-\infty}^{\infty} K(t) e^{itz} dt
$$

is an entire function either of order less than two or of order two and type less than or equal to $\frac{1}{1+\alpha}$.

**Proof.** By Theorem 1.17 $f(z)$ is an entire function. In order to estimate its growth, we bound $|f(z)|$ as follows,

$$
|f(z)| = \left| \int_{-\infty}^{\infty} K(t) e^{itz} dt \right|
$$

$$
\leq \left| \int_{-t_0}^{t_0} K(t) e^{itz} dt \right| + \left| \int_{-\infty}^{-t_0} K(t) e^{itz} dt + \int_{t_0}^{\infty} K(t) e^{itz} dt \right|
$$

$$
< \left| \int_{-t_0}^{t_0} K(t) e^{itz} dt \right| + 2A \int_{0}^{\infty} e^{-at^2} e^{zt} dt.
$$

(14)
The first term in (14) is the modulus of a function which we know to be of order one by the Paley-Wiener Theorem (Theorem 1.13). Thus, it is sufficient to estimate the second integral in (14). To this end, we consider, since $a > 0$,

$$\int_0^\infty e^{-at^2} e^{yt} dt = \frac{\sqrt{\pi}}{2\sqrt{a}} \exp\left\{\frac{y^2}{4a}\right\} \left(1 + \frac{2}{\sqrt{\pi}} \int_0^{y/(2\sqrt{\pi})} e^{-t^2} dt\right)$$

$$< \frac{\sqrt{\pi}}{\sqrt{a}} \exp\left\{\frac{y^2}{4a}\right\}.$$  \hfill (15)

Thus, by (14) and (15), $|f(z)|$ is bounded by a function of order two and type $1/4a$.

$\square$

**Remark 1.20.** Proposition 1.19 bounds the order and type of a Fourier transform, $f(z)$, for kernels of sufficiently fast decay. If we assume that if $f(z)$ is of order less than or equal to two and genus one, then, by Hadamard’s Factorization Theorem (Theorem 1.11),

$$f(z) = c e^{-az^2 + bz} z^m \prod_{k=1}^\omega \left(1 - \frac{z}{z_k}\right) \exp\left\{\frac{z}{z_k}\right\}, \quad (0 \leq \omega \leq \infty),$$  \hfill (16)

where the exponential factors in the canonical product may or may not be present.

In fact, we know that $a \geq 0$ since, by the Riemann-Lebesgue Lemma (Theorem 1.4), $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$.

The importance of this decay property, $a \geq 0$ if $f(z)$ is of order two as in (16), will become more clear in the next section.

### 1.2 The Laguerre-Pólya Class

Before introducing the Laguerre-Pólya class of real entire functions, we first recall Hurwitz’s theorem (see, for example, [30, p. 4]) and two of its consequences (cf. [47, Corollary 8.8]), as they are fundamental in the analysis of the distribution of zeros of entire functions.

**Theorem 1.21** (Hurwitz’s Theorem). Let $\{f_n(z)\}_{n=0}^\infty$ be a sequence of functions, analytic on a region $R \subset \mathbb{C}$, which converges uniformly to a function $f(z)$, $f(z) \not\equiv 0$, on a compact set, $D \subset R$. If $z_0 \in D$ is a limit point of zeros of the sequence $\{f_n(z)\}_{n=0}^\infty$, then $f(z_0) = 0$.

Conversely, if $z_0 \in D$, $f(z_0) = 0$, possibly of multiplicity greater than one, and $B \subset D$ is a neighborhood of $z_0$ which contains no other zeros of $f(z)$ inside of it or on its boundary, then there exists $N \in \mathbb{N}$ such that for $n \geq N$, $f_n(z)$ and $f(z)$ have the same number of zeros inside $B$ (counting multiplicities).

**Corollary 1.22.** Let $p(z) = \sum_{k=0}^n a_k z^k$ and $p_j(z) = \sum_{k=0}^n a_{k,j} z^k$. If $a_{k,j} \rightarrow a_k$ as $j \rightarrow \infty$ for all $k$, then $p_j(z) \rightarrow p(z)$ as $j \rightarrow \infty$ uniformly on compact subsets of the complex plane.

**Corollary 1.23.** If

$$p_j(z) = c \prod_{k=1}^n (z - z_{k,j}) = \sum_{k=0}^n a_{k,j} z^k,$$

where $z_{k,j} \rightarrow z_k$ as $j \rightarrow \infty$ and $\{z_k\}$ are the zeros of $p(z) = \sum_{k=0}^n a_k z^k$, then $a_{k,j} \rightarrow a_k$ for $k = 0, 1, \ldots, n$. 

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Definition 1.24. A real entire function \( f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k \) is in the \textit{Laguerre-Pólya} class, written \( f(z) \in \mathcal{L}-\mathcal{P} \), if

\[
f(z) = cz^m e^{-a z^2 + b z} \prod_{k=1}^{\omega} \left( 1 + \frac{z}{z_k} \right) e^{-z/z_k}
\]

where \( b, c, z_k \in \mathbb{R}, m \in \mathbb{N}_0, a \geq 0, 0 \leq \omega \leq \infty, \) and \( \sum_{k=1}^{\omega} \frac{1}{z_k} < \infty. \)

Remark 1.25. The significance of the Laguerre-Pólya class in the theory of entire function stems from the fact that functions in this class, and only these, are the uniform limits, on compact subsets of the complex plane, of polynomials with only real zeros (see [14] and [27, Ch. 8]).

Definition 1.26. Let \( f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k \) be an entire function. The \( n \)th \textit{Jensen polynomial associated with} \( f(z) \) is the polynomial

\[
g_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k.
\]

The Jensen polynomials play a prominent role in the investigation of the zeros of functions in the Laguerre-Pólya class, see, for example, G. Pólya and J. Schur [42].

Theorem 1.27. Let \( f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k \) be a real entire function. Then \( f(z) \in \mathcal{L}-\mathcal{P} \) if and only if each of the Jensen polynomials associated with \( f(z) \) has only real zeros.

Theorem 1.28 (cf. [13, Lemma 2.2]). Let \( f(z) \) be an entire function and let \( \{g_n(z)\}_{n=0}^{\infty} \) be its associated sequence of Jensen polynomials. Then the sequence of polynomials \( \{g_n(z/n)\}_{n=1}^{\infty} \) converges uniformly to \( f(z) \) on compact subsets of the complex plane.

We conclude this section by introducing two subclasses of the Laguerre-Pólya class.

Definition 1.29. A real entire function \( f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k \) is in \( \mathcal{L}-\mathcal{P}^+ \), if

\[
f(z) = cz^m e^{b z} \prod_{k=1}^{\omega} \left( 1 + \frac{z}{z_k} \right)
\]

where \( c, z_k > 0 \) for all \( k, m \in \mathbb{N}_0, b \geq 0, 0 \leq \omega \leq \infty, \) and \( \sum_{k=1}^{\omega} \frac{1}{z_k} < \infty. \)

Remark 1.30. An equivalent definition of \( \mathcal{L}-\mathcal{P}^+ \) is the following (cf. [10]): A function \( f(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k \) is in \( \mathcal{L}-\mathcal{P}^+ \) if \( f(z) \in \mathcal{L}-\mathcal{P} \) and \( \gamma_k \geq 0 \) for all \( k. \)

Theorem 1.31. Let \( f(z) \) be a real entire function. Then \( f(z) \in \mathcal{L}-\mathcal{P}^+ \) if and only if each of the Jensen polynomials has all of its zeros in the interval \((-\infty, 0]. \)

Another important subclass of \( \mathcal{L}-\mathcal{P} \), one seemingly quite similar to \( \mathcal{L}-\mathcal{P}^+ \), is \( \mathcal{L}-\mathcal{P}(\mathbb{R}, 0]. \) This class is defined as follows.
Definition 1.32. A function \( f(z) \in \mathcal{L}P \) is in \( \mathcal{L}P(-\infty,0] \) if all of its zeros are in the interval \( (-\infty,0] \).

The relations \( \mathcal{L}P^+ \subset \mathcal{L}P(-\infty,0] \subset \mathcal{L}P \) follow directly from the definitions above. That \( \mathcal{L}P(-\infty,0] \not\subset \mathcal{L}P \) may be easily illustrated by using polynomials as examples. It is less clear that \( \mathcal{L}P^+ \not\subset \mathcal{L}P(-\infty,0] \). However, \( 1/\Gamma(z) \) is a function which has all its zeros in the interval \( (-\infty,0] \), but has both positive and negative Taylor coefficients (cf. [8]).

1.3 Multiplier Sequences and Complex Zero Decreasing Sequences

In this section we introduce a class of linear operators which preserve the reality of zeros of functions in the Laguerre-Pólya class (cf. [14]).

Definition 1.33. A sequence \( T = \{\gamma_k\}_{k=0}^{\infty} \) of real numbers is called a multiplier sequence if, whenever the real polynomial \( p(z) = \sum_{k=0}^{n} a_k z^k \) has only real zeros, the polynomial \( T[p(z)] = \sum_{k=0}^{n} \gamma_k a_k z^k \) also has only real zeros.

Remark 1.34. Definition 1.33 is extended to the case when \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is a transcendental entire function in the Laguerre-Pólya class, provided the resultant infinite series, \( T[f(z)] = \sum_{k=0}^{\infty} \gamma_k a_k z^k \), converges.

If \( f(z) \in \mathcal{L}P \), \( \{g_n(z)\}_{n=0}^{\infty} \) is the sequence of Jensen polynomials associated with \( f(z) \), and \( \{\gamma_k\}_{k=0}^{\infty} \) is a multiplier sequence, then applying \( \{\gamma_k\}_{k=0}^{\infty} \) to the polynomials \( g_n(z/n) \) and taking \( n \to \infty \) shows, by Hurwitz’s Theorem (Theorem 1.21), when the result converges, \( f(z) \in \mathcal{L}P \). The following example shows that the restriction in Remark 1.34 to functions in the Laguerre-Pólya class is necessary.

Example 1.35. Let \( f(z) = e^{z^2}(z^2 - 1) \). The function, \( f(z) \), is clearly an entire function with only real zeros and, by Definition 1.24, \( f(z) \notin \mathcal{L}P \). It will be shown that \( T = \{k^2 + k + 1\}_{k=0}^{\infty} \) is a multiplier sequence (Corollary 1.41). Consider the Taylor series

\[
f(z) = e^{z^2}(z^2 - 1) = -1 + \sum_{k=1}^{\infty} \left( \frac{1}{(k-1)!} - \frac{1}{k!} \right) z^{2k}.
\]

Here we will use the fact that \( T = \{k^2 + k + 1\}_{k=0}^{\infty} \) is a multiplier sequence. Then

\[
T[f(z)] = -1 \sum_{k=0}^{\infty} (4k^2 + 2k + 1) \left( \frac{1}{(k-1)!} - \frac{1}{k!} \right) z^{2k}
= \left( z \frac{d}{dz} + z \frac{d}{dz} + 1 \right) \sum_{k=0}^{\infty} \left( \frac{1}{(k-1)!} - \frac{1}{k!} \right) z^{2k}
= e^{z^2}(4z^6 + 10z^4 + z^2 - 1),
\]

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where \((z \frac{d}{dz})^2 f(z) = z (z f'(z))'\). Setting \(u = z^2\),
\[
4z^6 + 10z^4 + z^2 - 1 = 4u^3 + 10u^2 + u - 1,
\]
and \(4u^3 + 10u^2 + u - 1\) is a polynomial with two negative zeros, implying \(4z^6 + 10z^4 + z^2 - 1\) has two conjugate pairs of non-real zeros. Thus, a multiplier sequence does not necessarily preserve the reality of the zeros of a transcendental entire function which is not in the Laguerre-Pólya class.

**Definition 1.36.** Suppose \(T = \{\gamma_k\}_{k=0}^\infty\) is a multiplier sequence. Let \(p(z)\) be a polynomial and \(Z_c(p(z))\) denote the number of non-real zeros of the polynomial \(p(z)\). If the inequality
\[
Z_c(T[p(z)]) \leq Z_c(p(z))
\]
holds for every polynomial \(p(z)\), then \(T = \{\gamma_k\}_{k=0}^\infty\) is a complex zero decreasing sequence (CZDS).

**Remark 1.37.** While every CZDS is a multiplier sequence, it is a non-trivial fact that there exist multiplier sequences which are not CZDS. The multiplier sequence in Example 1.35, \(\{k^2 + k + 1\}_{k=0}^\infty\), is not a CZDS. An illustration of this may be found in T. Craven and G. Csordas’ 1995 paper [10].

**Example 1.38.** An alternative multiplier sequence to \(\{k^2 + k + 1\}_{k=0}^\infty\), as appearing in Example 1.35, is \(T = \{k(k-1)\}\). This multiplier sequence is also a CZDS and yet
\[
T[e^{z^2}(z^2 - 1)] = 4z^4(z^2 + 2)e^{z^2}
\]
has non-real zeros. That is, just as was found with the multiplier sequence \(\{k^2 + k + 1\}_{k=0}^\infty\), if \(T\) is a CZDS and \(f(z) \notin \mathcal{L} \cdot \mathcal{P}\), then \(T[f(z)]\) does not necessarily have only real zeros.

We now introduce two theorems of great importance. As with many of the results in this and the previous section, proof of these results may be found in standard literature on the topic of multiplier sequences (see [27, Ch. VIII], [31, Satz 3.2], [42]).

The first result characterizes multiplier sequences in terms of both a transcendental function, \(e^x\), and in terms of the Jensen polynomials of \(e^x\).

**Theorem 1.39** (cf. [31, pp. 29-47]). A sequence \(T = \{\gamma_k\}_{k=0}^\infty\) is a multiplier sequence if and only if:

1. The function
\[
\varphi(z) = T[e^z] = \sum_{k=0}^\infty \gamma_k \frac{z^k}{k!} \in \mathcal{L} \cdot \mathcal{P}^+.
\]
2. The Jensen polynomials
\[
T[(1 + z)^n] = \sum_{k=0}^n \binom{n}{k} \gamma_k z^k \in \mathcal{L} \cdot \mathcal{P}^+ \text{ for all } n \in \mathbb{N}.
\]
Remark 1.40. It is possible that a sequence of real numbers \( \{\gamma_k\}_{k=0}^\infty \) may have properties similar to (1.) and (2.) in Theorem 1.39, except that either the sequence alternates in sign or the sequence is all negative, corresponding to the change of variable \( z \to -z \) and multiplication by \((-1)\), respectively (cf. [14]). We will not require this precision in what follows.

Corollary 1.41 (Example 1.35, ctd.). The operator \( T = \{k^2 + k + 1\}_{k=0}^\infty \) is a multiplier sequence.

Proof. Using the representation as a differential operator, as in (17),

\[
T[e^z] = \sum_{k=0}^\infty \frac{k^2 + k + 1}{k!} z^k
= \left( \left( z \frac{d}{dz} \right)^2 + z \frac{d}{dz} + 1 \right) e^z
= e^z (1 + z)^2,
\]

which has only real zeros and \( k^2 + k + 1 > 0 \) for all \( k \). Then, by Remark 1.30, \( T[e^z] \in \mathcal{L} \cdot \mathcal{P}^+ \) and so, by Theorem 1.39, \( \{k^2+k+1\}_{k=0}^\infty \) is a multiplier sequence. \( \square \)

An alternative characterization of multiplier sequences will be of use in subsequent results in this paper.

Corollary 1.42. Let \( T = \{\gamma_k\}_{k=0}^\infty \), with \( \gamma_k \geq 0 \) and

\[
\varphi(z) = T[e^z] = \sum_{k=0}^\infty \gamma_k \frac{z^k}{k!}.
\]

Then \( T \) is a multiplier sequence if and only if \( \varphi(-u^2) \in \mathcal{L} \cdot \mathcal{P} \).

Proof. For any branch, \( u = \sqrt{-z} \) is a real number if and only if \( z \in (-\infty, 0] \). \( \square \)

Remark 1.43. We note that with the change of variable \( z = -u^2 \), by the definition of the action of \( T = \{\gamma_k\}_{k=0}^\infty \),

\[
\sum_{k=0}^\infty \gamma_k \frac{z^k}{k!} = T[e^z] \neq T\left[e^{-u^2}\right] = \sum_{k=0}^\infty \gamma_{2k} \frac{(-1)^k u^{2k}}{k!}.
\]

The next result, due to Laguerre (cf. [31, Theorem 3.2]), shows the existence of non-trivial complex zero decreasing sequences.

Theorem 1.44 (Laguerre’s Theorem). Let \( f(z) = \sum_{k=0}^n a_k z^k \) be a real polynomial of degree \( n \).

1. If \( h(z) \) is a real polynomial with only real zeros (i.e. \( h(z) \in \mathcal{L} \cdot \mathcal{P} \cap \mathbb{R}[z] \)), none of which are in the interval \((0, n)\), then

\[
Z_c \left( \sum_{k=0}^n h(k) a_k z^k \right) \leq Z_c(f(z)).
\]

2. If \( \varphi(z) \in \mathcal{L} \cdot \mathcal{P} \) and \( \varphi(z) \) has no zeros in the interval \((0, n)\), then

\[
Z_c \left( \sum_{k=0}^n \varphi(k) a_k z^k \right) \leq Z_c(f(z)).
\]

3. If \( \varphi(z) \in \mathcal{L} \cdot \mathcal{P}(-\infty, 0] \), then \( \{\varphi(k)\}_{k=0}^\infty \) is a CZDS.
2 Locations of Zeros of Fourier Transforms

In the four sections in this chapter we illustrate known properties of a kernel which are sufficient conditions for its Fourier transform or Fourier cosine transform to have either only real zeros or some non-real zeros. We will work with the infinite and finite transforms separately.

We reiterate that (cf. Introduction): Today, there are no known necessary and sufficient conditions that a kernel must satisfy so that its Fourier transform, finite or infinite, will have only real zeros.

