Fractal Strings, Complex Dimensions and the Spectral Operator:
From the Riemann Hypothesis to Phase Transitions and Universality

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Background material: In collaboration with Machiel van Frankenhuysen (Helmut Maier and/or Carl Pomerance).

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Generalized Fractal Strings

Definition: A generalized fractal string \( \eta \) is a local positive or complex measure on \((0, +\infty)\) satisfying 
\[ |\eta|_{(0, x_0)} = 0, \] for some \( x_0 > 0 \), where 
\[ |\eta|_{(A)} = \sup \left\{ \infty \sum_{k=1}^{\infty} |\eta(A_k)| \right\}, \] 
and \( \{A_k\}_{k=1}^{\infty} \) ranges over all finite or countable partitions of \( A \) into measurable subsets of \((0, +\infty)\).

By "local measure" here, we mean that \( \eta \) is a set function on the Borel class of \((0, +\infty)\) (with values in \([0, +\infty]\) or \(\mathbb{C}\), respectively) whose restriction to any bounded Borel subset of \((0, +\infty)\) is a positive or complex measure, respectively.
A generalized fractal string $\eta$ is a local positive or complex measure on $(0, +\infty)$ satisfying $|\eta|(0, x_0) = 0$, for some $x_0 > 0$, where $|\eta|$ is the total variation measure of $A$ defined by

$$|\eta|(A) = \sup \left\{ \sum_{k=1}^{\infty} |\eta(A_k)| \right\},$$

and $\{A_k\}_{k=1}^{\infty}$ ranges over all finite or countable partitions of $A$ into measurable subsets of $(0, +\infty)$. 
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By “local measure” here, we mean that $\eta$ is a set function on the Borel class of $(0, +\infty)$ (with values in $[0, +\infty]$ or $\mathbb{C}$, respectively) whose restriction to any bounded Borel subset of $(0, +\infty)$ is a positive or complex measure, respectively.
Example

Consider the string consisting of the sequence of lengths \( \mathcal{L} = \{a^{-j}\}_{j=1}^\infty \), with typically nonintegers multiplicities \( w_j = b^j \), where \( 1 < b < a \).

Then, the measure associated to this string, called the generalized Cantor string (see [La-vF3, §10.1]) is given by

\[
\eta_{CS} = \sum_{j=0}^\infty b^j \delta_{\{a^{-j}\}}.
\]

Here, \( \delta_x \) is the Dirac measure concentrated at \( x \).
Example

Let $\Omega$ be a bounded open subset of $\mathbb{R}$. Write $\Omega = \bigcup_{j=1}^{\infty} I_j$ as a disjoint union of (bounded, open) intervals $I_j$ of lengths $\ell_j$ (repeated according to their multiplicity), and such that $\ell_j \downarrow 0$ as $j \to 0$. Then $\Omega$ (or $\mathcal{L} := \{\ell_j\}_{j=1}^{\infty}$) is called an ordinary fractal string. Furthermore,

$$\eta_{\mathcal{L}} := \sum_{j=1}^{\infty} \delta_{\ell_j^{-1}}$$

is the associated generalized fractal string.
The Cantor string
Definition

Let \( \eta \) be a generalized fractal string. Then its *dimension*, denoted by \( D_\eta \), is the abscissa of convergence of the Dirichlet integral \( \int_0^{+\infty} x^{-s} \eta(dx) \); that is,

\[
D_\eta = \inf \left\{ \sigma \in \mathbb{R} : \int_0^{+\infty} x^{-\sigma} |\eta|(dx) < \infty \right\}.
\]
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$$D_\eta = \inf \left\{ \sigma \in \mathbb{R} : \int_{0}^{+\infty} x^{-\sigma} |\eta|(dx) < \infty \right\}.$$

Definition

Given a generalized fractal string $\eta$, its *geometric zeta function*, denoted by $\zeta_\eta$, is its Mellin transform; namely,

$$\zeta_\eta(s) = \int_{0}^{+\infty} x^{-s} \eta(dx) \quad \text{for } \text{Re}(s) > D_\eta.$$
We will be interested in the meromorphic continuation of $s \mapsto \zeta_{\eta}(s)$. For this purpose, we define the \textit{screen} as the curve $S : S(t) + it$, $t \in \mathbb{R}$, and the \textit{window} $\mathcal{W}$ as the subset of the complex plane $\mathcal{W} := \{ s \in \mathbb{C} : \text{Re}(s) \geq S(\text{Im}s) \}$.