2.1 Infinite Fourier Transforms in \( \mathcal{L}-\mathcal{P} \)

We introduce this section with one of the most basic and important examples of a kernel for which the infinite Fourier transform has only real zeros (albeit, trivially so).

The Infinite Fourier Transform of the Gaussian

Definition 2.1. The function \( e^{-t^2} \) is called the Gaussian. It is sometimes written with the real parameter \( C \) and positive parameter \( \alpha \) as \( Ce^{-\alpha t^2} \).

Proposition 2.2. The infinite Fourier transform of the Gaussian,

\[
\int_{-\infty}^{\infty} e^{-t^2} e^{izt} dt,
\]

is itself a Gaussian and, thus, has no zeros.

Proof. Completing the square,

\[
\int_{-\infty}^{\infty} e^{-t^2} e^{izt} dt = \int_{-\infty}^{\infty} e^{izt-t^2} e^{-z^2/4-z^2/4} dt
\]

\[
= e^{-z^2/4} \int_{-\infty}^{\infty} e^{(z/2+it)^2} dt
\]

\[
= \sqrt{\pi} e^{-z^2/4}.
\]

In Section 2.3 and Chapter 3 we will revisit the Gaussian, addressing two distinct cases when it is the kernel of the finite Fourier transform.

The Zeros of the Fourier Transform \( \int_{0}^{\infty} e^{-t^\alpha} \cos zt dt \)

We now produce several results regarding the distribution of zeros of the infinite Fourier cosine transform which are included in G. Pólya’s 1923 paper [37]. In his paper G. Pólya examines the kernel \( e^{-t^\alpha}, \alpha > 0 \), with special consideration given to the cases \( \alpha = 2, \alpha > 2 \) an even integer, and \( \alpha > 1 \) not an even integer.

First, we justify the exclusion of the case when \( \alpha < 1 \).
Proposition 2.3. The integral
\[ \int_0^\infty e^{-t^\alpha} \cos zt \, dt \]
fails to converge absolutely off of the real axis for \( \alpha < 1 \).

Proof. Let \( z = x + iy \) and suppose \( y \neq 0 \). Without loss of generality, assume \( y > 0 \). Then,
\[
\int_0^\infty |e^{-t^\alpha} \cos zt| \, dt \geq \int_0^\infty e^{-|t|^\alpha} |\sinh yt| \, dt
\]
\[
= \frac{1}{2} \int_0^\infty |e^{-t^\alpha + yt} - e^{-t^\alpha - yt}| \, dt
\]
\[
\geq \frac{1}{2} \int_0^{t_0} |e^{-t^\alpha + yt} - e^{-t^\alpha - yt}| \, dt + \frac{1}{2} \int_{t_0}^\infty dt, \tag{18}
\]
for some \( t_0 > 0 \). It is clear the second integral in (18) does not converge.

We next deal with the case when \( \alpha \) is an even integer greater than two. To this end, two lemmas and a definition are required. The first lemma shows that the Fourier transform of a kernel of the form \( e^{-t^\alpha} \), \( \alpha \) strictly greater than two, will always have infinitely many zeros.

Lemma 2.4. If \( f(z) \) is the infinite Fourier transform of a kernel, \( K(t) \), where \( K(t) = O \left( e^{-|t|^\alpha} \right) \), as \( t \to \pm \infty \), for some \( \alpha > 2 \), then \( f(z) \) has infinitely many zeros.

Proof. By Theorem 1.18, \( f(z) \) is an entire function of order less than two.

We now argue by contradiction. Suppose \( f(z) \) has only finitely many zeros. By Hadamard’s Factorization Theorem (Theorem 1.11),
\[
f(z) = e^{az+b} z^m \prod_{k=1}^N \left( 1 + \frac{z}{z_k} \right). \tag{19}
\]
Since the Fourier cosine transform of any kernel is an even function, \( a = 0 \) in (19). This implies that \( f(z) \) is a polynomial. However, by the Riemann-Lebesgue Lemma (Theorem 1.4), \( \lim_{z \to \pm \infty} f(z) = 0 \). This is a contradiction, so the assumption that \( f(z) \) has only finitely many zeros must be false.

Remark 2.5. If the order of the entire function \( f(z) \) is not an integer, then it follows from the Hadamard Factorization Theorem that \( f(z) \) has an infinite number of zeros.

We now define the Gamma function and include an explanatory remark.

Definition 2.6. The function, for \( \Re z > 0 \),
\[
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt,
\]
is called the Gamma function. It is extended analytically, via its Prym decomposition (see [46, p. 201]), to the entire complex plane except its simple poles at the negative integers and zero.
Remark 2.7. We state the following properties of the function $\Gamma(z)$ without proof (cf. [29, pp. 304-325]):

1. $\Gamma(n) = (n - 1)!$ for positive integers $n$ and $z\Gamma(z) = \Gamma(z + 1)$ for $z \neq 0, -1, -2, \ldots$.
2. $\Gamma(z) \neq 0$ for all $z$, and
3. $1/\Gamma(z)$ is an entire function of order one and infinite type with zeros of multiplicity one at the negative integers and zero. Thus, $1/\Gamma(z)$ is a function in $L^\mathcal{P}(-\infty, 0]$.

Proposition 2.8. If $\alpha$ is an even integer greater than two, then

$$f(z) = \int_0^\infty e^{-t^\alpha} \cos zt \, dt$$

has infinitely many and only real zeros.

Proof. Since $\alpha > 2$, by Theorem 1.17, $f(z)$ is an entire function. By Lemma 2.4 $f(z)$ must have infinitely many zeros. Hence, it suffices to show that all of the zeros of $f(z)$ are real.

Let $\alpha$ be an even integer greater than 2. By the bounded convergence theorem and with the substitution $u = t^\alpha$,

$$\int_0^\infty e^{-t^\alpha} \cos zt \, dt = \sum_{k=0}^\infty (-1)^k \int_0^\infty e^{-u^{(2k+1)/\alpha}} du \frac{z^{2k}}{(2k)!}$$

$$= \frac{1}{\alpha} \sum_{k=0}^\infty (-1)^k \int_0^\infty e^{-u(2k+1)/\alpha} \frac{z^{2k}}{(2k)!}$$

$$= \frac{1}{\alpha} \sum_{k=0}^\infty (-1)^k \frac{(\Gamma(k+1) \Gamma(2k+1)/\alpha)}{k!} z^{2k}.$$  \hspace{1cm} (20)

Since $\alpha$ is an even integer, the quotient

$$h(z) = \frac{\Gamma(z+1) \Gamma(\frac{2z+1}{\alpha})}{\Gamma(2z+1)}$$  \hspace{1cm} (21)

has the set of poles of its numerator,

$$\{-1, -2, \ldots\} \cup \left\{-\frac{1}{2} + \frac{\alpha k}{2}\right\}_{k=0}^\infty,$$

canceled by the set of poles of its denominator,

$$\left\{-\frac{1}{2}, -1, -\frac{3}{2}, -2, \ldots\right\}.$$

Thus, by Remark 2.7 (3), $h(z)$ is an entire function of order not larger than one (see [23]). If $h(z)$ has any zeros, then those zeros must be in the interval $(-\infty, 0]$. That is,

$$h(z) = \frac{\Gamma(z+1) \Gamma(\frac{2z+1}{\alpha})}{\Gamma(2z+1)} \in L^\mathcal{P}(-\infty, 0].$$  \hspace{1cm} (22)
This implies, by part (iii) of Laguerre’s Theorem (Theorem 1.44), that \( h(z) \) interpolates a CZDS. Since \( \alpha \) is an even integer greater than or equal to two and \( \Gamma(x) \) is increasing for \( x > 0 \),

\[
\frac{\Gamma(k+1)\Gamma\left(\frac{2k+1}{\alpha}\right)}{\Gamma(2k+1)} < \frac{k!}{(2k)!},
\]

and \( \frac{k!}{(2k)!} \to 0 \) as \( k \to \infty \). Thus, (20) converges locally uniformly on the complex plane. Then, by the observation that \( e^{-z^2} = \sum_{k=0}^{\infty} \frac{(-z^2)^n}{n!} \) and Corollary 1.42,

\[
\int_{0}^{\infty} e^{-t^z} \cos zt \, dt \in \mathcal{L}-\mathcal{P}.
\]

We now mention two alternate methods of proving Proposition 2.8. One similar approach which yields considerable generalization will be presented in Chapter 6. We will also see that this result extends to the finite transform, but difficulty arises: In contrast to the Gamma function in equation (21), we must work with the incomplete Gamma function (defined in Chapter 6) which, in certain cases, has non-real zeros. This does not generally allow us to conclude, as we did in line (22), that we are working with a CZDS.

The second alternative approach, outlined in D. Dimitrov and P. Rusev’s paper [18, p. 72], was proposed by L. Ilieff in his dissertation [25]. Rather than using CZDS to prove the proposition, L. Ilieff uses an inductive argument employing Rolle’s Theorem to show that, for all \( m, n \in \mathbb{N} \),

\[
\int_{-m^{1/2q}}^{m^{1/2q}} (1 - t^{2q}/m)^m(1 + izt/n)^n \, dt \in \mathcal{L}-\mathcal{P}.
\]

L. Ilieff then takes the limit first as \( n \) then as \( m \to \infty \) and applies Hurwitz’s Theorem (Theorem 1.21).

An extension of L. Ilieff’s method of proof yields the following generalization (cf. [18]).

**Proposition 2.9.** If \( K(t) \) has the usual properties (see Remark 1.2), is non-constant, non-negative, and if \( K'(iz) \in \mathcal{L}-\mathcal{P} \), then

\[
f(z) = \int_{0}^{\infty} \exp\{-K(t)\} \cos zt \, dt \in \mathcal{L}-\mathcal{P}.
\]

As mentioned at the start of this section, Pólya also addresses the case when \( \alpha > 1 \) is not an even integer. This result, which shows that the integral

\[
\int_{0}^{\infty} e^{-t^z} \cos zt \, dt
\]

has non-real zeros, is postponed until the next section where we investigate kernels whose Fourier transforms have non-real zeros.
2.2 Infinite Fourier Transforms not in \( \mathcal{L}-\mathcal{P} \)

The results of this section focus on properties of a kernel such that its Fourier transform or Fourier cosine transform has non-real zeros. We will prove several results of this nature. These results will eventually lead to proving, as a corollary, the outstanding case from the G. Pólya’s paper [37]: when \( K(t) = e^{-t^\alpha} \) with \( \alpha > 1 \) and \( \alpha \) is not an even integer, as discussed in the previous section.

We often restrict our investigations to even kernels. In particular, even differentiable functions have the property that \( K'(0) = 0 \). Without this property, when \( K'(0) \neq 0 \) and \( K(t), K'(t) \) have the decay property of Theorem 1.18, the infinite Fourier cosine transform has non-real zeros, as we will now show.

**Proposition 2.10.** Suppose \( K(t) \in C^2([0, \infty)) \) and that for some \( \alpha > 2 \),

\[
K(t) = \mathcal{O}(e^{-|t|^\alpha}), \quad \text{as} \ t \to \pm \infty.
\]

Suppose, additionally, that, for \( z \in \mathbb{C} \),

\[
\int_0^\infty K^{(n)}(t) \cos z t \, dt
\]

converges for \( n = 1, 2 \). If \( K'(0) \neq 0 \), then the Fourier cosine transform,

\[
f(z) = \int_0^\infty K(t) \cos z t \, dt,
\]

(23)

has infinitely many non-real zeros and only finitely many real zeros.

**Proof.** By Theorem 1.18 \( f(z) \) is an entire function. Integrating (23) by parts twice

\[
z^2 f(z) = -z \int_0^\infty K'(t) \sin z t \, dt
\]

\[
= K'(0) - \int_0^\infty K''(t) \cos z t \, dt.
\]

By the Riemann-Lebesgue Lemma (Theorem 1.4), there exists \( z_0 > 0 \) such that for \( |z| > z_0, \ z \in \mathbb{R} \),

\[
\left| \int_0^\infty K''(t) \cos z t \, dt \right| < |K'(0)|
\]

since \( |K'(0)| > 0 \). That is, all of the real zeros of \( f(z) \) must lie in the interval \([ -z_0, z_0] \). Since \( K(t) \) cannot be the zero function by \( K'(0) \neq 0 \), \( f(z) \) has only finitely many zeros in this interval.

By Lemma 2.4, \( f(z) \) has infinitely many zeros, hence \( f(z) \) must have infinitely many non-real zeros. This completes the proof. \( \square \)

Note that the preceding proposition proves part of the final outstanding case of the integral \( \int_0^\infty e^{-t^\alpha} \cos z t \, dt \), when \( \alpha > 2 \) is not an even integer (cf. Propositions 2.3 and 2.8). With a little more work, and an additional assumption about the “niceness” of the kernel \( K(t) \), we now show that for any kernel which is not even the Fourier cosine transform will have non-real zeros.
Theorem 2.11. Let $K(t)$ be real analytic and suppose
\[ \int_0^\infty K^{(n)}(t) \cos zt \, dt \]
exists for all $n = 0, 1, 2, \ldots$. If $K(t)$ is not an even function and
\[ f(z) = \int_0^\infty K(t) \cos zt \, dt \]
is an entire function, then $f(z)$ has only finitely many real zeros and infinitely
many non-real zeros.

Proof. Since $K(t)$ is real analytic we may write, for real $t$,
\[ K(t) = \sum_{n=0}^{\infty} K^{(n)}(0) \frac{t^n}{n!}. \]
In light of this Taylor expansion, since $K(t)$ is not even, there exists $N, N$ an
odd integer, such that $K^{(N)}(0) \neq 0$. Let $N$ be the least such odd integer. Then,
integrating $f(z)$ by parts $N + 1$ times,
\[ f(z) = \int_0^\infty K(t) \cos zt \, dt = \frac{1}{z} K(t) \sin (zt) \bigg|_{t=0}^{\infty} + \frac{1}{z^2} K'(t) \cos (zt) \bigg|_{t=0}^{\infty} + \cdots \]
\[ + \frac{1}{z^N} K^{(N)}(t) \cos (zt) \bigg|_{t=0}^{\infty} + \frac{1}{z^N} \int_0^\infty K^{(N)}(t) \sin zt \, dt \]
\[ = \frac{1}{z^N} K^{(N)}(0) + \frac{1}{z^N} \int_0^\infty K^{(N)}(t) \sin zt \, dt. \]

Then, for real $x$,
\[ x^N f(x) = K^{(N)}(0) + \int_0^\infty K^{(N)}(t) \sin xt \, dt \] (24)
and observe, $K^{(N)}(0) \neq 0$ and, by the Riemann-Lebesgue Lemma (Theorem 1.4),
\[ \int_0^\infty K^{(N)}(t) \sin xt \, dt \rightarrow 0 \text{ as } |x| \rightarrow \infty. \] Hence, $z^N f(z)$ and, thus, $f(z)$ have only
finitely many real zeros.

To complete the proof, we must illustrate that $f(z)$ has infinitely many zeros. We proceed by contradiction. Suppose $f(z)$ has only finitely many zeros. Then, by Hadamard’s Factorization Theorem (Theorem 1.11),
\[ f(z) = p(z) e^{q(z)}, \]
where both $p(z)$ and $q(z)$ are polynomials. Again by the Riemann-Lebesgue Lemma, we know that $f(x)$ tends toward zero along both directions of the real
axis. Then $q(z)$ must be of even degree and have a negative leading coefficient. This implies, for all $N \in \mathbb{N}$, $x^N f(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Then, taking limits, (24) implies
\[ -K^{(N)}(0) = \lim_{|x| \rightarrow \infty} \int_0^\infty K^{(N)}(t) \sin xt \, dt, \]
contradicting the previous conclusion $\int_0^\infty K^{(N)}(t) \sin xt \, dt \rightarrow 0$ as $|x| \rightarrow \infty$. \qed

We now prove the remaining result from G. Pólya’s paper [37].
Proposition 2.12. If \( \alpha > 1 \) and \( \alpha \) is not an even integer, then
\[
f(z) = \int_0^\infty e^{-t^\alpha} \cos zt \, dt
\]
is an entire function with only finitely many real zeros and infinitely many non-real zeros.

Proof. Let \( \alpha > 1 \) and \( z = x + iy \) be in the open right half-plane. Consider
\[
f(x) = \int_0^\infty e^{-t^\alpha} \cos xt \, dt
\]
which may be rewritten
\[
x^{\alpha+1} f(x) = x^\alpha \Im \left( \int_0^\infty x^{\alpha-1} e^{-t^\alpha} e^{it \theta} \, dt \right).
\]
Let \( u = x^{\alpha} t^\alpha \), then
\[
x^{\alpha+1} f(x) = \Im \left( \int_0^\infty e^{-ux^{-\alpha}} e^{iu^{1/\alpha}} \, du \right).
\]
By taking \( \theta > 0 \) small enough so that \( e^{i\theta} \) is in the principle branch of the \( \alpha \)th root,
\[
\lim_{r \to \infty} ire^{i\theta} \int_0^\theta e^{-(re^{i\theta})x^{-\alpha}} e^{i(re^{i\theta})^{1/\alpha}} \, du = 0,
\]
such that, by Cauchy’s Theorem,
\[
0 = \int_0^\infty e^{-ux^{-\alpha}} e^{iu^{1/\alpha}} \, du + e^{i\theta} \int_0^\infty e^{-(re^{i\theta})x^{-\alpha}} e^{i(re^{i\theta})^{1/\alpha}} \, du.
\]
Then, by increasing the angle \( \theta \to \alpha \pi/2 \) (using the branch of the \( \alpha \)th root such that \( (e^{i\alpha \pi/2})^{1/\alpha} = e^{i\pi/2} \)),
\[
\lim_{x \to \infty} x^{\alpha+1} f(x) = \lim_{x \to \infty} \Im \left( e^{i\alpha \pi/2} \int_0^\infty e^{-(re^{i\alpha \pi/2})x^{-\alpha}} e^{i(re^{i\alpha \pi/2})^{1/\alpha}} \, du \right)
\]
\[
= \Im \left( e^{i\alpha \pi/2} \int_0^\infty e^{i(r^{1/\alpha} e^{i\pi/2})} \, du \right)
\]
\[
= \sin \left( \frac{\alpha \pi}{2} \right) \int_0^\infty e^{-r^{1/\alpha}} \, du,
\]
which, employing the substitution \( s = r^{1/\alpha} \), yields
\[
\lim_{x \to \infty} x^{\alpha+1} f(x) = \Gamma(\alpha + 1) \sin \left( \frac{\alpha \pi}{2} \right).
\]
By the hypothesis that $\alpha > 0$ and $\alpha \neq 2, 4, 6, \ldots$,
\[
\lim_{x \to \infty} x^{\alpha+1} f(x) = \Gamma(\alpha + 1) \sin \left( \frac{\alpha \pi}{2} \right) \neq 0
\] (25)

and, thus, $f(x)$ has all of its zeros in some compact subset of the real line. This implies $f(z) \neq 0$ has only finitely many real zeros.

As in the proof of Theorem 2.11, if $f(z)$ has only finitely many zeros then
\[
f(z) = p(z) e^{q(z)},
\]
where both $p(z)$ and $q(z)$ are polynomials. Thus, again, by the Riemann-Lebesgue Lemma, $q(z)$ must be a polynomial of even degree with negative leading coefficient. Since this would contradict (25), we must conclude $f(z)$ has infinitely many zeros.

\[\square\]

Infinite Fourier Transforms Which Are Positive on the Real Axis

Lemma 2.4 tells us that whenever a kernel decays sufficiently quickly its Fourier transform will have an infinite number of zeros. Then kernels which have only finitely many real zeros and decay sufficiently quickly along the real axis will have infinitely many non-real zeros.

The results in this subsection show cases where the Fourier transform is always positive on the real axis. That is, transforms which have no real zeros.

We begin with another result of G. Pólya which he presented at the Berkeley Symposium in 1949 [36].

**Proposition 2.13.** Suppose $K(t) \in C^2(0, \infty)$ and
\[
\int_0^\infty K^{(n)}(t) \cos zt \, dt
\]
exists for $n = 0, 1, 2$. If $K(t) \geq 0$, $K'(t) \leq 0$, and $K''(t) \geq 0$ for $t \in [0, \infty)$, then
\[
f(z) = \int_0^\infty K(t) \cos zt \, dt > 0, \text{ for } z \in \mathbb{R}
\]
or $f(z)$ is everywhere zero.

**Proof.** If $K(t)$ is identically zero, then so is $f(z)$ and we are done. Suppose otherwise, then $K(0) > 0$. By $f(z)$ even and $K(0) > 0$, it is sufficient to show $f(x) > 0$ for $x > 0$. Let $x > 0$. Integrating $f(x)$ by parts,
\[
f(x) = \frac{1}{x} K(t) \sin(zt) \bigg|_{t=0}^{t=x} - \frac{1}{z} \int_0^\infty K'(t) \sin xt \, dt
\]
\[
= -\frac{1}{x} \int_0^\infty K'(t) \sin xt \, dt.
\]

Breaking up the integral into intervals of $\pi/x$,
\[
x f(x) = -\sum_{n=0}^{\infty} \int_0^{\pi/x} K'(t + \frac{n\pi}{x}) [(-1)^n \sin xt] \, dt
\]
\[
= \sum_{n=0}^{\infty} \int_0^{\pi/x} (-1)^{n+1} K'(t + \frac{n\pi}{x}) \sin xt \, dt.
\] (26)
By hypothesis, $K'(t)$ is non-positive and non-decreasing for $t > 0$, and, since $K(0) > 0$, $K'(t)$ must, in fact, be negative and non-decreasing when $t > 0$ for $f(x)$ to exist. Hence,

$$-\int_{0}^{\pi/x} K'(t + \frac{2n\pi}{x}) \sin xt \, dt + \int_{0}^{\pi/x} K'(t + \frac{(2n+1)\pi}{x}) \sin xt \, dt \geq 0$$

for all $n \in \mathbb{N}_0$. Since (26) converges by hypothesis $K'(t) \to 0$, then, for $x > 0$, there exists some $n \in \mathbb{N}$ such that

$$-\int_{0}^{\pi/x} K'(t + \frac{2n\pi}{x}) \sin xt \, dt + \int_{0}^{\pi/x} K'(t + \frac{(2n+1)\pi}{x}) \sin xt \, dt > 0.$$  

Thus the series in (26) is strictly positive for all $x > 0$ and so $f(x) > 0$ for all $x \in \mathbb{R}$.

A result of R. Boas and K. Andersen (see [3]) shows another kernel for which its infinite Fourier cosine transform is always positive on the real axis. We now reproduce this result.

**Proposition 2.14.** There exists a positive number $\sigma$, $0 < \sigma < 1/2$, such that the Fourier cosine transform of a positive function defined on the real axis, $K(t)$,

$$f(z) = \int_{0}^{\infty} K(t) \cos zt \, dt$$

is positive for $z \in \mathbb{R}$ provided the function $t^\mu K(t)$ is decreasing, when $t > 0$, for some $\mu$, $\mu > \sigma$.

To show the sharpness of the result, we prove that if $t^\mu K(t)$ is non-increasing when $\mu < \sigma$ but not for any $\mu > \sigma$, then $f(z)$ is not necessarily positive for real values of $z$.

**Proof.** For $0 \leq \mu < 1$, let

$$g(\mu) = \int_{0}^{\frac{\pi}{2}} t^{-\mu} \cos t \, dt.$$  

(27)

Then $g(\mu)$ is a continuous function on $[0, 1)$ and, thus, by Leibniz’s Integral Rule,

$$g'(\mu) = \int_{0}^{\frac{\pi}{2}} (-\ln t) t^{-\mu} \cos t \, dt$$

$$= \left( \int_{0}^{2/\pi} + \int_{2/\pi}^{1} + \int_{1}^{\pi/2} + \int_{\pi/2}^{3\pi/2} \right) (-\ln t) t^{-\mu} \cos t \, dt$$

$$> \left( \int_{0}^{2/\pi} + \int_{1}^{\pi/2} \right) (-\ln t) t^{-\mu} \cos t \, dt$$

$$> \left( \ln \frac{\pi}{2} \right) \left( \int_{0}^{2/\pi} - \int_{1}^{\pi/2} \right) \cos t \, dt$$

$$= \left( \ln \frac{\pi}{2} \right) \left( \sin \frac{\pi}{2} + \sin 1 - 1 \right)$$

$$> 0,$$
where the first inequality holds since we have deleted only portions of the integral where the integrand is positive and the second by the particular decreasing nature of $t^{-\mu}$ for $\mu \in [0,1)$. This inequality shows that $g(\mu)$ is strictly increasing for $\mu \in [0,1)$.

By the definition of $g(\mu)$ in (27), we can see $g(0) < 0$ and $g(1/2) > 0$. Thus, there exists $\sigma \in (0,1/2)$ such that $g(\sigma) = 0$. Now suppose $K(t)$ is a positive function which has a convergent Fourier cosine transform. Suppose, additionally, that there exists $\mu > \sigma$ such that $t^\mu K(t)$ is non-increasing for $t > 0$. Let

$$f(z) = \int_0^\infty K(t) \cos z t \, dt.$$ 

Then,

$$f(1) = \left( \int_0^{3\pi/2} + \int_{3\pi/2}^\infty \right) K(t) \cos t \, dt \geq \int_0^{3\pi/2} K(t) \cos t \, dt \quad (28)$$

$$= \left( \int_0^{\pi/2} + \int_{\pi/2}^{3\pi/2} \right) t^\mu K(t) t^{-\mu} \cos t \, dt \geq (\pi/2)^\mu K(\pi/2) \int_0^{3\pi/2} t^{-\mu} \cos t \, dt \quad (29)$$

$$= (\pi/2)^\mu K(\pi/2) g(\mu) > 0.$$ 

Line (28) follows from the observation that the second integral in the previous line is positive and line (29) by replacing the decreasing function $t^\mu K(t)$ with $(\pi/2)^\mu K(\pi/2)$ in both integrals. Now, letting $K_s(t) = K(t/s)$, for $s > 0$, $K_s(t)$ satisfies the same hypothesis, above, as $K(t)$. Then observing that

$$f_s(z) = \int_0^\infty K(t/s) \cos z t \, dt = s \int_0^\infty K(u) \cos s z u \, du,$$

via the substitution $u = t/s$, $f(s) = s^{-1} f_s(1)$ (known as the scaling property of the Fourier transform). Thus, for $s > 0$, $f(s) > 0$, but by the evenness of the Fourier cosine transform this must hold for all $s < 0$. By $K(t) > 0$, $f(0) > 0$, too. That is, $f(z) > 0$ for every $s \in \mathbb{R}$.

We now develop a counter example when $t^\mu$ is non-increasing only when $\mu \leq \sigma$. Let $\mu \leq \sigma$ and

$$K(t) = \begin{cases} t^{-\mu}, & t < \frac{3\pi}{2} \\ 0, & t \geq \frac{3\pi}{2} \end{cases}.$$ 

Then $t^\mu K(t)$ is non-increasing, but

$$f(1) = \int_0^\infty K(t) \cos t \, dt = \int_0^{3\pi/2} t^{-\mu} \cos t \, dt = g(\mu),$$

which is less than or equal to zero by the hypothesis that $\mu \leq \sigma$.  

$\square$
After producing this solution, Boas notes that he has produced the approximate value of $\sigma$ as 0.30844. This result is not proved in [3].

**Remark 2.15.** If $\mu \in (0, 1)$ and $t^\mu K(t)$ is decreasing for $t > 0$, then $K(t) \to \infty$ as $t \to 0^+$. We conclude this section with the following, related, question.

**Question 2.16.** What sort of extension, if any, does Proposition 2.14 have to the case of the finite Fourier transform of $K(t)$?

### 2.3 Finite Fourier Transforms in $L-P$

We begin this section by again noting that an analogue of Theorem 2.8 holds in the case of the finite Fourier transform when the interval of integration is properly restricted. The particulars of the extension to the case of the finite transform will be addressed when we revisit the result in a considerably more general form in Chapter 6 (cf. Theorem 6.4).

**Finite Fourier Cosine Transform of Non-Decreasing Kernels**

Certainly, $K(t)$ non-decreasing and not identically zero is a sort of kernel for which the infinite Fourier transform is not convergent. However, it turns out that this sort of kernel will always have a finite Fourier transform with only real zeros. The results found in this subsection, unless otherwise noted, are from the first section of G. Pólya’s paper, *Über die Nullstellen gewisser ganzer Funktionen* [35].

To show that the finite transform of a non-decreasing kernel has only real zeros, we must begin with some preliminary theorems regarding the location of the zeros of certain polynomials and trigonometric polynomials. The first result is the Eneström-Kakeya Theorem (see [21]). This shows that the zeros of certain polynomials occur only on the closed unit disk.

**Theorem 2.17 (Eneström-Kakeya Theorem).** If $0 \leq a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n$, with $a_n > 0$, then the polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ has all of its zeros in the closed unit disk.

**Proof.** Consider the polynomial

$$q(z) = (1 - z) p(z) = -a_n z^{n+1} + \left( \sum_{k=1}^{n} (a_k - a_{k-1}) z^k \right) + a_0.$$

Now, arguing by contradiction, suppose $z_0$, $|z_0| > 1$, is a zero of $q(z)$. Then

$$|a_n z_0^{n+1} = \left| \left( \sum_{k=1}^{n} (a_k - a_{k-1}) z_0^k \right) + a_0 \right| \leq \left( \sum_{k=1}^{n} (a_k - a_{k-1}) |z_0|^k \right) + a_0.$$
\[ \sum_{k=1}^{n} (a_k - a_{k-1}) |z_0|^n + a_0 |z_0|^n = a_n |z_0|^n, \]

which is the required contradiction.

We use this result to show that trigonometric polynomials with coefficients satisfying the relation in the Eneström-Kakeya Theorem, \( 0 \leq a_0 \leq a_1 \leq \cdots \leq a_n \), \( a_n > 0 \), have only real zeros. This is an easy corollary of the following result.

**Theorem 2.18.** Suppose \( p(z) = a_0 + a_1 z + \cdots + a_n z^n \) is a polynomial of degree \( n \) which has all its zeros in the closed unit disk. Then the trigonometric polynomials

\[
\begin{align*}
p_c(z) &= a_0 + a_1 \cos z + a_2 \cos 2z + \cdots + a_n \cos nz \\
p_s(z) &= a_1 \sin z + a_2 \sin 2z + \cdots + a_n \sin nz
\end{align*}
\]

have only real zeros.

**Proof.** We begin by assuming all of the zeros of \( p(z) \) lie in the interior of the unit disk. Then we proceed using a counting argument, showing first that \( p_c(z) \) and \( p_s(z) \) have exactly \( 2n \) zeros for \( z \in [2n\pi, 2(n+1)\pi) \), \( n \in \mathbb{Z} \).

Let \( w_1, w_2, \ldots, w_n \) be the \( n \) zeros of \( p(z) \), all in the interior of the unit disk, let \( x \) be real, and define

\[ r_k e^{i\theta_k} = e^{ix} - w_k \quad \text{for} \quad k = 1, 2, \ldots, n, \]

where \( r_k \) and \( \theta_k \) are real. The values of \( r_k \) and \( \theta_k \) vary with both \( x \) and \( w_k \). In particular, given a fixed \( w_k \), then \( r_k > 0 \) and as \( x \) increases from 0 to \( 2\pi \) the value of \( \theta_k \) also increases by \( 2\pi \). In addition, the sum

\[ \theta = \theta_1 + \theta_2 + \cdots + \theta_n \]

increases by \( 2n\pi \) as \( x \) increases from 0 to \( 2\pi \). By factoring \( p(z) \) and replacing \( z \) with \( e^{ix} \),

\[
p(e^{ix}) = a_n (e^{ix} - w_1)(e^{ix} - w_2) \cdots (e^{ix} - w_n) = a_n r_1 r_2 \cdots r_n e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)} = Re^{i\theta},
\]

where \( R \) and \( \theta \) are, of course, dependent upon \( x \). The real and imaginary parts are given by

\[
\Re p(e^{ix}) = R \cos \theta \quad \text{and} \quad \Im p(e^{ix}) = R \sin \theta
\]

respectively. Both \( \Re p(e^{ix}) \) and \( \Im p(e^{ix}) \) have \( 2n \) zeros for \( x \in [0, 2\pi) \) as, correspondingly, \( \theta \) ranges between \( \theta_0 \) and \( 2n\pi + \theta_0 \), for some \( \theta_0 \in \mathbb{R} \). In addition, by looking at the real part of \( p(e^{ix}) \),

\[
\Re p(e^{ix}) = R \left( a_0 + a_1 e^{ix} + a_2 e^{2ix} + \cdots + a_n e^{nix} \right) = a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx = p_c(x)
\]

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Thus, by (30) we know that \( p_c(x) \) and, by a similar argument, \( p_s(x) \) have at least \( 2n \) real zeros in each interval of the form \([2n\pi + c, (2n+1)\pi + c]\), \( c \in \mathbb{R} \).

We now prove that the \( 2n \) zeros just found account for all of the zeros of the trigonometric polynomials \( p_c(z) \) and \( p_s(z) \). The case of \( p_c(z) \) is illustrated since the case of \( p_s(z) \) is essentially the same. From (31),

\[
p_c(z) = a_0 + a_1 \cos z + a_2 \cos 2z + \cdots + a_n \cos nz
\]

and, letting \( u = e^{iz} \),

\[
= a_0 + a_1 \frac{u + u^{-1}}{2} + a_2 \frac{u^2 + u^{-2}}{2} + \cdots + a_n \frac{u^n + u^{-n}}{2},
\]

so that with a final change of variable, \( u = s^n \),

\[
= a_0 + a_1 \frac{s^{n+1} + s^{n-1}}{2} + a_2 \frac{s^{n+2} + s^{n-2}}{2} + \cdots + a_n \frac{s^{2n} + 1}{2},
\]

a polynomial of degree \( 2n \). Thus, since \( 2n \) real zeros were found above, we are done when \( p(z) \) has all of its zeros inside the unit disk.

If \( p(z) \) has zeros on the boundary and inside of the unit disk, then, by Corollary 1.23, we may form a sequence of polynomials \( p_n(z) \) which have all of their zeros inside of the unit disk and which converge uniformly to \( p(z) \). The associated sequences of trigonometric polynomials will also converge uniformly to \( p_c(z) \) and \( p_s(z) \), and, thus, \( p_c(z) \) and \( p_s(z) \) will have only real zeros.

**Corollary 2.19.** Let \( p(z) = \sum_{k=0}^{n} a_k z^k \) be a real polynomial such that \( 0 \leq a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n, \ a_n > 0 \). Then the trigonometric polynomials

\[
p_c(z) = \sum_{k=0}^{n} a_k \cos k z \quad \text{and} \quad p_s(z) = \sum_{k=1}^{n} a_k \sin k z
\]

have only real zeros.

**Proof.** This result follows directly from the Eneström-Kakeya Theorem (Theorem 2.17) and Theorem 2.18.

**Proposition 2.20.** Suppose \( K(t) \geq 0 \) is bounded, non-decreasing, and not identically zero for \( t \in [0, r] \). Then the finite Fourier cosine transform of \( K(t) \),

\[
f(z) = \int_{0}^{r} K(t) \cos z t \, dt \in L-\mathcal{P}.
\]

**Proof.** Consider the approximating lower Riemann sum,

\[
\int_{0}^{r} K(t) \cos z t \, dt = \lim_{N \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} K \left( \frac{r k}{N+1} \right) \cos \left( z \frac{r k}{N+1} \right).
\]

For each \( N \), this is a trigonometric polynomial with positive and non-decreasing coefficients. By Corollary 2.19, for each \( N \in \mathbb{N} \) the polynomial has only real zeros.