We assume that $s \mapsto \zeta_{\eta}(s)$ has a meromorphic continuation to some neighborhood of $\mathcal{W}$, and we define the set of \textit{visible complex dimensions} of $\eta$ as

$$D_{\eta}(\mathcal{W}) = \{ \omega \in \mathcal{W} : \zeta_{\eta} \text{ has a pole at } \omega \}.$$
The Cantor string: \( CS = \left\{ \frac{a_j}{3^j} \right\}_{j=1}^{\infty} \), where \( a_j = 3^{-j-1} \) and \( \omega_j = 2j \).

Geometric zeta function: \( \xi(s) = \sum_{j=1}^{\infty} \omega_j \cdot \frac{1}{3^j} = \frac{1}{1 - 2 \cdot 3^s} \).

Complex dimensions: \( \log 2 + i \pi \cdot \frac{2\pi}{\log 2} \cdot \frac{2^n}{n} \), for \( n \in \mathbb{Z} \).
Example

Harmonic string:

\[ h = \sum_{j=1}^{\infty} \delta_{\{j-1\}} \]
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**Prime harmonic string:**

\[ h_p = \sum_{j=1}^{\infty} \delta_{\{p-j\}}, \]

where \( p \in \mathcal{P} := \text{the set of all primes.} \)
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Prime harmonic string:

\[ h_p = \sum_{j=1}^{\infty} \delta_{\{p-j\}}, \]

where \( p \in \mathcal{P} \):= the set of all primes.

Note that \( h \) and \( h_p \) are both generalized fractal strings. Moreover, for any \( p \in \mathcal{P} \), we have

\[ h = \ast_{p \in \mathcal{P}} h_p, \]

where \( \ast \) is the multiplicative convolution of measures on \((0, +\infty)\).
Let $\eta$ and $\eta'$ be two generalized fractal strings. Then we have

$$\zeta_{\eta * \eta'}(s) = \zeta_\eta(s) \cdot \zeta_{\eta'}(s).$$
Let \( \eta \) and \( \eta' \) be two generalized fractal strings. Then we have

\[
\zeta_{\eta \ast \eta'}(s) = \zeta_{\eta}(s) \cdot \zeta_{\eta'}(s).
\]

The geometric zeta function associated to \( h_p \) is

\[
\zeta_{h_p}(s) = \frac{1}{1 - p^{-s}},
\]

the \( p^{th} \) Euler factor of \( \zeta(s) \).
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\zeta_{h_p}(s) = \frac{1}{1 - p^{-s}},
\]

the \( p^{th} \) Euler factor of \( \zeta(s) \).

Hence, we have for \( \text{Re}(s) > 1 \),

\[
\zeta_h(s) = \zeta_{\ast h_p}(s) = \zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}} = \prod_{p \in \mathcal{P}} \zeta_{h_p}(s).
\]

We thus recover the well-known Euler product for \( \zeta \).
Definition

Given a generalized fractal string $\eta$, the spectral measure $\nu$ associated to $\eta$ is defined by

$$\nu(A) = \sum_{k=1}^{+\infty} \eta \left( \frac{A}{k} \right),$$

for each bounded Borel subset $A \subset (0, +\infty)$.

We define the spectral zeta function of $\eta$ to be the geometric zeta function associated to $\nu$; we denote it by $\zeta_{\nu}$. 

Definition

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We define the spectral zeta function of \( \eta \) to be the geometric zeta function associated to \( \nu \); we denote it by \( \zeta_\nu \).

Lemma

Let \( \eta \) be a generalized fractal string. Then the spectral measure associated to \( \eta \) is the convolution of \( \eta \) (the harmonic string) with \( \eta \). That is,

\[
\nu = \eta \ast \mathcal{H}.
\]
The spectral zeta function of $\eta$, denoted by $\zeta_\nu$, is obtained by multiplying $\zeta_\eta$ by the Riemann zeta function