These polynomials converge uniformly on compact subsets of the complex plane to the function \( f(z) \) and so, by Hurwitz’s Theorem (Theorem 1.21), we know that \( f(z) \in L-\mathcal{P} \).
Finite Fourier Transform of a Concave Kernel

We now show that every finite Fourier transform of a kernel which is twice continuously differentiable, positive, decreasing, and concave over the interval of integration has infinitely many and only real zeros (cf. [39, p. 136] and [40, p. 65]).

**Lemma 2.21.** If, for some $C > 0$, $f(z)$ satisfies the inequality

$$|f(z)| < Ce^{|y|}, \text{ for all } z = x + iy \in \mathbb{C},$$

then

$$\lim_{N \to \infty} \sum_{k=-n}^{n} \frac{(-1)^{k+1}f(k\pi)}{(z-k\pi)^2} = \frac{d}{dz} \left( \frac{f(z)}{\sin z} \right).$$

(32)

Moreover, the convergence is uniform on compact subsets of the complex plane punctured at the poles.

**Proof.** Let $\pi/2 > \epsilon > 0$. Consider the disk

$$S_n = \{ z \in \mathbb{C} : |z| \leq (n + 1/2)\pi \}$$

for $n \in \mathbb{N}_0$ and the closed and bounded region formed by removing, from $S_n$, the open $\epsilon$-balls centered at each singularity of the function $f(z)/\sin z$. Let

$$D_{n,\epsilon} = S_n \setminus \bigcup_{k=-n}^{n} B(k\pi, \epsilon).$$

Fix $z \in D_{n,\epsilon}$. By the residue theorem and Cauchy’s formula,

$$\oint_{\partial S_n} \frac{f(\zeta)}{\sin \zeta} \frac{d\zeta}{(\zeta - z)^2} = \frac{d}{dz} \left( \frac{f(z)}{\sin z} \right) + \sum_{k=-n}^{n} \frac{f(k\pi)}{\cos k\pi} \frac{1}{(k\pi - z)^2}$$

(33)

holds for any non-negative integer $n$.

By Euler's formula $|\sin(z)| \leq M^{-1}e^{|y|}$ for some $M > 0$. By hypothesis, $|f(z)| < Ce^{|y|}$. Then

$$\left| \oint_{\partial S_n} \frac{f(\zeta)}{\sin \zeta} \frac{d\zeta}{(\zeta - z)^2} \right| \leq \oint_{\partial S_n} \frac{|f(\zeta)|}{|\sin \zeta|} \frac{|d\zeta|}{|\zeta - z|^2} \leq CM \oint_{\partial S_n} \frac{|d\zeta|}{(|(n + \frac{1}{2})\pi| - |z|)^2} \leq 2CM(n + \frac{1}{2})\pi^2 \left( \frac{((n + \frac{1}{2})\pi) - |z|}{((n + \frac{1}{2})\pi - |z|)^2} \right),$$

and this converges to zero as $n \to \infty$. That is, by taking limits, equation (33) yields (32).

It is clear, for $\pi/2 > \epsilon > 0$ and $n \in \mathbb{N}$, that this convergence is uniform on compact subsets of the set $D_{n,\epsilon}$. That is, integrating the expression term by term is permissible and, thus, we arrive at the following corollary.
Corollary 2.22. Let \( \pi/2 > \epsilon > 0 \). If \( f(z) \) and \( D_{n,\epsilon} \) are as in Lemma 2.21, then for \( z \in D_{n,\epsilon} \), for each \( n \in \mathbb{N} \), and some complex number \( c \),

\[
\frac{f(z)}{\sin z} = \sum_{k=-\infty}^{\infty} \frac{(-1)^k f(k\pi)}{(z-k\pi)} + c.
\]

Remark 2.23. The Paley-Wiener Theorem shows that the finite Fourier transform over \([-r, r] \), \(0 < r \leq 1\), of a function continuous on \([-r, r]\) satisfies the hypothesis of Lemma 2.21.

We are now in the position to prove that a real, positive, decreasing, and concave kernel has a finite Fourier transform with infinitely many and only real zeros (cf. [40, p. 65]).

Theorem 2.24. Let \( K(t) \) be a twice differentiable and real-valued for \( t \in [0, 1] \). If \( K(t) > 0 \), \( K'(t) \leq 0 \) and \( K''(t) < 0 \) for \( t \in [0, 1) \), then the finite Fourier cosine transform of \( K(t) \),

\[
f(z) = \int_0^1 K(t) \cos z \, dt,
\]

has infinitely many and only real zeros.

Proof. Integrating \( f(z) \) by parts twice yields,

\[
\int_0^1 K(t) \cos z \, dt = \frac{1}{z} K(1) \sin z - \frac{1}{z^2} \int_0^1 K'(t) \sin z \, dt \\
= \frac{1}{z} K(1) \sin z + \frac{1}{z^2} (K'(1) \cos z - K'(0)) \\
- \frac{1}{z^2} \int_0^1 K''(t) \cos z \, dt \\
= \frac{1}{z} K(1) \sin z - \frac{1}{z^2} K'(0)(1 - \cos z) \\
+ \frac{1}{z^2} \int_0^1 K''(t)(\cos z - \cos z t) \, dt.
\]

By examining the representation in line (35), it is possible to determine the sign of \( f(z) \) when \( z = k\pi \), when \( k \) is an integer. In particular, it is apparent that \( f(2k\pi) < 0 \), when \( k \neq 0 \), and \( f((2k+1)\pi) > 0 \) for all integers \( k \). Since \( K(t) > 0 \) for \( t \in [0, 1) \), \( f(0) > 0 \).

Let \( n \in \mathbb{N} \). From the above, it follows that the rational function

\[
\sum_{k=-n}^{n} \frac{(-1)^k f(k\pi)}{(z-k\pi)}
\]

has poles at \( z = 0, \pm \pi, \ldots, \pm n\pi \) and \( 2n - 2 \) zeros, one zero between the poles, with the possible exception of the two intervals \((-\pi, 0)\) and \((0, \pi)\). In addition, such a rational function has no more than \( 2n \) zeros, so, for each \( n \), it must have no more than two non-real zeros.

Now let \( n \to \infty \). Then, by Corollary 2.22 and Hurwitz’s Theorem (Theorem 1.21), we infer that the function \( f(z)/\sin z \) has at most two non-real zeros and at
least one real zero (possibly of multiplicity two) on each of the intervals \([k\pi, (k + 1)\pi]\) except, perhaps, the intervals \([-\pi, 0]\) and \([0, \pi]\). Since multiplication by \(\sin z\) has no effect on the location of these zeros and certainly yields no additional zeros, this statement also holds for \(f(z)\).

It has now been established that \(f(z)\) has infinitely many real zeros (as well as the approximate location of these real zeros). The proof will be completed by demonstrating that \(f(z)\) cannot have any non-real zeros. We proceed by contradiction.

Suppose \(f(z)\) has at least one non-real zero. Since \(f(z)\) is a real entire function, its zeros appear in conjugate pairs. By our previous work, \(f(z)\) has no more than two non-real zeros. Since \(f(z)\) is even, these two non-real zeros must lie on the imaginary axis, but

\[
f(iy) = \int_0^1 K(t) \cosh(yt) \, dt
\]

which, by \(K(t) > 0\), is positive for any real \(y\).

Remark 2.25. By the changes of variables \(s = rt\) and \(w = z/r\),

\[
\int_0^1 K(t) \cos zt \, dt = \int_0^r rK\left(\frac{s}{r}\right) \cos ws \, ds,
\]

Theorem 2.24 can be generalized to a Fourier transform over any interval \([0, r]\), \(0 < r < \infty\).

We conclude this subsection with two final observations. The first is about the zeros of any finite Fourier transform which is continuously differentiable over its interval of integration and non-zero at the endpoints of the interval. The second observation is an example corresponding to Theorem 2.24.

By the proof of Theorem 2.24, integrating \(f(z)\) by parts once we get (34). Then recalling the Riemann-Lebesgue lemma (Theorem 1.4), equation (34), upon multiplying by \(z\), yields the following corollary.

**Corollary 2.26.** The finite Fourier transform of a kernel that is continuously differentiable over its interval of integration and non-zero at the endpoints of integration has infinitely many real zeros.

**Remark 2.27.** Looking at the parallel methods of proof, Corollary 2.26 is, in some sense, in contrast to Theorem 2.11.

As an application of Theorem 2.24 and Corollary 2.26, we obtain the following proposition.

**Proposition 2.28.** For \(r \in (0, \sqrt{1/2}]\), the finite Fourier transform of the Gaussian,

\[
\int_{-r}^{r} e^{-t^2} e^{izt} \, dt,
\]

has infinitely many and only real zeros.
Proof. First note that \( e^{-t^2} \) is positive, even, infinitely differentiable, decreasing, and concave for \( t \in (0, \sqrt{1/2}] \). Then
\[
\int_{-r}^{r} e^{-t^2} \cos zt \, dt = 2 \int_{0}^{r} e^{-t^2} \cos zt \, dt.
\]
With the changes of variables \( u = t/r \) and \( w = zr \), the integral becomes
\[
2 \int_{0}^{1} e^{-(ur)^2} \cos wu \, du.
\]
Then the kernel of this cosine transform, \( e^{-(ur)^2} \), is also positive, infinitely differentiable, decreasing, and concave over the region of integration when \( r \in (0, \sqrt{1/2}] \). Then, by Theorem 2.24 and Corollary 2.26, this transform has infinitely many and only real zeros. \( \square \)

Other Finite Fourier Transforms with Only Real Zeros

The following results illustrate another way that the rate of growth (or decay) of a kernel is related to its Fourier transform having only real zeros. The proof of this proposition is postponed until Chapter 4.

**Proposition 2.29.** Let \( r > 0 \) and suppose \( K(t) \in C^1[0, r] \) is even. If \( K(r) \neq 0 \) and
\[
\int_{0}^{1} \frac{|K'(r t)|}{K(r)} \, dt \leq \frac{1}{r},
\]
than the finite Fourier transform of \( K(t) \)
\[
\int_{0}^{r} K(t) \cos zt \, dt
\]
has only real zeros. In particular, for \( r = 1 \),
\[
\int_{0}^{1} \frac{|K'(t)|}{K(1)} \, dt \leq 1,
\]
implies
\[
\int_{0}^{1} K(t) \cos zt \, dt
\]
has only real zeros.

**Corollary 2.30.** Let \( r > 0 \) and suppose \( K(t) \in C^1[0, r] \) is even, \( K(t) > 0 \), and \( K'(t) \leq 0 \). If \( 2K(r) \geq K(0) \) then
\[
\int_{0}^{r} K(t) \cos zt \, dt \in \mathcal{L} \cdot \mathcal{P}.
\]

**Proof.** By \( K(t) \in C^1[0, r], K(t) > 0, \) and \( K'(t) \leq 0, \)
\[
\int_{0}^{1} \frac{|K'(r t)|}{K(r)} \, dt = \frac{-1}{r K(r)} \int_{0}^{r} K'(u) \, du
\]
\[
= \frac{1}{r} \left( \frac{K(0) - K(r)}{K(r)} \right).
\]
with the substitution \( u = rt \). Then \( 2K(t) \geq K(0) \) if and only if
\[
\int_0^1 \left| \frac{K'(rt)}{K(r)} \right| dt \leq \frac{1}{r},
\]
and, by Proposition 2.29, the proof is complete. \( \square \)

**Remark 2.31.** An application of Corollary 2.30 to the kernel \( e^{-t^2} \) shows that
\[
f(z) = \int_r^{-r} e^{-t^2} e^{zt} dt
\]
has only real zeros when \( 0 < r < \sqrt{\log 2} \). This provides a slightly better bound than that given in Proposition 2.28.

**Example 2.32.** Another example of an application of Corollary 2.30 shows that for kernels other than the Gaussian, as in Remark 2.31, Theorem 2.24 provides the better bound. Let \( K(t) = e^{-t^4} \). Then \( K(t) \) satisfies the hypothesis of Theorem 2.24 for \( r \) in the interval \( (0, \frac{\sqrt{3}}{4}] \), the hypothesis of Corollary 2.30 for \( r \) in the interval \( (0, \frac{\sqrt{\log 2}}{4}] \), and \( \log 2 < 3/4 \).

Other results which appear in D. Dimitrov and P. Rusev’s survey of Fourier transforms (cf. [18]) include the following several, which we present without proof. First, by N. Obrechkoff, we have two necessary conditions for a kernel to have a finite Fourier transform with only real zeros (cf. [32]).

**Proposition 2.33.** Let \( K(t) \) have the usual properties (Remark 1.2). Suppose
\[
f(z) = \int_0^1 K(t) \cos zt dt \in \mathcal{L} \cdot \mathcal{P}.
\]
If \( p(z) \) is a real polynomial, then the polynomial
\[
q(z) = \int_{-1}^1 K(t)p(z - it) dt
\]
has at least as many real zeros as \( p(z) \).

**Proposition 2.34.** Let \( K(t) \) have the usual properties. Suppose
\[
f(z) = \int_0^1 K(t) \cos zt dt \in \mathcal{L} \cdot \mathcal{P}.
\]
If \( q(z) \) has all of its zeros in \( (-\infty, 0] \), then the polynomial
\[
q(z) = \int_{-1}^1 K(t)p(z - it) dt \in \mathcal{L} \cdot \mathcal{P}.
\]

D. Dimitrov and P. Rusev also note a further extension of Ilieff’s Proposition 2.9 by A. Rényi.

**Notation 2.35.** We use the convention that \( f(z) \in C^n[a, b] \), \( a < b \), implies that \( f(z) \) is \( n \) times continuously differentiable on the open interval \( (a, b) \) and that \( f(a) = \lim_{x \to a^+} f(x) \) is finite (and a similar condition holds at \( f(b) \)).
Theorem 2.36 (see [44]). Let \( m \) and \( n \) be positive integers such that \( m + n \) is odd. If \( K(t) \) is a real-valued function with \( K(t) \in C^m[0,1] \), \( K^{(j)}(1) = 0 \) for positive integers \( j \) less than \( n \), \( K^{(j)}(0) = 0 \) for odd and positive integers less than \( n \), and \( t^m K^{(n)}(t) \) is integrable, non-negative, and non-decreasing on \((0,1)\), then

\[
f(z) = \int_0^1 K(t) \cos zt \, dt \in \mathcal{L}-\mathcal{P}
\]

and

\[
g(z) = \int_0^1 K(t) \sin zt \, dt \in \mathcal{L}-\mathcal{P}.
\]

Many other results of this sort may be found in the aforementioned paper by D. Dimitrov and P. Rusev (cf. [18]).

2.4 Finite Fourier (and Related) Transforms not in \( \mathcal{L}-\mathcal{P} \)

The results in this section are, somewhat unsurprisingly, sparse. For the finite Fourier transform, there are fewer known necessary conditions that a kernel must satisfy for its transform to have only real zeros. We present only a few results, without proof, which serve to bound the zeros of the finite transform in regions of the complex plane, but not necessarily to the real axis.

From Pólya’s 1918 paper [38], we have the following result which allows us to bound all of the zeros of a finite Laplace-like transform within a strip around the real imaginary axis.

Proposition 2.37. Suppose \( K(t) \in C^1[a,b] \). If

\[
\alpha \leq - (\ln K(t))' \leq \beta, \quad -\infty \leq \alpha \leq \beta \leq \infty,
\]

and \( - (\ln K(t))' \) is not a constant, then the transform

\[
\int_a^b K(t) e^{zt} \, dt
\]

has all of its zeros in the strip \( \alpha < \Re(z) < \beta \).

Other related results presented in D. Dimitrov and P. Rusev’s paper (cf. [18]) include the two related results by K. Doćev, which define a class of kernels for which an associated finite transform, akin to a finite Fourier transform, has zeros in a half-plane (cf. [19]).

Definition 2.38. Let \( L_\alpha(\lambda) \), \( \lambda > 0 \), denote the class of functions, \( K(t) \), integrable on \([0,1]\) which have the property that for \( \delta > 0 \) there exists a sufficiently large positive integer \( n \) such that all of the zeros of the polynomial

\[
p_n(K;z) = \sum_{j=0}^n K(j/n) z^j
\]

are in the open disk \( \{ z \in \mathbb{C} : |z| < 1 + (\lambda + \delta)n^{-\alpha} \} \).
Proposition 2.39. If \( K(t) \in L_1(\lambda) \), then the zeros of the transform
\[
\int_0^1 K(t) e^{izt} \, dt
\]
are in the half plane \( \{ z \in \mathbb{C} : \Im z \geq -\lambda \} \).

Proposition 2.40. If \( K(t) \in L_\alpha(\lambda) \), with \( \alpha > 1 \), then the zeros of the transform
\[
\int_0^1 K(t) e^{izt} \, dt
\]
are in the right-half plane, \( \{ z \in \mathbb{C} : \Im z \geq 0 \} \).
3 The Finite Fourier Transform of the Gaussian Has Non-Real Zeros

This chapter is an extension of Section 2.4. Due to the length of the proof and the required preliminaries, it has been separated.

By Proposition 2.28 and Remark 2.31, for sufficiently small intervals of integration, the finite Fourier transform of the Gaussian has infinitely many and only real zeros. In this section we show that if the interval of integration is larger, then the finite Fourier transform of the Gaussian has non-real zeros.

3.1 The Main Result

We illustrate what we believe to be a new result: The finite Fourier transform of the Gaussian has non-real zeros whenever the interval of integration is larger than \([-2.2036, 2.2036]\) (i.e. \(2.2036 < r < \infty\) in Definition 1.1). In fact, there is experimental evidence that for smaller intervals of integration the finite Fourier transform has non-real zeros, but we postpone these results until Section 3.2.