$$\zeta_\nu(s) = \zeta_\eta(s) \cdot \zeta(s).$$
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<td><strong>Definition</strong></td>
<td>A bounded open subset of $\mathbb{R}$</td>
<td>A Borel (complex or positive) measure $\mathcal{M}$ on $(0, +\infty)$ s.t.</td>
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<tr>
<td></td>
<td>$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \cdots$</td>
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<td><strong>Counting Function</strong></td>
<td>$N_\mathcal{L}(x) = \sum_{y \geq 1} \frac{1}{y^{d-1}} I_{y^{d-1} \leq x}$</td>
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<tr>
<td>Characteristics</td>
<td>Standard Fractal String $\zeta$</td>
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<td>$\zeta_\mathcal{L}(s) = \sum_{j=1}^{+\infty} p_j s^j$</td>
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<td>$N_\mu(x) = \sum_{k=1}^{+\infty} N_\mathcal{L}(\frac{x}{k})$</td>
<td>$\mu(A) = \sum_{k=1}^{+\infty} \eta(\frac{A}{k})$</td>
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</table>
| Spectral Zeta Function   | $\zeta_{\mathcal{L}}(s) = \sum f^{(\nu)} f^{-s}$  \[
= \zeta_\mathcal{L}(s) \cdot \zeta_{\eta}(s)
\] | $\zeta_{\eta}(s) = \int_0^{+\infty} x^{-s} \mu(dx)$  \[
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\] |
The Distributional Explicit Formulas

Theorem (See [La-vF3, Ch.5]). Let \( \eta \) be a languid generalized fractal string. Then, for any \( k \in \mathbb{Z} \), its \( k \)th distributional primitive (or anti-derivative) \( P[k] \eta \) is given by

\[
P[k] \eta (x) = \sum_{\omega \in D \eta (W)} \text{res} \left( x^{s+k-1} \right) \eta (s) + 1 \left( k-1 \right) \sum_{j=0}^{k-1} C_{k-1}^j (-1)^j x^{k-1-j} \eta (-j) + R[k] \eta (x),
\]

where \( R[k] \eta (x) = \frac{1}{2\pi i} \int_{S} x^{s+k-1} \eta (s) \, ds \) is the error term and can be suitably estimated as \( x \to +\infty \).

In addition, if \( \eta \) is strongly languid, then we may choose \( W = \mathcal{C} \) and \( R \eta (x) \equiv 0 \).
The Distributional Explicit Formulas

Theorem

(See [La-vF3, Ch.5]). Let η be a languid generalized fractal string. Then, for any $k \in \mathbb{Z}$, its $k$th distributional primitive (or anti-derivative) $\mathcal{P}^{[k]}_\eta$ is given by

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$$

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In addition, if η is strongly languid, then we may choose $\mathcal{W} = \mathbb{C}$ and $\mathcal{R}_\eta(x) \equiv 0$. 
When we apply the distributional explicit formulas at level $k = 0$, assuming that $\eta$ is a languid generalized fractal string whose complex dimensions are simple and satisfies certain mild additional conditions, we obtain that, as a distribution, the measure $\eta$ is given by the following **density of geometric states** (or **density of lengths**) formula:

$$
\eta = \sum_{\omega \in \mathcal{D}_\eta(W)} \text{res}(\zeta_\eta(s); \omega) x^{\omega - 1}.
$$
We also obtain for the spectral measure, by applying the previous theorem to $\nu = \mathcal{P}[0]$, the following density of spectral states (or density of frequencies) formula:

$$\nu = \zeta_{\eta}(1) + \sum_{\omega \in \mathcal{D}_{\eta}(\mathcal{W})} \text{res}(\zeta_{\eta}(s)\zeta(s)x^{s-1}; \omega)x^{\omega-1}$$

$$= \zeta_{\eta}(1) + \sum_{\omega \in \mathcal{D}_{\eta}(\mathcal{W})} \zeta(\omega)\text{res}(\zeta_{\eta}(s); \omega)x^{\omega-1},$$

for simple poles.
The Inverse Spectral Problem

The inverse problem for fractal strings considered in [LaMa1, LaMa2] (and later revisited in [La-vF2, La-vF3]) is the following, for any fixed $D \in (0, 1)$:

\[
\text{(IVS) } D
\]

Given any fractal string $L$ of dimension $D$ such that for some constants $C_D$ and $\delta > 0$,

\[
N_\nu(x) = W(x) - c_D x + O(x^{\delta - \delta}),
\]

as $x \to +\infty$, is it true that $L$ is Minkowski measurable?

(Here, the leading term $W(x)$ is the Weyl term, given by $W(x) := \text{vol}_1(L_x) x$, and below, $M$ is the Minkowski content of $L$.)

Remark: It follows from results in [LaPo2] and [La2, La3] that if such a nonzero constant $c_D$ exists, then $c_D > 0$ and is given by

\[
c_D = 2 - (1 - D)(1 - D)(-\zeta(D)) M.
\]

where $M$ is the Minkowski content of $L$. 
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\[N_\nu(x) = W(x) - c_D x^D + O(x^{D-\delta}), \]

as $x \to +\infty$, is it true that $\mathcal{L}$ is Minkowski measurable?

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It follows from results in [LaPo2] and [La2, La3] that if such a nonzero constant $c_D$ exists, then $c_D > 0$ and is given by

$$c_D = 2^{-(1-D)}(1 - D)(-\zeta(D))\mathcal{M}.$$
As a consequence, the main result of [LaMa2] can be stated as follows:

For any given $D \in (0, 1)$, the inverse spectral problem $(IVS)_D$ has an affirmative answer if and only if $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ such that $\text{Re}(s) = D$. Hence, $(IVS)_D$ is not true in the "mid-fractal" case when $D = \frac{1}{2}$, and it holds everywhere else (i.e., for every $D \in (0, 1)$, $D \neq \frac{1}{2}$) if and only if the Riemann hypothesis is true.

This spectral reformulation was revisited in [La-vF2, La-vF3] by using the then rigorously developed theory of complex dimensions and the associated explicit formulas. (See [La-vF3, Ch.9].)
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Using the distributional explicit formula as a motivation, the spectral operator will be defined at level $k = 0$ as the map $\eta \mapsto -\nu$ and at level $k = 1$ as the map $N_\eta(x) \mapsto -\nu(N_\eta(x)) = N_\nu(x) = \sum_{k=1}^{\infty} N_\eta(x_k)$. 
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$$\eta \mapsto \nu$$
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\[ \eta \mapsto \nu \]

and at level $k = 1$ as the map

\[ N\eta(x) \mapsto \nu(N\eta)(x) = N\nu(x) = \sum_{k=1}^{\infty} N\eta \left( \frac{x}{k} \right). \]
For every prime number $p$, we also define the $p$-factor of $\nu$ by

\[ N_{\eta}(x) \mapsto \nu_p(N_{\eta})(x) = N_{\nu_p}(x) = N_{\eta*h_p}(x) = \sum_{k=0}^{\infty} N_{\eta}(x p^{-k}), \]

where the terms in the sum necessarily vanish when $p^k \geq x$. 
For every prime number $p$, we also define the \textit{p-factor} of $\nu$ by

$$N_\eta(x) \mapsto \nu_p(N_\eta)(x) = N_{\nu_p}(x) = N_{\eta \ast h_p}(x) = \sum_{k=0}^{\infty} N_\eta(xp^{-k}),$$

where the terms in the sum necessarily vanish when $p^k \geq x$.

The operators $\nu_p$ commute with each other and their composition gives the Euler product for $\nu$:

$$N_\eta(x) \mapsto \nu(N_\eta)(x) = \left( \prod_{p \in \mathcal{P}} \nu_p \right)(N_\eta).$$
Making the change of variable \( x = e^t \) (\( x > 0 \)) or equivalently, \( t = \log x \) (and hence, \( t \in \mathbb{R} \)), and writing \( f(t) = N_\eta(x) \), we obtain the \textit{additive version} of the spectral operator

\[
f(t) \mapsto a(f)(t) = \sum_{k=1}^{\infty} f(t - \log k),
\]
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f(t) \mapsto a(f)(t) = \sum_{k=1}^{\infty} f(t - \log k),
\]

and of its operator-valued Euler factors (for each prime \( p \in \mathcal{P} \))

\[
f(t) \mapsto a_p(f)(t) = \sum_{k=0}^{\infty} f(t - k \log p).
\]
The spectral operator $a$ and its Euler factors $a_p$ are also related by an operator-valued Euler product

$$f(t) \mapsto a(f)(t) = \bigg( \prod_{p \in \mathcal{P}} a_p \bigg)(f)(t),$$

where the product is given in the sense of the composition of operators.
If we denote by $\partial := \frac{d}{dt}$ the first order differential operator with respect to $t$, the Taylor series associated to $f$, a smooth function, can be written as