To begin, we introduce notation for the error function and the double factorial, both of which appear in the investigation of the finite Fourier transform of the Gaussian (cf. Lemma 3.3).

Notation 3.1. The error function is defined as

\[ \text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \]

It is worth noting that the error function is an odd function which is positive for positive arguments, converges rapidly to one for real and positive \(z\), and is strictly increasing for real \(z\). In particular, \(\text{erf}(1) \approx 0.843\) and \(\text{erf}(2) \approx 0.995\).

Notation 3.2. The double factorial is defined as

\[ (2k + 1)!! := \prod_{j=0}^k (2j + 1) \quad \text{or} \quad (2k)!! := \prod_{j=1}^k (2j), \]

where \(0!!\) is defined to be 1.

We now include a lemma which outlines the straightforward, albeit elementary, calculation of the Taylor series which will be required in this section’s main result (Theorem 3.6).

Lemma 3.3. Let

\[ e^zh(z) = e^z \int_{-r}^r e^{-t^2} \cos(-i\sqrt{z}t) \ dt = \sum_{\ell=0}^{\infty} \gamma_\ell \frac{z^\ell}{\ell!}. \]  

Then the sequence \(\{\gamma_\ell\}_{\ell=0}^{\infty}\) is given by

\[ \gamma_0 = \sqrt{\pi} \ \text{erf}(r), \]

\[ \gamma_\ell = \ell! \sum_{n=0}^{\ell} \frac{(2n - 1)!!}{2^n (2n)! (\ell - n)!} \left( \sqrt{\pi} \ \text{erf}(r) - e^{-r^2} \sum_{k=0}^{n-1} \frac{2^{n-k} r^{2n-2k-1} (2n-2k-1)!!}{(2n-2k-1)!!} \right), \ell \geq 1. \]
Proof. Consider the general finite Fourier transform of the Gaussian,

\[ f(z) = \int_{-r}^{r} e^{-t^2} \cos zt \, dt. \]

It has the Taylor series expansion

\[
f(z) = \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{(2n)!} \int_{-r}^{r} e^{-t^2} t^{2n} \, dt.
\]

(by uniform convergence)

\[
= \sqrt{\pi} \text{erf}(r) + \left( \frac{\sqrt{\pi}}{2} \text{erf}(r) - e^{-r^2} r \right) \frac{(iz)^2}{2}
\]

\[
+ \left( \frac{3\sqrt{\pi}}{4} \text{erf}(r) - e^{-r^2} (r^3 + \frac{3}{2} r) \right) \frac{(iz)^4}{4!}
\]

\[
+ \sum_{n=3}^{\infty} \frac{(iz)^{2n}}{(2n)!} \int_{-r}^{r} e^{-t^2} t^{2n} \, dt
\]

\[
= \sqrt{\pi} \text{erf}(r) + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n} \frac{(iz)^{2n}}{2n(2n)!} \left( \sqrt{\pi} \text{erf}(r) - e^{-r^2} \sum_{k=0}^{n-1} 2^{n-k} r^{2n-2k-1} \frac{1!!}{(2n-2k-1)!!} \right),
\]

obtained by repeated integration by parts. We note that the sum in (39) is the sum of the partial moments of the Gaussian over an interval of length 2r symmetric about the origin.

For brevity, we write \( f(z) = \sum_{n=0}^{\infty} \alpha_{2n} (iz)^{2n} / (2n)! \), where

\[
\alpha_0 = \sqrt{\pi} \text{erf}(r)
\]

\[
\alpha_{2n} = \frac{(2n-1)!!}{2^n} \left( \sqrt{\pi} \text{erf}(r) - e^{-r^2} \sum_{k=0}^{n-1} 2^{n-k} r^{2n-2k-1} \frac{1!!}{(2n-2k-1)!!} \right), \quad n \geq 1.
\]

Then making the substitution \( z \to -i\sqrt{z} \) and multiplying by \( e^z \), as in equation (36),

\[
e^{z}f(-i\sqrt{z}) = \sum_{\ell=0}^{\infty} \frac{z^{\ell}}{\ell!} \sum_{n=0}^{\infty} \frac{\alpha_{2n}}{(2n)!} z^n
\]

\[
= \sum_{\ell=0}^{\infty} \left( \sum_{n=0}^{\ell} \frac{\alpha_{2n}}{(2n)!(\ell-n)!} \right) z^{\ell}.
\]

Multiplying by \( \ell! \) gives the formulae in equations (37) and (38) for the \( \gamma_\ell \)'s.

We now show that the discriminant of a particular cubic polynomial, the third Jensen polynomial, \( g_3(z) \) of \( e^z h(z) \) in (36) is negative for \( r \geq 2.2036 \). We first recall the formula for the discriminant of a cubic, \( \Delta \). If \( \Delta < 0 \), then the cubic polynomial has non-real zeros.

**Definition 3.4.** The discriminant of a real cubic \( p(x) = ax^3 + bx^2 + cx + d \) is

\[
\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd.
\]
We denote the discriminant by $\Delta(r)$ to indicate that it depends upon the endpoints, $\pm r$, of integration of the finite Fourier transform of the Gaussian.

**Lemma 3.5.** The discriminant of the cubic polynomial $g_3(z) = \sum_{\ell=0}^{3} (\ell) \gamma_\ell z^\ell$, where $\gamma_\ell$ is defined by (37) and (38), is

$$25600 \Delta(r) = -3 \pi r^2 \text{erf}^2(r) e^{-2r^2} (4r^4 - 20r^2 + 15)^2 + 20 r^3 \text{erf}(r) \sqrt{\pi} e^{-3r^2} (8r^6 + 108r^4 - 270r^2 + 135) + 180 r^4 e^{-4r^2} (4r^4 + 20r^2 - 15)$$

(40)

and is zero for $r_0 = 2.2035\ldots$, and negative for $r > r_0$.

**Proof.** By Definition 3.4, (37) and (38), and letting $a, b, c,$ and $d$ be as in (40),

$$a = \gamma_3, \quad b = 3\gamma_2, \quad c = 3\gamma_1, \quad \text{and} \quad d = \gamma_0.$$

Then, by (40),

$$b^2e^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

$$= 81\gamma_2^2\gamma_1^4 - 108\gamma_3\gamma_1^3 - 108\gamma_3^3\gamma_0 - 27\gamma_3^2\gamma_0^2 + 162\gamma_3\gamma_2\gamma_1\gamma_0,$$

which, after a great deal of simplification, may be rewritten as

$$25600 \Delta(r) = -3 \pi r^2 \text{erf}^2(r) e^{-2r^2} (4r^4 - 20r^2 + 15)^2 + 20 r^3 \text{erf}(r) \sqrt{\pi} e^{-3r^2} (8r^6 + 108r^4 - 270r^2 + 135) + 180 r^4 e^{-4r^2} (4r^4 + 20r^2 - 15).$$

Using Mathematica, we observe

$$25600 \Delta \left( \frac{22035}{10000} \right) = 0.00037 05357 66894 07450 \ldots$$

$$25600 \Delta \left( \frac{22036}{10000} \right) = -0.00024 69444 73589 70548 \ldots$$

By the continuity of $\Delta(r)$, this establishes the existence of a zero in the interval $(2.2035, 2.2036)$. Multiplying $\Delta(r)$ by $e^{2r^2}$ has no effect on its sign and

$$12800 \frac{d}{dr} e^{2r^2} \Delta(r)$$

$$= -15\pi r \text{erf}^2(r) (16r^8 - 128r^6 + 312r^4 - 240r^2 + 45)$$

$$- 4\sqrt{\pi} r^2 \text{erf}(r) e^{-r^2} (64r^8 + 120r^6 - 2460r^4 + 3150r^2 - 675)$$

$$- 20r^3 e^{-2r^2} (64r^6 + 108r^4 - 540r^2 + 135).$$

Let

$$p_1(r) = 16r^8 - 128r^6 + 312r^4 - 240r^2 + 45,$$

$$p_2(r) = 64r^8 + 120r^6 - 2460r^4 + 3150r^2 - 675, \quad \text{and}$$

$$p_3(r) = 64r^6 + 108r^4 - 540r^2 + 135.$$
We now establish that \( \frac{d}{dr} e^{2r^2} \Delta(r) \) is negative for \( r > 2.11 \). Observe

\[
12800 \frac{d}{dr} e^{2r^2} \Delta(r) = \left[ -15\pi r \operatorname{erf}^2(r) \right] p_1(z) + \left[ -4\sqrt{\pi} r^2 \operatorname{erf}(r) e^{-r^2} \right] p_2(z) + \left[ -20r^3 e^{-2r^2} \right] p_3(z),
\]

and each of the bracketed terms is a function which is negative for \( r > 0 \). Then it is sufficient to show that \( p_k(r) > 0 \), \( k = 1, 2, 3 \), for \( r > 2.11 \).

Each \( p_k(r) \), \( k = 1, 2, 3 \), has a positive leading coefficient. Then, if all of the real roots of \( p_k(r) \) occur inside the interval \( [-2.11, 2.11] \) we are done. Using Sturm’s Theorem (see [43, pp. 336-339]), we find that \( p_1(z) \) has eight zeros in the interval \( [-2.11, 2.11] \), \( p_2(z) \) has six real zeros, all of which lie in the interval \( [-2.11, 2.11] \), and \( p_3(z) \) has four real zeros, all of which lie in the interval \( [-2.11, 2.11] \).

The following plot is of the function \( e^{2r^2} \Delta(r) \).

![Plot of function](image)

We now prove this section’s main result.

**Theorem 3.6.** For \( r > 2.036 \), the finite Fourier transform of the Gaussian,

\[
f(z) = \int_{-r}^{r} e^{-t^2} e^{izt} dt = \int_{-r}^{r} e^{-t^2} \cos z t dt,
\]

has non-real zeros.

**Proof.** Let

\[
h(z) = f(-i\sqrt{z}) = \int_{-r}^{r} e^{-t^2} \cos (-i\sqrt{zt}) dt.
\]  

(41)

We note that \( f(z) \) has only real zeros if and only if \( e^z h(z) \) has all of its zeros in the interval \( (-\infty, 0] \). In addition, since \( f(z) \) is even and order one by the Paley-Wiener Theorem (Theorem 1.13), \( h(z) \) is of order \( 1/2 \). That is, \( f(z) \) has only real zeros if and only if \( e^z h(z) \notin L-P(-\infty, 0] \).

By Lemma 3.3 the cubic polynomial

\[
g_3(z) = \sum_{\ell=0}^{3} \binom{3}{\ell} \gamma_{\ell} z^\ell,
\]

with \( \gamma_{\ell} \) as in (37) and (38), is the third Jensen polynomial of \( e^z h(z) \). By Lemma 3.5, \( g_3(z) \) has non-real zeros when \( r > 2.036 \). Then, by Theorem 1.27, \( e^z h(z) \notin L-P \) and, thus, \( f(z) \) has non-real zeros.

\( \square \)
3.2 Zeros of the Finite Fourier Transform of the Gaussian when $r < 2.2036$

By Proposition 2.28, for $r \leq \sqrt{1/2}$, the finite transform of $e^{-t^2}$ has only real zeros.

We now know that for sufficiently large $r$, $r$ greater than approximately 2.2036, the finite transform has non-real zeros. These facts lend themselves to the natural question:

**Question 3.7.** What is the infimum of the set of $r$'s for which the finite Fourier transform,

$$
\int_{-r}^{r} e^{-t^2} e^{izt} dt = \int_{-r}^{r} e^{-t^2} \cos zt dt,
$$

has non-real zeros?

We assault this question with the aid of numerical methods. To this end, we state two results which allow us to narrow our search for the real zeros of the Jensen polynomials of the finite Fourier transform of the Gaussian (cf. [12, Lemma 2.2 and Theorem 2.3]).

**Lemma 3.8.** Let $f(z) = \sum_{k=0}^{\infty} \gamma_k z^k/k!$ be a transcendental entire function in $\mathcal{L}-\mathcal{P}$. Suppose that the product representation of $f(z)$ has the form

$$
f(z) = cz^m e^{bz} \prod_{k=1}^{\omega} \left(1 + \frac{z}{z_k}\right), \quad 0 \leq \omega \leq \infty,
$$

where $b \geq 0$, $z_k > 0$, $c > 0$ and $\sum_{k=0}^{\omega} 1/z_k < \infty$ and where $s$ is a non-negative integer. Then $b \geq 1$ if and only if $0 \leq \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$.

**Remark 3.9.** If $f(z)$, as in (42), has only real zeros then the function $e^zh(z)$ as defined in (36) is of the form (43) with $b = 1$ and so has positive, non-decreasing coefficients by Lemma 3.8.

**Theorem 3.10.** Let $f(z) = \sum_{k=0}^{\infty} \gamma_k z^k/k!$ be a transcendental entire function in $\mathcal{L}-\mathcal{P}^+$. For each $n = 1, 2, 3, \ldots$, let $g_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k$ denote the $n^{th}$ Jensen polynomial associated with $f(z)$. Then, for all $n$, the zeros of $g_n(z)$ all lie in the interval $[-1, 0]$ if and only if $0 \leq \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$.

These results imply that all of the zeros of the Jensen polynomials of $e^zh(z)$, as defined in equation (41) of Theorem 3.6, must occur in the interval $[-1, 0]$ if the original function $f(z) \in \mathcal{L}-\mathcal{P}$. Based on this fact, numerical evidence seems to support a lower bound of $r$ which guarantees the existence of non-real zeros that is somewhat less than the bound given by the work in the previous section. Some of this evidence appears in the following plots of the behavior of real zeros of the the Jensen polynomials of the transform (42) as they depend upon the bound of integration, $r$. 

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Where the horizontal axis represents the endpoints of the region of integration, ±r, and the vertical axis the location of the real zeros of \( g_4(z) \). Based on this plot, it appears that, by analysis of the quartic Jensen polynomial associated with \( e^zf(z) \), it appears that the transform has non-real zeros for \( r \geq 2 \).

While the quartic polynomial may be solved exactly, it is, of course, not necessarily the case that this will hold for higher degree polynomials. However, again, numerical evidence seems to support the claim that, at least, for \( r > 1.7 \) the transform will have non-real zeros. Looking at the location of the real zeros of the 16th Jensen polynomial of \( e^zh(z) \):

it can be seen that for \( r > 1.7 \), only 14 or fewer zeros fall in the interval \([-1,0]\).

We conclude this section by recalling, the still open, Problem 1.6 from [16] in the manner originally stated. This particular statement of the problem is related to the results that appear in Chapter 6.

**Question 3.11.** For each positive integer \( n \), set \( K_n(t) = \exp\{-t^{2n}\} \). Let

\[
  f(z, n) := \int_0^\infty K_n(t) \cos z t \, dt \quad \text{and} \quad f_r(z, n) := \int_0^r K_n(t) \cos z t \, dt.
\]

Then the Taylor series expansion of the even entire function, \( f(z, n) \), is given by

\[
  f(z, n) = \frac{1}{2n} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\frac{2k+1}{2n})}{\Gamma(2k+1)} z^{2k},
\]
where $\Gamma(z)$ denotes the Gamma function. Using the theory of multiplier sequences, Pólya [34, Theorem 2] has shown $f(z, n) \in \mathcal{L-P}$ for $n = 1, 2, 3, \ldots$. Thus, the question is: For which values of $r > 0$ is the finite Fourier transform $f_r(z, n) \in \mathcal{L-P}$?
4 On the Reality of the Zeros of Two Classes of Functions Closely Related to the Fourier Transform

We begin with Problem 174 in Chapter V, Part 3 of G. Pólya and G. Szegő (see [40, p. 65]). This result introduces Theorem 1 in R. Duffin and A. Schaeffer’s 1938 paper [20], which is the main focus of this section. Both results are related to functions of the form

\[ \sin z - f(z) \quad \text{or} \quad \cos z - f(z) \]

where \( f(z) \) is a transcendental real entire function of sufficiently slow growth (made more precise in Proposition 4.3 and Theorem 4.6).

We now define what it means for an entire function to be of exponential type.

**Definition 4.1.** An entire function \( f(z) \) is said to be of exponential type \( \alpha \), if there exist positive constants \( A \) and \( z_0 \) such that

\[ |f(z)| \leq Ae^{\alpha|z|}, \]

whenever \( |z| > z_0 \) (cf. Definitions 1.8 and 1.9).

### 4.1 The Function \( \sin z - f(z) \) Has Only Real Zeros When \( f(z) \) is a Finite Fourier Transform of \( K(t) \), \( |K(t)| \leq 1 \)

We begin G. Pólya and G. Szegő’s Problem 174 with a lemma which is akin to their Problem 27 of the same section (see [40, p. 39]).

**Lemma 4.2.** Let \( a_0, a_1, \ldots, a_n \in \mathbb{R} \) such that \( |a_0| + |a_1| + \cdots + |a_{n-1}| \leq |a_n| \). Then the trigonometric polynomials

\[
\begin{align*}
  p_c(z) &= \sum_{k=0}^{n} a_k \cos k\zeta \\
  p_s(z) &= \sum_{k=1}^{n} a_k \sin k\zeta
\end{align*}
\]

have only real zeros.