$$f(t + h) = f(t) + \frac{hf'(t)}{1!} + \frac{h^2f''(t)}{2!} + ...$$

$$= e^{hd} (f)(t) = e^{h\partial} (f)(t);$$

that is, $\partial = \frac{d}{dt}$ is the infinitesimal generator of the (one-parameter) group of shifts on the real line.
This gives a new representation for the spectral operator and its prime factors:

\[
\alpha(f)(t) = \sum_{k=1}^{\infty} e^{-(\log k) \partial}(f)(t)
\]

\[
= \sum_{k=1}^{\infty} \left( \frac{1}{k^\partial} \right)(f)(t)
\]

\[
= \zeta(\partial)(f)(t)
\]

\[
= \prod_{p \in \mathcal{P}} (1 - p^{-\partial})^{-1}(f)(t),
\]
and for any prime $p$,

$$
a_p(f)(t) = \sum_{k=0}^{\infty} f(t - k \log p)
\quad = \sum_{k=0}^{\infty} e^{-k(\log p)\partial} f(t)
\quad = \sum_{k=0}^{\infty} \left(p^{-k\partial}\right) f(t)
\quad = (1 - p^{-\partial})^{-1}(f)(t)
\quad = \zeta_{h_p}(\partial)(t).
$$
The Weighted Hilbert Space $\mathcal{H}_c$

For $c \geq 0$, let

$$\mathcal{H}_c := \left\{ f \in C^\infty(\mathbb{R}) : \text{supp}(f) \subset (0, +\infty) \text{ and } \int_0^{+\infty} |f(t)|^2 e^{-2ct} dt < \infty \right\}.$$ 

$\mathcal{H}_c$ is a pre-Hilbert space for the natural inner product indicated below. Its completion is a Hilbert space and is denoted by $\mathcal{H}_c$. It is equipped with the following inner product

$$\langle f, g \rangle_c = \int_0^{+\infty} f(t) g(t) e^{-2ct} dt$$

and the associated Hilbert norm $\| \cdot \|_c$ (so that $\|f\|^2_c = \int_0^{+\infty} |f(t)|^2 e^{-2ct} dt$).
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For $c \geq 0$, let

$$\mathcal{H}_c := \left\{ f \in C^\infty(\mathbb{R}) : \text{supp}(f) \subset (0, +\infty) \text{ and } \int_0^{+\infty} |f(t)|^2 e^{-2ct} \, dt < \infty \right\}$$

$\mathcal{H}_c$ is a pre-Hilbert space for the natural inner product indicated below.

Its completion is a Hilbert space and is denoted by $\mathbb{H}_c$. It is equipped with the following inner product

$$< f, g >_c = \int_0^{+\infty} f(t)\overline{g(t)} e^{-2ct} \, dt$$

and the associated Hilbert norm $\| \cdot \|_c = \sqrt{< \cdot, \cdot >_c}$ (so that $\|f\|_c^2 = \int_0^{+\infty} |f(t)|^2 e^{-2ct} \, dt$).
The Differentiation Operator $A = \partial_c$

Given $c \geq 0$, we define $A := \partial = \partial_c = \frac{d}{dt}$ as the unbounded linear operator from $H^c$ to itself with domain $D(A)$ consisting of all the functions $f \in H^c$ that are (locally) absolutely continuous on $\mathbb{R}$ (i.e., $f \in C_{\text{abs}}(\mathbb{R})$) and such that $f' \in H^c$ (where $f'$ denotes the pointwise derivative of $f$, which exists Lebesgue almost everywhere on $\mathbb{R}$). Furthermore, for $f \in D(A)$, we let $Af = \partial f := f'$, for all $f \in D(A)$. 
Given $c \geq 0$, we define $A := \partial = \partial_c = \frac{d}{dt}$ as the unbounded linear operator from $\mathbb{H}_c$ to itself with domain $D(A)$ consisting of all the functions $f \in \mathbb{H}_c$ that are (locally) absolutely continuous on $\mathbb{R}$ (i.e., $f \in C_{abs}(\mathbb{R})$) and such that $f' \in \mathbb{H}_c$ (where $f'$ denotes the pointwise derivative of $f$, which exists Lebesgue almost everywhere on $\mathbb{R}$). Furthermore, for $f \in D(A)$, we let
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$$Af = \partial f := f',$$

for all $f \in D(A)$. 

It follows from the definition of $\mathbb{H}_c$, $D(A)$ and a well-known lemma (about absolutely continuous functions) that every $f \in D(A)$ naturally satisfies the following boundary conditions at $-\infty$ and $+\infty$, respectively:

1. (Boundary condition at $-\infty$) $f(t) = 0$ for all $t \leq 0$; in particular, we have $f(0) = 0$.
2. (Boundary condition at $+\infty$) $\lim_{t \to +\infty} f(t) e^{-tc} = 0$.

Remark: Intuitively, condition (2) means that the corresponding fractal strings have (Minkowski) dimension $D \leq c$. (See [LapPo2].)
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Intuitively, condition (2) means that the corresponding fractal strings have (Minkowski) dimension $D \leq c$. (See [LapPo2].)
Normality of the unbounded operator \( \partial_c \)

Theorem

For every \( c \geq 0 \), \( A = \partial_c \) is an unbounded normal linear operator on \( H_c \). Moreover, its adjoint \( A^* \) is given by \( A^* = 2c - A \), with \( \mathcal{D}(A^*) = \mathcal{D}(A) \).
Theorem

For every $c \geq 0$, $A = \partial_c$ is an unbounded normal linear operator on $\mathbb{H}_c$.

Moreover, its adjoint $A^*$ is given by $A^* = 2c - A$, with $D(A^*) = D(A)$. 
Theorem
For every $c \geq 0$, the spectrum $\sigma(A)$ of the differentiation operator $A = \partial_c$ is the closed vertical line of the complex plane passing through $c$. Furthermore, it is equal to the essential spectrum $\sigma_e(A)$ of $A$: $\sigma(A) = \sigma_e(A) = \{ \lambda \in \mathbb{C} | \text{Re}(\lambda) = c \}$. Moreover, the point spectrum of $A$ is empty (i.e., $A$ does not have any eigenvalues).
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Moreover, the point spectrum of $A$ is empty (i.e., $A$ does not have any eigenvalues).
The strongly continuous group $e^{-t\partial_c}$
The strongly continuous group $e^{-t\partial_c}$

**Lemma**

For every $c \geq 0$, \( \{ e^{-t\partial_c} \}_{t \in \mathbb{R}} \) is a strongly continuous group of (bounded linear) operators and

\[
\| e^{-t\partial_c} \| = e^{-ct}
\]

for any $t \in \mathbb{R}$.

Moreover, its adjoint group \( \{ (e^{-t\partial_c})^* \}_{t \in \mathbb{R}} \) is given by

\[
\{ e^{-t\partial_c^*} \}_{t \in \mathbb{R}} = \{ e^{-t(2c-\partial_c)} \}_{t \in \mathbb{R}} = e^{-2ct} \{ e^{t\partial_c} \}_{t \in \mathbb{R}}.
\]
The strongly continuous group of operators $\{e^{-t\partial}\}_{t \in \mathbb{R}}$ is a translation (or shift) group. That is, for every $t \in \mathbb{R}$,

$$(e^{-t\partial})(f)(u) = f(u - t)$$

for all $f \in \mathcal{H}_c$ and $u \in \mathbb{R}$. (For a fixed $t \in \mathbb{R}$, this equality holds for elements in $\mathcal{H}_c$ and hence, for a.e. $u \in \mathbb{R}$.)
Lemma

The strongly continuous group of operators \( \{ e^{-t\partial} \}_{t \in \mathbb{R}} \) is a translation (or shift) group. That is, for every \( t \in \mathbb{R} \),

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for all \( f \in \mathbb{H}_c \) and \( u \in \mathbb{R} \). (For a fixed \( t \in \mathbb{R} \), this equality holds for elements in \( \mathbb{H}_c \) and hence, for a.e. \( u \in \mathbb{R} \).)

Remark: As a result, \( \partial = \partial_c \), the infinitesimal generator of the shift group \( \{ e^{-t\partial} \}_{t \in \mathbb{R}} \), is called the infinitesimal shift of the real line (with parameter \( c \geq 0 \)).
We can now define the spectral operator as follows: $a = \zeta(\partial)$, via the measurable functional calculus for unbounded normal operators. (If, for simplicity, we assume $c \neq 1$ in order to avoid the pole of $\zeta$ at $s = 1$, then $\zeta$ is holomorphic in a neighborhood of $\sigma(\partial)$. If $c = 1$ is allowed, then we may also use the meromorphic functional calculus for sectorial operators; see [Haase].)