**Proof.** Let \( p(z) = \sum_{k=0}^{n} a_k z^k \) and suppose \( |z| > 1 \). Then

\[
\begin{align*}
|a_n||z|^n &> |a_0||z|^n + |a_1||z|^n + \cdots + |a_{n-1}||z|^n \\
&\geq |a_0| + |a_1||z| + \cdots + |a_{n-1}||z|^{n-1} \\
&\geq |a_0 + a_1z + \cdots + a_{n-1}z^{n-1}|.
\end{align*}
\]

So that

\[
|a_n z^n + a_0 + a_1z + \cdots + a_{n-1}z^{n-1}| \geq |a_n z^n| - |a_0 + a_1z + \cdots + a_{n-1}z^{n-1}| > 0.
\]

That is, all of the zeros of \( p(z) \) must be on the closed unit disk. Then, by Theorem 2.18, the trigonometric polynomials \( p_c(z) \) and \( p_s(z) \) have all of their zeros on the real axis. \( \square \)
Proposition 4.3 (G. Pólya and G. Szegő [40, Problem 173]). Suppose \( K(t) \in C[0,1] \). If
\[
\int_0^1 |K(t)| \, dt \leq 1,
\]
then the entire function
\[
f(z) = \sin z - \int_0^1 K(t) \sin z \, dt
\]
has only real zeros.

Proof. Consider the Riemann sums approximating the integral in (44),
\[
\int_0^1 |K(t)| \, dt = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left| K\left( \frac{j+1}{n} \right) \right|. \tag{46}
\]
Each term in the sum in (46) is positive and \( \int_0^1 |K(t)| \, dt \leq 1 \), so
\[
\frac{1}{n} \sum_{j=0}^{n-1} \left| K\left( \frac{j+1}{n} \right) \right| \leq 1
\]
for every \( n \). Now,
\[
f(z) = \sin z - \int_0^1 K(t) \sin z \, dt
\]
\[
= \sin z - \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} K\left( \frac{j+1}{n} \right) \sin \left( \frac{z(j+1)}{n} \right), \tag{47}
\]
and the convergence is uniform on compact subsets of the complex plane. By Lemma 4.2 and the substitution \( u = z/n \), the trigonometric polynomials in (47),
\[
\sin nu - \frac{1}{n} \sum_{j=0}^{n-1} K\left( \frac{j+1}{n} \right) \sin \left( u(j+1) \right), \text{ with } n = 1, 2, 3, \ldots,
\]
have only real zeros. This implies
\[
\sin z - \frac{1}{n} \sum_{j=0}^{n-1} K\left( \frac{j+1}{n} \right) \sin \left( \frac{z(j+1)}{n} \right)
\]
has only real zeros for all \( n \in \mathbb{N} \). Then taking \( n \to \infty \), by Hurwitz’s Theorem (Theorem 1.21), \( f(z) \) has only real zeros. \( \square \)

By factoring a negative one out of \( K(t) \) in (45), the following corollary follows immediately.

Corollary 4.4. Suppose \( K(t) \in C[0,1] \). If
\[
\int_0^1 |K(t)| \, dt \leq 1,
\]
then the entire function
\[
f(z) = \sin z \pm \int_0^1 K(t) \sin z \, dt
\]
has only real zeros.
Corollary 4.4 allows us to prove Proposition 2.29 from Section 2.3.

*Proof of Proposition 2.29.* Multiplying by $z$ and integrating by parts once,

$$z \int_0^r K(t) \cos zt \, dt = K(r) \sin rz - \int_0^r K'(t) \sin zt \, dt.$$  

Dividing both sides by $K(r)$ and substituting $rs = t$,

$$z \int_0^r \frac{K(t)}{K(r)} \cos zt \, dt = \sin rz - \int_0^r \frac{K'(t)}{K(r)} \sin zt \, dt$$

$$= \sin rz + r \int_0^1 \frac{K'(rs)}{K(r)} \sin zrs \, ds$$

$$= \sin u + r \int_0^1 \frac{K'(rs)}{K(r)} \sin us \, ds, \quad (u = rz).$$

(48)

Then, by Corollary 4.4, if

$$\int_0^1 \left| \frac{K'(rt)}{K(r)} \right| \, dt \leq \frac{1}{r},$$

then (48) and, thus, $\int_0^r K(t) \cos zt \, dt$ have only real zeros. \(\square\)

### 4.2 The Function $\cos z - f(z)$ Has Only Real Zeros When $f(z)$ is a Finite Fourier Transform of $K(t)$, $|K(t)| \leq 1$

The result of R. Duffin and A. Schaeffer [20] is more general than the title of this section indicates. We will prove it in its full generality and then apply this result to the case when $f(z)$ is a finite Fourier transform.

We begin by recalling Rouché’s Theorem.

**Theorem 4.5** (Rouché’s Theorem [5, p. 125]). If $R$ is an open simply connected region in the complex plane and if $f(z)$ and $g(z)$ are two functions which are analytic on $R$ and, on the boundary of $R$, satisfy the inequality

$$|g(z)| < |f(z)|,$$

then the function $f(z) + g(z)$ has the same number of zeros as $f(z)$ inside of $R$.

We now prove R. Duffin and A. Schaeffer’s main result.

**Theorem 4.6.** Let $f(z)$ be a real entire function. If $f(z)$ is of exponential type $\sigma$ for some $\sigma > 0$, then for every real number $a$ the function

$$\cos(\sigma z + a) \pm f(z)$$

has only real zeros or vanishes identically. Moreover, all of the zeros are simple except, perhaps, at points on the real axis where $f(x) = \pm 1$.

*Proof.* We consider the case when $a = 0$. Let $1 > \epsilon > 0$, $z = x + iy \neq 0$, and

$$f_\epsilon(z) = \frac{\sin \sigma \epsilon z}{\sigma \epsilon z} (1 - \epsilon) f((1 - \epsilon) z).$$

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Then
\[ |f_\epsilon(z)| = \left| \frac{\sin \sigma \epsilon z}{\sigma \epsilon z} (1 - \epsilon) f((1 - \epsilon)z) \right| \]
\[ \leq \frac{e^{\sigma|y|}}{\sigma \epsilon |z|} (1 - \epsilon) e^{\sigma|y|(1 - \epsilon)} \]
\[ < \frac{e^{\sigma|y|}}{\sigma \epsilon |z|}. \]  
(49)

Let \( y_0 = 1/(\sigma \epsilon) \), then for \( |y| > y_0 \)
\[ |f_\epsilon(z)| < |\cos \sigma z|. \]

In particular, this inequality holds on any horizontal line \( y, |y| > y_0 \), in the complex plane. Let \( \beta > y_0 \). Also by the inequality in (49),
\[ |f_\epsilon(z)| < |\cos \sigma z| \]  
(50)
when \( x = n\pi/\sigma \) and \( |x| > y_0 \). The inequality in (50) also holds on a vertical lines in the complex plane which pass through such \( x = n\pi/\sigma \). Let \( n_0 \in \mathbb{N} \) and \( \alpha = n_0 \pi/\sigma \). Then we form the rectangle \( R \) with the sides at the vertical lines \( x = \pm \alpha \) and top and bottom at the horizontal lines \( y = \pm \beta \). All around the boundary of this rectangle
\[ |f_\epsilon(z)| < |\cos \sigma z|. \]

Thus, by Rouché’s Theorem, \( f_\epsilon(z) \pm \cos \sigma z \) has the same number of zeros inside \( R \) as \( \cos \sigma z \), namely \( 2n \). However, by \( |f_\epsilon(z)| < 1 \) on the real axis,
\[ f_\epsilon(z) \pm \cos \sigma z \]
alters the sign when
\[ z = \ldots, \frac{-2\pi}{\sigma}, \frac{-\pi}{\sigma}, 0, \frac{\pi}{\sigma}, \frac{2\pi}{\sigma}, \ldots \]
Since \( f_\epsilon(z) \pm \cos \sigma z \) is real-valued for real \( z \), this function must have \( 2n \) real zeros between \( x = -\beta \) and \( x = \beta \). Then, since \( f_\epsilon(z) \pm \cos \sigma z \) has at most \( 2n \) zeros, we have shown that the function has only real zeros. Taking the limit as \( \epsilon \to 0 \),
\[ \lim_{\epsilon \to 0} f_\epsilon(z) \pm \cos \sigma z = f(z) \pm \cos \sigma z, \]
which is either identically zero or has only real zeros by Hurwitz’s Theorem (Theorem 1.21).

This result extends to \( a \neq 0 \) by adjusting the the sides of \( R \) as necessary and applying the same argument. \( \Box \)

R. Duffin and A. Schaeffer state their result somewhat differently. They begin with a lemma which extends Theorem 4.6 to functions which are bounded in absolute value by one on the real axis and of exponential type. As the following results will show, these conditions are sufficient to show that a function satisfies the hypotheses of the version of Theorem 4.6 stated above.

This extension follows from the Phragmén-Lindelöf Principle, which we state now (cf. [27, p. 50]).
Theorem 4.7 (Phragmén-Lindelöf Principle). Let $\alpha$ be a real number. Suppose $f(z)$ is analytic in the half-plane $\alpha < \arg z < \alpha + \pi$ and continuous on the boundary of the half-plane. If $|f(z)| \leq 1$ for $z$ on the boundary and $|f(z)| \leq e^{\sigma|z|}$ for $z$ inside the half-plane, then

$$|f(z)| \leq e^{\sigma|y|}$$

for all $z = x + iy$ inside or on the boundary of the half-plane.

We will also need the following corollary [27, p. 51].

Corollary 4.8. If the entire function $f(z)$ is, at most, of order one and minimal type and if its modulus is bounded on a line, then it is a constant.

Proposition 4.9. If $f(z)$ is of exponential type $\sigma$ for some $\sigma > 0$, and if $|f(x)| \leq 1$ for real $x$, then

$$|f(z)| \leq e^{\sigma|y|}, \quad z = x + iy.$$ 

In other words, $f(z)$ satisfies the hypothesis of Theorem 4.6.

Proof. By the Phragmén-Lindelöf Principle, $|f(z)| \leq e^{\sigma|y|}$ in the closed upper-half plane. Similarly, the result shows the same bound for $|f(z)|$ in the lower-half plane.

Corollary 4.10. If $\int_0^r |K(t)| \ dt \leq 1$ and $f(z)$ is the finite Fourier transform of $K(t)$ over the interval $[-r, r]$, then $\cos(rz + a) \pm f(z)$ is either identically zero or has only real zeros for every real number $a$.

Proof. By the hypothesis that $\int_0^r |K(t)| \ dt \leq 1$, then, for real $x,

$$|f(x)| = \left| \int_0^r K(t) \cos xt \ dt \right| \leq \int_0^r |K(t)| \ dt \leq 1.$$ 

By the Paley-Wiener Theorem (Theorem 1.13), a finite Fourier transform over the interval $[-r, r]$ may be bounded by

$$|f(z)| \leq e^{\sigma|y|}, \quad z = x + iy.$$ 

This shows that $f(z)$ satisfies the hypothesis of Theorem 4.6, and, thus,

$$\cos rz \pm f(z)$$

is either identically zero or has only real zeros.

Remark 4.11. The following proposition shows that the infinite Fourier transforms grow at least as quickly as functions of order one and infinite type. Since it was fundamental to Theorem 4.6 that $f(z)$ be of order one and finite type, it appears that Corollary 4.10 will not directly extend to the infinite Fourier transform.

Proposition 4.12. If the infinite Fourier transform of $K(t)$,

$$f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} \ dt,$$

is an entire function and $K(t)$ is not of compact support on the real line, then $f(z)$ is either of order greater than one or is of order one and infinite type.

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Proof. The Fourier transform $f(z)$ is bounded on the real axis by the Riemann-Lebesgue Lemma (Theorem 1.4). Then, by Corollary 4.8, it is a constant if it is of order one and type zero or of order less than one. This is impossible as, again by the Riemann-Lebesgue lemma, $f(z)$ tends toward zero along the real line and is not identically zero by the hypothesis that $K(t)$ is not identically zero. Then $f(z)$ must be of at least order one and non-zero type or of type greater than one.

If $f(z)$ is of order one and type $r$, $0 < r < \infty$ then $K(t)$ is supported on the interval $[-r, r]$ by the Paley-Wiener Theorem (Theorem 1.13), contrary to the hypothesis that $K(t)$ is not of compact support on the real axis. That is, $f(z)$ must be of order one and infinite type or of order greater than one. \qed
5 The Finite Fourier Transform and Universal Factors

In 1927 G. Pólya [34] introduced a class of functions he termed *universal factors*. Let $K(t)$ be an even and real-valued function that is absolutely integrable over $\mathbb{R}$. Also, suppose, for $b > 2$,

$$K(t) = O\left(e^{-|t|^b}\right), \text{ as } t \to \pm \infty.$$  (51)

Universal factors are the collection of functions, $\{\varphi(t)\}$, such that if the integral

$$\int_{-r}^{r} K(t) e^{itz} dt, \quad r = \infty,$$  (52)

is an entire function with only real zeros then the integral

$$\int_{-r}^{r} \varphi(t) K(t) e^{itz} dt, \quad r = \infty,$$  (53)

is also an entire function with only real zeros.

G. Pólya was able to completely characterize the functions, $\varphi(t)$ in (53), that comprise this class. In this section we generalize this result, using a method of proof similar to G. Pólya’s, but using the weaker assumptions that $0 < r \leq \infty$ in expressions (52) and (53) and by allowing the requirement in (51) to be relaxed slightly.

5.1 Infinite Order Differential Operators

Naturally, we begin with some preliminary results. We proceed by defining an infinite order differential operator and stating several theorems regarding the location of zeros of functions acted upon by these operators.

**Definition 5.1.** Let $\varphi(z) = \sum_{k=0}^{\infty} \gamma_k z^k$ be a formal power series with $\gamma_k \in \mathbb{R}$ and let $D := \frac{d}{dz}$, be differentiation with respect to $z$. The action of the *infinite order differential operator*, $\varphi(D)$, on a function $f(z)$ is defined as

$$\varphi(D)f(z) = \sum_{k=0}^{\infty} \gamma_k f^{(k)}(z),$$  (54)

when the right hand side represents a analytic function in a neighborhood of the origin.

**Definition 5.2.** A *finite order differential operator* is defined as in (54) with the restriction that $\varphi(z) = \sum_{k=0}^{n} \gamma_k z^k$ be a polynomial.

For the results of interest in this chapter, we must first dispense with the rather unsatisfying phrase in Definition 5.1, “... when the right hand side represents an analytic function in the neighborhood of the origin.” In particular, we wish to have conditions under which (54) is an entire function. Lemma 3.1 of [11] provides just such conditions. We state this result now.
Lemma 5.3. Let \( \varphi(z) = e^{\alpha_1 z^2} \varphi_1(z) \) and \( f(z) = e^{\alpha_2 z^2} f_1(z) \) where \( \alpha_i \in \mathbb{R} \) such that \( |\alpha_1 \alpha_2| < 1/4 \) and \( \varphi_1(z) \) and \( f_1(z) \) are of genus no larger than one. Then \( \varphi(D)f(z) \) is an entire function of order less than or equal to two.

Remark 5.4 (cf. [11, p. 7], [33, p. 243]). The assumption that \( |\alpha_1 \alpha_2| < 1/4 \) in Lemma 5.3 is necessary. Let \( \varphi(z) = e^{-\alpha_1 z^2} \) and \( f(z) = e^{-\alpha_2 z^2} \) where \( \alpha_1, \alpha_2 > 0 \). Then

\[
\varphi(D)f(z)|_{z=0} = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2} (\alpha_1 \alpha_2)^k,
\]

which converges if and only if \( |\alpha_1 \alpha_2| < 1/4 \).

Now, briefly setting aside the issue of convergence, we recall the Hermite-Poulain Theorem (cf. [31, p. 4]). Stated here in terms of finite differential operators, it provides a sufficient condition for such an operator to preserve the reality of zeros of a polynomial with only real zeros.

Notation 5.5. Recall that \( Z_c(f(z)) \) denotes the number of non-real zeros of the function \( f(z) \).

Theorem 5.6 (Hermite-Poulain). If \( f(z) \) is a real polynomial and

\[
g(x) = \sum_{k=0}^{n} \gamma_k x^k
\]

is a real polynomial with only real zeros, then for

\[
g(D)f(z) = \sum_{k=0}^{n} \gamma_k f^{(k)}(z),
\]

we have

\[
Z_c(g(D)f(z)) \leq Z_c(f(z)).
\]

Granting convergence, this result may be extended to \( f(z) \) and \( \varphi(z) \) real entire functions with only real zeros. The following result is an adaptation of Lemma 3.2 in [11].

Theorem 5.7. Suppose \( \varphi(z) = \sum_{k=0}^{\infty} \gamma_k z^k \) and \( f(z) \in \mathcal{L-P} \). Then the differential operator \( \varphi(D) \), given by Definition 5.1, when acting on \( f(z) \) preserves the reality of zeros where the result converges. That is, \( \varphi(D)f(z) \) is a function, real-valued for real \( z \), with only real zeros on any compact subset of the region on which \( \varphi(D)f(z) \) defines an analytic function.

Proof. Let \{\( \varphi_n(z) \)\} and \{\( f_j(z) \)\} denote the sequences of Jensen polynomials associated with \( \varphi(z) \) and \( f(z) \), respectively, and let

\[
\varphi_n(z) = \sum_{k=0}^{n} \gamma_{n,k} z^k.
\]

Since \( \varphi(z) \) and \( f(z) \in \mathcal{L-P} \), \( \varphi_n(z) \) and \( f_j(z) \) are real polynomials with only real zeros by Theorem 1.27. Then, by Theorem 5.6, \( \{\varphi_n(D)f_j(z)\}_{j=0}^{\infty} \) is a sequence of
polynomials with only real zeros for all \( n \in \mathbb{N} \). Then, since \( f_j^{(k)}(z) \to f^{(k)}(z) \) as \( j \to \infty \) uniformly on compact subsets of \( \mathbb{C} \) for each \( k \in \mathbb{N} \) so does the finite sum

\[
\varphi_n(D)f_j(z) = \sum_{k=0}^{n} \gamma_{n,k} f_j^{(k)}(z) \to \varphi_n(D)f(z)
\]

uniformly on compact subsets of \( \mathbb{C} \) as \( j \to \infty \). Thus, by Hurwitz’s Theorem (Theorem 1.21), we know that \( \varphi_n(D)f(z) \in \mathcal{L}P \), for each \( n \in \mathbb{N}_0 \).

Suppose that \( \varphi(D)f(z) \) defines an analytic function on an open set \( R, R \subset \mathbb{C} \). If we can show that \( \varphi_n(D)f(z) \to \varphi(D)f(z) \) uniformly on compact subsets of \( R \), then, by Hurwitz’s Theorem, \( \varphi(D)f(z) \) has only real zeros. By Definition 5.1,

\[
\varphi(D)f(z) = \sum_{k=0}^{\infty} \gamma_k f^{(k)}(z).
\]

Let \( \epsilon > 0 \) and \( C \subset R \), \( C \) compact. By the assumption that \( \varphi(D)f(z) \) defines an analytic function on \( R \), for \( z \in C \), there exists \( N_1 \in \mathbb{N} \) such that for all \( N \geq N_1 \), \( N \in \mathbb{N} \),

\[
\sum_{k=N}^{\infty} |\gamma_k| |f^{(k)}(z)| < \frac{\epsilon}{3}, \quad (56)
\]

By the definition of the Jensen polynomials associated with \( \varphi(z) \)

\[
\varphi_n \left( \frac{D}{n} \right) f(z) = \gamma_0 f(z) + \gamma_1 f'(z) + \sum_{k=2}^{n} \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) \gamma_k f^{(k)}(z),
\]

since \( \gamma_{n,0} = \gamma_0, \gamma_{n,1} = n \gamma_1 \), and \( \gamma_{n,k} = (n-1) \cdots (n-(k-1)) \gamma_k \) when \( k > 2 \). Let \( N \geq N_1 \). Then (56) also indicates, when \( n > N \),

\[
\sum_{k=N}^{n} \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) |\gamma_k| |f^{(k)}(z)| < \frac{\epsilon}{3}, \quad (57)
\]

since \( 0 < \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) < 1 \). Finally, there exists \( N_2 \in \mathbb{N} \), \( N_2 > N \), such that, for \( n \geq N_2 \) and \( z \in C \),

\[
\left| \sum_{k=2}^{n} \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) \gamma_k f^{(k)}(z) - \sum_{k=2}^{n} \gamma_k f^{(k)}(z) \right| < \frac{\epsilon}{3}, \quad (58)
\]

Putting these three inequalities together, (56), (57), and (58), when \( n > N_2 \) and \( z \in C \),

\[
\left| \varphi_n \left( \frac{D}{n} \right) f(z) - \varphi(D)f(z) \right|
\]

\[
= \left| \sum_{k=2}^{n} \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) \gamma_k f^{(k)}(z) - \sum_{k=2}^{\infty} \gamma_k f^{(k)}(z) \right|
\]

\[
= \left| \sum_{k=2}^{N} \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) \gamma_k f^{(k)}(z) - \sum_{k=2}^{N} \gamma_k f^{(k)}(z) \right|
\]

\[
+ \left| \sum_{k=N+1}^{n} \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) \gamma_k f^{(k)}(z) \right|
\]

\[
- \sum_{k=N}^{\infty} \gamma_k f^{(k)}(z) \right|
\]

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\[ \leq \left| \sum_{k=2}^{N} \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) \gamma_k f^{(k)}(z) - \sum_{k=2}^{N} \gamma_k f^{(k)}(z) \right| \\
+ \left| \sum_{k=N+1}^{n} \left( 1 - \frac{1}{n} \right) \cdots \left( 1 - \frac{k-1}{n} \right) \gamma_k f^{(k)}(z) \right| \\
+ \left| \sum_{k=N}^{\infty} \gamma_k f^{(k)}(z) \right| \\
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \]

\[5.2\] Characterization of Pólya’s Universal Factors

In this section we break Pólya’s characterization of universal factors into Theorems 5.10 and 5.18, with an extension which shows that the characterization of universal factors holds not only for the infinite Fourier transform, but also the finite Fourier transform. We also show that Pólya’s growth restriction on the kernel, given by (51), is slightly more than necessary (as he notes in his paper).

Our extensions require a slight augument to the definition of universal factor given at the start of Section 5.2. We begin by defining our term universal factors of class \( r \).

**Notation 5.8.** Let \( 0 < r \leq \infty \) and \( \mathcal{F}_r \) denote the class of functions \( \{ K(t) \} \) such that \( K(t) : [-r, r] \rightarrow \mathbb{R} \) is absolutely integrable, and

\[ f(z) = \int_{-r}^{r} K(t) e^{izt} \, dt \in L^1. \tag{59} \]

If \( r = \infty \), then we require, in addition, \( K(t) = O\left( e^{-|t|^b} \right) \), as \( t \rightarrow \pm \infty \), for some \( b > 2 \).

**Definition 5.9.** We say a real analytic function \( \varphi(t) \) is a universal factor of class \( r \) if \( K(t) \in \mathcal{F}_r \) implies

\[ \int_{-r}^{r} \varphi(t)K(t) e^{izt} \, dt \in L^1. \]

The characterization of functions \( \varphi(t) \) that are universal factors comes, in part, from the recasting of these functions as differential operators which are known to preserve the reality of zeros of entire functions via Theorem 5.7.

**Theorem 5.10.** Let \( 0 < r \leq \infty \). If \( \varphi(iz) \in L^1 \), then \( \varphi(t) \) is a universal factor of class \( r \).

**Proof.** Let \( \varphi(z) = \sum_{n=0}^{\infty} a_n z^n \). Note that, by Definition 5.1 and Leibniz’s integral
\[ \varphi(-iD) \int_{-r}^{r} K(t) e^{izt} dt = \sum_{n=0}^{\infty} a_n \left( -i \frac{d}{dz} \right)^n \int_{-r}^{r} K(t) e^{izt} dt \]  
\[ = \sum_{n=0}^{\infty} \int_{-r}^{r} a_n \left( -i \frac{d}{dz} \right)^n e^{izt} K(t) dt \]  
\[ = \sum_{n=0}^{\infty} \int_{-r}^{r} a_n t^n K(t) e^{izt} dt \]  
\[ = \int_{-r}^{r} \varphi(t) K(t) e^{izt} dt. \]  

When \( r = \infty \), Theorem 1.18 tells us that \( f(z) \), in (59), is of order strictly less than two and so \( \varphi(-iD)f(z) \) exists by Lemma 5.3. If \( r < \infty \), then the Paley-Wiener Theorem (Theorem 1.13) implies that \( f(z) \) is of order one and so, again, Lemma 5.3 ensures \( \varphi(-iD)f(z) \) converges.

By the hypothesis that \( \varphi(-iz) \in \mathcal{L} \cdot \mathcal{P} \), Theorem 5.7 allows us to conclude the function in (60) has only real zeros.

The hypothesis
\[ |K(t)| = \mathcal{O}\left( e^{-|t|^b} \right), \quad \text{as} \ t \to \pm \infty, \quad \text{for some} \ b > 2, \]
in the case when \( r = \infty \), as in Notation 5.8, is precisely the hypothesis G. Pólya used to bound the growth of \( K(t) \). We now weaken this requirement slightly by adding additional constraints on the growth of the functions \( \varphi(z) \).

**Corollary 5.11.** Let \( \varphi(iz) \in \mathcal{L} \cdot \mathcal{P} \) and set
\[ \varphi(z) = e^{\alpha \varphi t^2} \varphi^*(z), \]
where \( \alpha \varphi \) is real and \( \varphi^*(z) \) is a function of order less than two. Let \( K(t) \) be a real-valued function which is integrable over the real line and suppose
\[ K(t) = \mathcal{O}\left( e^{-\alpha_K t^2} \right), \quad \text{as} \ t \to \pm \infty, \]
for some \( \alpha_K > 0 \). If
\[ f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} dt \in \mathcal{L} \cdot \mathcal{P} \]
and \( \alpha \varphi / \alpha_K < 1 \), then
\[ \int_{-\infty}^{\infty} \varphi(t) K(t) e^{izt} dt \in \mathcal{L} \cdot \mathcal{P}. \]

**Proof.** By Proposition 1.19 and Remark 1.20, there exists \( \alpha_f \geq 0 \) real such that \( f(z) = e^{-\alpha_f t^2} f^*(t) \) and \( f^*(t) \) is a function of order strictly less than two. Again by Proposition 1.19,
\[ |\alpha \varphi \alpha_f| \leq \frac{|\alpha \varphi|}{4 \alpha_K} < 1/4, \]
which implies the conditions of Lemma 5.3 are satisfied and, thus, \( \varphi(-iD)f(z) \) exists. The remainder of the proof is exactly as in Theorem 5.10. \( \square \)
Remark 5.12. Let $\alpha \geq 1/2$ and $\varphi(t) = e^{\alpha t^2}$. Then, by
\[ e^{-z^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} e^{izt} dt, \]
and (55) in Remark 5.4, $\varphi(D)f(z)$ does not converge. Thus the requirement in Corollary 5.11 that $|\alpha_\varphi/\alpha_K| < 1$ is sharp. Alternatively,
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) e^{-t^2/2} e^{izt} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\alpha-t)^2/2} e^{izt} dt, \]
which does not converge.

In addition, now looking at the finite transform, we have established the following corollary.

Corollary 5.13. The finite transform of any function such that $\varphi(iz) \in \mathcal{L}\cdot\mathcal{P}$, as in Theorem 5.10, is an entire function with only real zeros.

Proof. Note that the characteristic function of the interval $[-r,r]$, $0 < r < \infty$, satisfies the hypothesis required of $K(t)$ as
\[ \int_{-r}^{r} e^{izt} dt = \frac{2 \sin(rz)}{z}. \]  
Then, by Theorem 5.6, $\varphi(iD)\frac{2\sin(rz)}{z} \in \mathcal{L}\cdot\mathcal{P}$. \qed

Theorem 5.10 also implies that, in the case of the infinite Fourier transform, when the kernel is a function which decays on the real axis in the manner indicated by (51) and has only imaginary zeros then the infinite Fourier transform has only real zeros. We now show that Theorem 5.10 yields another class of functions for which their infinite Fourier has only real zeros.

Theorem 5.14. If $K(t) = e^{-\alpha_1 t^2} K_0(t)$ where $\alpha_1 > 0$ and $K_0(t)$ is of the form
\[ K_0(z) = ce^{\alpha_0 z^2 + b z} z^m \prod_{k=1}^{\omega} \left(1 + \frac{z}{z_k}\right) e^{-z/z_k}, \]
where $b, c, iz_k \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $\alpha_1 > \alpha_0$, then
\[ f(z) = \int_{-\infty}^{\infty} K(t) e^{izt} dt \in \mathcal{L}\cdot\mathcal{P}. \]

Proof. Without loss of generality, we assume that $\alpha_0 = 0$. Otherwise we simply replace $\alpha_1$ with $\alpha_1 + \alpha_0$ and $\alpha_0$ with zero. Then
\[ f(z) = \int_{-\infty}^{\infty} K_0(t) e^{-\alpha_1 t^2} e^{izt} dt \]
\[ = K_0(iD) \int_{-\infty}^{\infty} e^{-\alpha_1 t^2} e^{izt} dt \]
\[ = K_0(iD) \frac{\sqrt{\pi}}{\sqrt{\alpha_1}} \exp \left\{ \frac{-z^2}{4\alpha_1} \right\}. \]
Since $K_0(z)$ is of order less than two by Lemma 5.3, $f(z)$ is an entire function of order less than or equal to two and by Theorem 5.7 has only real zeros, so $f(z) \in \mathcal{L}\cdot\mathcal{P}$. \qed
Stated a different way this yields another, known, necessary condition for a function to be in $\mathcal{L}$-$\mathcal{P}$.

**Corollary 5.15.** If $f(z) \in \mathcal{L}$-$\mathcal{P}$, then there exists $\alpha > 0$ such that
\[
\int_{-\infty}^{\infty} f(iz) e^{-at^2} e^{izt} \, dt \in \mathcal{L}$-$\mathcal{P}.
\]

**Remark 5.16.** We note that the converses of Corollaries 5.13 and 5.15 both fail. Consider, for example, the polynomial $(iz)^4 + (iz)^2 + 1$, which has non-real zeros. Since $t^4 + t^2 + 1$ is an even and increasing kernel for $t > 0$, by Proposition 2.20, the finite Fourier transform of $t^4 + t^2 + 1$,
\[
\int_{-r}^{r} (t^4 + t^2 + 1) e^{izt} \, dt,
\]
has only real zeros for all $r$, $0 < r < \infty$. In addition, for the infinite transform,
\[
\int_{-\infty}^{\infty} (t^4 + t^2 + 1) e^{-t^2} e^{izt} \, dt = \sqrt{\frac{n}{16}} e^{-z^2/4} (z^4 - 16z^2 + 36),
\]
which clearly has only real zeros.

In order to prove the converse of Theorem 5.10, we require a short lemma related to polynomials of a very specific form.

**Lemma 5.17.** Suppose that all of the zeros of the polynomial
\[
p(z) = a_0 + a_1 z + \cdots + a_j z^j + \cdots + a_n z^n
\]
depend on the real axis, let
\[
a_0 K(t+(n-2j)\delta) + a_1 K(t+(n-2j)\delta) + \cdots + a_j K(t+(2j-n)\delta) + \cdots + a_n K(t+n\delta)
\]
be $K^*(t)$.

If $r = \infty$, then the Fourier transform of $K^*(t)$ has only real zeros. On the other hand, if $r < \infty$ and $0 < a < r$, then there exists $\delta_0 > 0$ such that, for all $K(t)$ supported on $[-a, a]$, the Fourier transform of $K^*(t)$ has only real zeros.

**Proof.** If $p(z)$ has only zeros of modulus one, then the associated trigonometric polynomial,
\[
a_0 e^{iz\delta} + a_1 e^{iz(n-2j)\delta} + \cdots + a_k e^{iz(n-2k)\delta} + \cdots + a_n e^{iz(n-2n)\delta}
\]
has only real zeros.

By the hypothesis that $K(t)$ is integrable,
\[
\int_{-r}^{r} K^*(t) e^{izt} \, dt = \int_{-r}^{r} \left( \sum_{j=0}^{n} a_j K(t-(n-2j)\delta) \right) e^{izt} \, dt
\]
\[
= \sum_{j=0}^{n} a_j \int_{-r}^{r} K(t-(n-2j)\delta) e^{izt} \, dt
\]
\[
= \sum_{j=0}^{n} a_j \int_{-r-(n-2j)\delta}^{r-(n-2j)\delta} K(u) e^{iz(u+(n-2j)\delta)} \, du
\]
\[
= \left( \sum_{j=0}^{n} a_j e^{iz(n-2j)\delta} \right) \left( \int_{-r-(n-2j)\delta}^{r-(n-2j)\delta} K(u) e^{izu} \, du \right). \quad (62)
\]

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Now, if \( r = \infty \), then (62) becomes
\[
\left( \sum_{j=0}^{n} a_j e^{iz(n-2j)\delta} \right) \left( \int_{-r}^{r} K(u) e^{izu} du \right),
\]
which is the product of two functions, both of which have only real zeros. On the other hand, if \( r < \infty \), then \( K(t) \) is supported on \([-a, a]\) and we may take \( \delta_0 \) sufficiently small such that \( r - n \delta_0 > a \). Thus, (62) again becomes
\[
\left( \sum_{j=0}^{n} a_j e^{iz(n-2j)\delta} \right) \left( \int_{-r}^{r} K(u) e^{izu} du \right).
\]

\[ \square \]

**Theorem 5.18.** If the real analytic function \( \varphi(t) \) is a universal factor, then \( \varphi(iz) \in L-P \).

**Proof.** Let the power series expansion of \( \varphi(t) \), \( t \in \mathbb{R} \), be given by
\[
\varphi(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k.
\]
This uniquely determines an analytic continuation on the complex plane, \( \varphi(z) \).

Let \( n \in \mathbb{N} \) and let \( a \) be a positive number less than \( r \). Define \( K(t) = \frac{1}{2a} \chi_{[-a,a]}(t) \), where \( \chi_{A} \) denotes the characteristic function of the set \( A \subseteq \mathbb{C} \). By (61),
\[
\int_{-r}^{r} K(t) e^{itz} dt
\]
has only real zeros. With the observation that
\[
(1 - z)^n = 1 - nz + \cdots + (-1)^n \binom{n}{n} z^n
\]
has all of its zeros on the unit circle, by Lemma 5.17, there exists \( \delta_0 > 0 \) such that when \( 0 < \delta < \delta_0 \),
\[
\int_{-r}^{r} \left( \sum_{j=0}^{n} (-1)^j \binom{n}{j} K(t - (n-2j)\delta) \right) e^{itz} dt
\]
also has only real zeros. Then, by the hypothesis that \( \varphi(t) \) is a universal factor, the form of \( K(t) \), and the restriction on \( \delta \),
\[
\int_{-r}^{r} \varphi(t) \left( \sum_{j=0}^{n} (-1)^j \binom{n}{j} K(t - (n-2j)\delta) \right) e^{itz} dt
\]
\[
= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{2a} \int_{-a+(n-2j)\delta}^{a+(n-2j)\delta} \varphi(t) e^{itz} dt
\]

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converges and has only real zeros. Again, by Hurwitz’s Theorem (Theorem 1.21) and Lemma 5.17,

$$\lim_{a \to 0} \left( \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{2a} \int_{-a+(n-2j)\delta}^{a+(n-2j)\delta} \varphi(t) e^{izt} \, dt \right) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \varphi((n-2j)\delta) e^{iz(n-2j)\delta}$$

has only real zeros, so

$$\lim_{\delta \to 0} \frac{1}{(2\delta)^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \varphi((n-2j)\delta) e^{iz(n-2j)\delta} = \frac{d^n}{dt^n} \varphi(t) e^{izt} \bigg|_{t=0} = \sum_{k=0}^{n} \varphi^{(k)}(0) \binom{n}{k} (iz)^{n-k} = \sum_{k=0}^{n} a_k \binom{n}{k} (iz)^{n-k},$$

also has only real zeros. Then, making the change of variable $z \to z/n$, by Theorem 1.28 the sequence of polynomials, indexed by $n$,

$$\sum_{k=0}^{n} a_k \binom{n}{k} \left( \frac{iz}{n} \right)^k$$

has only real zeros for every $n \in \mathbb{N}$ and converges locally uniformly to $\varphi(iz)$ as $n \to \infty$. Thus, by Hurwitz’s Theorem, $\varphi(iz)$ has only real zeros.