Theorem
Assume that $c > 1$. Then, for any $f \in D(a)$, we have $a(f)(t) = \sum_{k=1}^{\infty} f(t - \log k) = \zeta(\partial)(f)(t) = \sum_{n=1}^{\infty} n^{-\partial}(f)(t)$. 

The Spectrum of $\alpha$
We can now define the *spectral operator* as follows:

\[ a = \zeta(\partial), \]

via the measurable functional calculus for unbounded normal operators. (If, for simplicity, we assume \( c \neq 1 \) in order to avoid the pole of \( \zeta \) at \( s = 1 \), then \( \zeta \) is holomorphic in a neighborhood of \( \sigma(\partial) \). If \( c = 1 \) is allowed, then we may also use the meromorphic functional calculus for sectorial operators; see [Haase].)
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**Theorem**

*Assume that \( c > 1 \). Then, for any \( f \in D(a) \), we have*

\[ a(f)(t) = \sum_{k=1}^{\infty} f(t - \log k) = \zeta(\partial)(f)(t) = \sum_{n=1}^{\infty} n^{-\partial}(f)(t). \]
Remark:

For any $c > 0$, we also show that the above equation holds for all $f$ in a suitable dense subspace of $D(\alpha)$. 
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**Theorem**

Assume that $c \geq 0$. Then

$$
\sigma(\alpha) = \overline{\zeta(\sigma(\partial))} = cl\left(\zeta(\{\lambda \in \mathbb{C} | \text{Re}(\lambda) = c\})\right),
$$

where $\sigma(\alpha)$ is the spectrum of $\alpha$ and $\overline{N} = cl(N)$ is the closure of $N \subset \mathbb{C}$. 
Assume that $c \geq 0$. Then, the spectral operator $a = \zeta(\partial)$ is quasi-invertible if and only if the Riemann zeta function does not vanish on the vertical line $\{s \in \mathbb{C}: \text{Re}(s) = c\}$. 
Assume that \( c \geq 0 \). Then, the spectral operator \( \alpha = \zeta(\partial) \) is quasi-invertible if and only if the Riemann zeta function does not vanish on the vertical line \( \{ s \in \mathbb{C} : \text{Re}(s) = c \} \).
Corollary

The spectral operator $\alpha$ is quasi-invertible for all $c \in (0, 1) - \frac{1}{2}$ if and only if the Riemann hypothesis is true.
Corollary

The spectral operator $\alpha$ is quasi-invertible for all $c \in (0, 1) - \frac{1}{2}$ if and only if the Riemann hypothesis is true.

Remarks:

- The notion of quasi-invertibility will be defined in the next part.
Corollary

The spectral operator $\alpha$ is quasi-invertible for all $c \in (0, 1) - \frac{1}{2}$ if and only if the Riemann hypothesis is true.

Remarks:

- The notion of quasi-invertibility will be defined in the next part.
- It suffices to require $c \in (0, \frac{1}{2})$ (or $c \in (\frac{1}{2}, 1)$) in the above corollary. This follows from the functional equation for $\zeta$ connecting $\zeta(s)$ and $\zeta(1 - s)$. 
We show in [HerLa1] that for any $c \geq 0$, the infinitesimal shift $\partial = \partial_c$ is given by

$$\partial = c + iV,$$

where $V$ is an unbounded self-adjoint operator such that $\sigma(V) = \mathbb{R}$. Thus, given $T \geq 0$, we define the \textit{truncated infinitesimal shift} as follows:

$$A^{(T)} = \partial^{(T)} := c + iV^{(T)},$$

where

$$V^{(T)} := \phi^{(T)}(V)$$

(in the sense of the functional calculus),
and $\phi^{(T)}$ is a suitable (i.e., $T$-admissible) continuous (if $c \neq 1$) or meromorphic (if $c = 1$) cut-off function (chosen so that $\phi^{(T)}(\mathbb{R}) = \sigma(A^{(T)}) = c + i[-T, T]$).
and $\phi^{(T)}$ is a suitable (i.e., $T$-admissible) continuous (if $c \neq 1$) or meromorphic (if $c = 1$) cut-off function (chosen so that $\phi^{(T)}(\mathbb{R}) = \sigma(A^{(T)}) = c + i[-T, T]$).