We conclude this section with a question that stems from the proof of Theorem 5.18 and a remark which suggests a related paper.

**Question 5.19.** In the proof of Theorem 5.18, we required a very carefully selected class of functions, $K^*(t)$, from Lemma 5.17, in order to show that $\varphi(iz) \in \mathcal{L}-\mathcal{P}$. The question is, the answer to which could have profound impacts: What, if any, other kernels, $K(t)$, have the property that if $\varphi(t)$ is a real analytic function and

$$f(z) = \int_{-r}^{r} \varphi(t) K(t) e^{izt} \, dt \in \mathcal{L}-\mathcal{P},$$

then $\varphi(iz) \in \mathcal{L}-\mathcal{P}$.

**Remark 5.20.** We conclude this chapter by mentioning the related paper of N. G. de Bruijn [4], in which he studies the conditions under which two Fourier transforms with only real zeros,

$$f(z) = \int_{-\infty}^{\infty} e^{-t^2/2} K_f(t) e^{itz} \, dt \in \mathcal{L}-\mathcal{P} \text{ and } g(z) = \int_{-\infty}^{\infty} e^{-t^2/2} K_g(t) e^{itz} \, dt \in \mathcal{L}-\mathcal{P},$$

給予的頁面內容為數學證明和推導，涉及複變函數和對象的極限性質。文中使用了Hurwitz的定理和相關的引理來支持這些推導。結果表明，在某些條件下，對象有實零點，並且可以通過變換得到實零點的序列。這為相關問題和建議提供了基礎，並引導出進一步的討論和研究方向。
imply that their composition,

$$\varphi(z) = \int_{-\infty}^{\infty} e^{-t^2/2} K_f(t) K_g(t) e^{itz} dt,$$

is also in $L-\mathcal{P}$. In his paper, de Bruijn uses the theory of differential operators in a manner similar to that seen in the proof of Theorem 5.10.
6 Multiplier Sequences and the Distribution of Zeros of Fourier Transforms

Our final result is a second from Pólya’s 1927 paper (see Theorem 2 in [34]). This result is the previously mentioned extension of Proposition 2.8. Here, Pólya examines the kernel, $K(t)$, of a Fourier transform through the lens of the Mellin transform, but uses a very different connection than the change of variable in Proposition 6.2. Instead, he decomposes the Fourier transform into a linear operator, $T_{\gamma_k}$, acting on the Gaussian. Pólya uses the power of Laguerre’s Theorem (Theorem 1.44) to determine sufficient conditions on the Mellin transform of $K(t)$ so that, after a slight change of variable, $t \to t^{2q}$ for $q \in \mathbb{N}$, the Fourier transforms of $K(t^{2q})$, $q \in \mathbb{N}$, will have only real zeros.

In this section, we extended Pólya’s result in three ways. First, the required rate of decay of the kernel is slightly relaxed (cf. [34, Satz 2, p. 297]). Second, we allow the zeros of the related Mellin transform to lie in an interval which is slightly larger than Pólya’s, with, albeit, an additional restriction on the allowable integers $q$. While making these augments, we prove the result not just for the infinite Fourier transform (as in Pólya’s paper), but for both the infinite and finite Fourier transform.

We begin by defining the Mellin transform, a transform closely related to the Fourier transform. We then show some of the linkage between the two transforms.

**Definition 6.1.** Let $K(t) : [0, \infty) \to \mathbb{R}$ be integrable on $[0, \infty)$. We call

$$H(z) = \int_0^\infty K(t) t^{z-1} dt$$

the **Mellin transform** of $K(t)$ when the integral converges.

**Proposition 6.2.** Suppose the Mellin transform of $K(t)$ exists. Then

$$\int_0^\infty K(t) t^{z-1} dt = \int_{-\infty}^{\infty} K(e^s) e^{zs} ds$$

**Proof.** Use the substitution $\ln t = s$.\qed

**Remark 6.3.** In particular, this connection between the Mellin and Fourier transforms implies that the Fourier transform of $K(t)$ has only real zeros if and only if the associated Mellin transform of $K(\ln t)$ has only purely imaginary zeros.

Unfortunately, while the above relationship is quite nice for the case of the infinite Fourier transform, no such easy connection seems to exist when the Mellin or Fourier transform is over a finite interval.

We now proceed to show Pólya’s association between the two transforms by employing multiplier sequences.

**Theorem 6.4.** Let $0 < r \leq \infty$, let the function $K(t)$ be absolutely integrable on $[0, r]$ and suppose

$$|K(t)| = O\left(\exp\{-t^2\}\right), \text{ as } t \to \pm\infty. \quad (63)$$
Suppose, additionally, that the function $K(t)$ is analytic in an open region, $R$, around the point $t = 0$. Then, for $q \in \mathbb{N}$,

$$H(z) = \int_0^{r^{2q}} t^{z-1}K(t) \, dt$$

is a meromorphic function. If the function $H(z)$ has all of its zeros in the interval $(-\infty, 1/(2q_0)]$, for some $q_0 \in \mathbb{N}$, then

$$\int_{-r}^r K(t^{2q}) e^{itz} \, dt$$

is an entire function with only real zeros when $q \leq q_0$.

**Proof.** Let $D \subset R$ denote a closed disk of radius $\epsilon$, $0 < \epsilon < 1$. Since $K(t)$ is real and analytic on $R$,

$$K(t) = \sum_{n=0}^{\infty} c_n t^n, \quad c_n \in \mathbb{R},$$

which converges uniformly on $D$. Then $H(z)$ may be recast as

$$H(z) = \int_0^{\epsilon} \sum_{n=0}^{\infty} c_n t^{n+z-1} \, dt + \int_{\epsilon}^{r^{2q}} K(t)t^{z-1} \, dt$$

or, again by the fact that $K(t)$ is analytic on $R$, we may integrate the series term by term and use the substitution $\log t = s$,

$$H(z) = \epsilon^z \sum_{n=0}^{\infty} c_n \frac{\epsilon^n}{z+n} + \int_{\ln \epsilon}^{2q \ln r} K(e^s) e^{zs} \, ds$$

on all of $\mathbb{C}$ excluding the simple poles at $z = 0, -1, -2, \ldots$. That is, $H(z)$ is a meromorphic function.

We now wish to estimate the growth of $H(z)$. We begin with the integral term, the second term in (66). If $r < \infty$, then we are done since this is a finite Fourier transform and thus may be estimated by the Paley-Wiener Theorem (Theorem 1.13). If $r = \infty$, then, by the Fourier transform associated via Proposition 6.2, the integral is bounded. That is, by (63), there exists $t_0 > 0$ such that for some $\beta$ and $A > 0$, when $s > t_0$

$$|K(e^s)| < A \exp\{-|e^s|^\beta\}$$

and so, by perhaps restricting $s$ to even larger values, for any $\alpha > 0$,

$$|K(e^s)| < A \exp\{-|e^s|^\beta\} < A \exp\{-|s|^{2+\alpha}\}. \quad (67)$$

Since this holds for every $\alpha > 0$, by Theorem 1.18 and Proposition 4.12, the integral on the right in (66) is an entire function of order one and infinite type.

Now, addressing the growth of the sum in (66), let $1/2 > \delta > 0$ and let $B(k, \delta)$ denote the ball of radius $\delta$ centered at the integer $k$. Then let

$$P_\delta = \mathbb{C} \setminus \{B(k, \delta) : -k \in \mathbb{N}_0\}$$
denote the punctured plane. Then for \( z \in P_\delta, |z| > 1 \), the sum on the left in (66) may be bounded by a constant,

\[
\epsilon^z \sum_{n=0}^{\infty} \frac{c_n \epsilon^n}{z + n} \leq \sum_{n=0}^{\infty} \frac{c_n \epsilon^n}{\delta} = \frac{|K(\epsilon)|}{\delta}. \tag{68}
\]

Let \( q \neq 0 \) and

\[
h_q(z) = \frac{\Gamma \left( 1 + \frac{z}{2} \right) H \left( \frac{1+z}{2q} \right)}{\Gamma(1+z)}. \tag{69}
\]

Since the Gamma function has poles at the negative integers and zero and, by \( H(z) \)'s definition in (66), the numerator in (69) has simple poles when, for \( n \in \mathbb{N}, \)

\[ z = -2, -4, \ldots, -2n, \ldots, \]

corresponding to the factor \( \Gamma(1 + z/2) \), and when

\[ z = -1, -(2q + 1), -(4q + 1), \ldots, -(2nq + 1), \ldots, \]

for the factor \( H \left( \frac{1+z}{2q} \right) \). For \( q \in \mathbb{N} \), all of these points are also simple poles of \( \Gamma(1 + z) \), the denominator in (69), so \( h_q(z) \) defines an entire function.

By the definition of \( H(z) \) in (66), by (67) and Theorem 1.18, (68), and by the order of \( \Gamma(1 + z/2) / \Gamma(1 + z) \), the ratio of two functions of order one, the function \( h_q(z) \) is of order less than or equal to one (cf. [23]).

Note, additionally, that if our function \( h_q(z) \) is to have any zeros outside of the interval \( (-\infty, 0] \), these zeros must correspond to zeros of \( H(z) \) since \( \Gamma(1 + z/2)/\Gamma(1 + z) \) has all of its zeros in \( (-\infty, 0] \).

Let \( q \in \mathbb{N} \) and \( q \leq q_0 \). By the hypothesis that \( H(z) \) has no zeros outside the interval \( (-\infty, 1/(2q_0)] \), \( h_q(z) \in L-P \{ -\infty, 0 \} \). Thus, by Laguerre’s Theorem (Theorem 1.44), \( \{ h_q(k) \} \) is a complex zero decreasing sequence (cf. Definition 1.36). Then

\[
\sum_{n=0}^{\infty} \frac{(-1)^n h_q(2n)}{n!} z^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(1 + n) H \left( \frac{1+2n}{2q} \right)}{n! \Gamma(1 + 2n)} z^{2n} \tag{70}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n \int_0^{\frac{1}{2q}} t^{\frac{1+2n}{2q}-1} K(t) \, dt}{(2n)!} z^{2n}
\]

\[
= 2q \sum_{n=0}^{\infty} \frac{(-1)^n \int_0^{\frac{1}{2q}} u^{2n} K(u) \, du}{(2n)!} z^{2n}
\]

\[
= 2q \int_0^{\frac{1}{2q}} K(u) \sum_{n=0}^{\infty} \frac{(-1)^n (uz)^{2n}}{(2n)!} \, du
\]

\[
= q \int_{-r}^{r} K(u^2) e^{uz} \, du,
\]

via the substitution \( t^{1/2q} = u \), has only real zeros, completing the proof.

We now state the contrapositive of Theorem 6.4.

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Corollary 6.5. Let $0 < r \leq \infty$, let the function $K(t)$ be absolutely integrable on $[0, r]$ and suppose

$$|K(t)| = \mathcal{O} \left( \exp \{ -t^\beta \} \right), \text{ as } t \to \pm \infty.$$  

Suppose, additionally, that the function $K(t)$ is analytic in an open region, $R$, around the point $t = 0$. Then

$$H(z) = \int_0^{r^2} t^{z-1} K(t) \, dt$$

is a meromorphic function. If the Fourier transform

$$\int_{-r}^r K(t^{2q}) e^{izt} \, dt$$

is an entire function with non-real zeros, then $H(z)$ has zeros outside the interval $(-\infty, 1/2q_0]$.

Remark 6.6. To state Corollary 6.5 differently, if the entire function

$$f(z) = \int_{-r}^r K(t^{2q}) e^{izt} \, dt$$

has non-real zeros, then, by (70), $\{h_q(k)\}_{k=0}^{\infty}$ is not a multiplier sequence. By Theorem 1.44, this means $h_q(z)$ cannot be in $\mathcal{L}P(-\infty, 0]$, which can only be the case if $H(1+\frac{z}{2q})$ has zeros outside of $(-\infty, 0]$.

This remark, along with Theorem 3.6 and the subsequent estimates, implies the incomplete Gamma function,

$$\Gamma(z, r^2) = \int_0^{r^2} e^{-t} t^{z-1} \, dt, \quad (71)$$

which is of the form (64), with $K(t) = e^{-t}$ having all of the necessary hypotheses, has the associated Fourier transform, from (65) with $q = 1$,

$$f_r(z) = \int_{-r}^r e^{-t^2} e^{izt} \, dt,$$

which has non-real zeros for $r \geq 2$. While no proof is necessary beyond the work already completed, we show this result in detail to better illustrate the proof of Theorem 6.4.

Proposition 6.7. The incomplete Gamma function,

$$\Gamma(z, r^2) = \int_0^{r^2} e^{-t} t^{z-1} \, dt$$

has non-real zeros for $r \geq 2$.  

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Proof. Let

\[ H_r(z) = \Gamma(z, r^2) = \int_0^{r^2} t^{z-1} e^{-t} \, dt \]

and

\[ h_r(z) = \frac{\Gamma\left(1 + \frac{z}{2}\right) H_r\left(\frac{1+z}{2}\right)}{\Gamma(1+z)} \]

and \( T_r = \{ h_r(k) \}_{k=0}^{\infty} \). Since, by our work in Chapter 3, \( f(z) = \int_0^r e^{-t^2} \cos zt \, dt \) has non-real zeros for \( r \geq 2 \) and, by (70),

\[ f(z) = \int_0^r e^{-t^2} \cos zt \, dt = T_r[e^{-z^2}], \]

\( T_r \) is not a multiplier sequence for \( r \geq 2 \). This implies \( h_r(z) \) has zeros outside of the interval \((-\infty, 0]\) when \( r \geq 2 \). Moreover, since \( h_r(z) > 0 \) for \( z > 0 \), we must have that \( h_r(z) \) and, thus, \( H_r\left(\frac{1+z}{2}\right) = \Gamma(z, r^2) \), has non-real zeros for \( r \geq 2 \).

This result, while, perhaps not well known, is not new. The incomplete Gamma function, \( \Gamma(z, r) \) in (71), has zeros for non-real values of \( z \) when the parameter \( r > 0.3081 \ldots \) (cf. [24], [26], and [28]). This does, however, mean that the sequence \( \{ h(k) \}_{k=0}^{\infty} \) where

\[ h_r(z) = \frac{\Gamma\left(1 + \frac{z}{2}\right) H_r\left(\frac{1+z}{2}\right)}{\Gamma(1+z)}, \]

and

\[ H_r(z) = \Gamma(z, r^2) = \int_0^{r^2} e^{-t} t^{z-1} \, dt \]

with \( r \in (\sqrt{r_0}, \sqrt{\log 2}] \), \( \sqrt{r_0} = 0.5551 \ldots \) and \( \sqrt{\log 2} = 0.8325 \ldots \), is an example of a multiplier sequence which is interpolated by a function that is not in \( L^{-P}(-\infty, 0] \). This example shows that the converse of Theorem 6.4 is false.

Closing Remarks and Questions

In conclusion, we cite [9], [13], [17], and [18] for additional information and recent developments in the study of entire functions with only real zeros. Many of these results focus, as this paper has, on entire functions which may be represented as a Fourier transform, though the results do not necessarily relate to the specific properties of a kernel which produce a Fourier transform with only real zeros.

For the sake of convenience, we now restate the questions posed in the previous chapters:

**Question 6.8** (cf. Question 2.16). What sort of extension, if any, does Proposition 2.14 have to the case of the finite Fourier transform of \( K(t) \)?

**Question 6.9** (cf. Question 3.7). What is the infimum of the set of \( r \)’s for which the finite Fourier transform,

\[ \int_{-r}^{r} e^{-t^2} e^{izt} \, dt = \int_{-r}^{r} e^{-t^2} \cos zt \, dt, \]

has non-real zeros?
More generally, we restate the following question, which is closely related to
the results in this section.

**Question 6.10** (cf. Question 3.11, [16, Problem 1.6]). For each positive integer
$n$, set $K_n(t) = \exp\{-t^{2n}\}$. Let

$$f(z, n) := \int_0^\infty K_n(t) \cos z t \, dt \quad \text{and} \quad f_r(z, n) := \int_0^r K_n(t) \cos z t \, dt.$$  

Then the Taylor series expansion of the even entire function, $f(z, n)$, is given by

$$f(z, n) = \frac{1}{2n} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma\left(\frac{2k+1}{2n}\right)}{\Gamma(2k+1)} z^{2k},$$

where $\Gamma(z)$ denotes the Gamma function. Using the theory of multiplier sequences,
Pólya [34, Theorem 2] has shown $f(z, n) \in \mathcal{L}-\mathcal{P}$ for $n = 1, 2, 3, \ldots$. Thus, the
question is: For which values of $r > 0$ is the finite Fourier transform $f_r(z, n) \in \mathcal{L}-\mathcal{P}$?

**Question 6.11** (cf. Question 5.19). In the proof of Theorem 5.18, we required
a very carefully selected class of functions, $K^*(t)$, from Lemma 5.17, in order to
show that $\varphi(iz) \in \mathcal{L}-\mathcal{P}$. The question is, the answer to which could have profound
impacts: What, if any, other kernels, $K(t)$, have the property that if $\varphi(t)$ is a real
analytic function and

$$f(z) = \int_{-r}^r \varphi(t) K(t) e^{izt} \, dt \in \mathcal{L}-\mathcal{P},$$

then $\varphi(iz) \in \mathcal{L}-\mathcal{P}$.
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