Similarly, the *truncated spectral operator* is defined (also for $c \geq 0$) by

$$a^{(T)} := \zeta \left( \partial^{(T)} \right).$$
The spectrum of $A(T)$
More precisely, the $T$-admissible function $\phi(T)$ is chosen as follows:

(i) If $c \neq 1$, $\phi(T)$ is any continuous function such that $\phi(R) = [-T, T]$. (For example, $\phi(T) = \tau$ for $0 \leq \tau \leq T$ and $= T$ for $\tau \geq T$; also, $\phi(T)$ is odd.)

(ii) If $c = 1$ (which corresponds to the pole of $\zeta(s)$ at $s = 1$), then $\phi(T)$ is a suitable meromorphic analog of (i). (For example, $\phi(T) = 2\pi \tan^{-1}(s)$, so that $\phi(R) = [-T, T]$.)

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(ii) If $c = 1$ (which corresponds to the pole of $\zeta(s)$ at $s = 1$), then $\phi^{(T)}$ is a suitable meromorphic analog of (i). (For example, $\phi^{(T)}(s) = \frac{2T}{\pi} \tan^{-1}(s)$, so that $\phi^{(T)}(\mathbb{R}) = [-T, T]$.)
One then uses the measurable functional calculus and an appropriate (continuous or meromorphic, for $c \neq 1$ or $c = 1$, respectively) version of the Spectral Mapping Theorem (SMT) for unbounded normal operators in order to define both $\partial(T)$ and $\alpha(T) = \zeta(\partial(T))$, as well as to determine their spectra.

\[
\textbf{SMT : } \sigma(\psi(L)) = \overline{\psi(\sigma(L))}
\]

if $\psi$ is a continuous (resp., meromorphic) function on $\sigma(L)$ (resp., on a neighborhood of $\sigma(L)$) and $L$ is an unbounded normal operator.
One then uses the measurable functional calculus and an appropriate (continuous or meromorphic, for $c \neq 1$ or $c = 1$, respectively) version of the Spectral Mapping Theorem (SMT) for unbounded normal operators in order to define both $\partial(T)$ and $\alpha(T) = \zeta(\partial(T))$, as well as to determine their spectra.

**SMT**: \[ \sigma(\psi(L)) = \psi(\sigma(L)) \]

if $\psi$ is a continuous (resp., meromorphic) function on $\sigma(L)$ (resp., on a neighborhood of $\sigma(L)$) and $L$ is an unbounded normal operator.

Note that for $c \neq 1$ (resp., $c = 1$), $\partial(T)$ and $\alpha(T)$ are then continuous (resp., meromorphic) functions of the normal (and sectorial, see [Haa]) operator $\partial$. An entirely analogous statement is true for the spectral operator $\alpha = \zeta(\partial)$. 
The above construction can be generalized as follows:

Given $0 \leq T_0 \leq T$, one can define a $(T_0, T)$-admissible cut-off function $\phi(T_0, T)$ exactly as above, except with $[-T, T]$ replaced with $\{\tau \in \mathbb{R} : T_0 \leq |\tau| \leq T\}$.

Correspondingly, one can define $V(T_0, T) = \phi(T_0, T)(V)$,

$$A(T_0, T) = \partial(T_0, T) := c + iV(T_0, T)$$

and

$$a(T_0, T) = \zeta(\partial(T_0, T)),$$

where $\partial(T_0, T)$ is the $(T_0, T)$-infinitesimal shift and $a(T_0, T)$ is the $(T_0, T)$-truncated spectral operator.
The above construction can be generalized as follows:

Given \(0 \leq T_0 \leq T\), one can define a \((T_0, T)\)-admissible cut-off function \(\phi^{(T_0, T)}\) exactly as above, except with \([-T, T]\) replaced with \(\{\tau \in \mathbb{R} : T_0 \leq |\tau| \leq T\}\).
Correspondingly, one can define \(V^{(T_0, T)} = \phi^{(T_0, T)}(V)\),

\[A^{(T_0, T)} = \partial^{(T_0, T)} := c + iV^{(T_0, T)}\]

and

\[\alpha^{(T_0, T)} = \zeta(\partial^{(T_0, T)})\]

where \(\partial^{(T_0, T)}\) is the \((T_0, T)\)-infinitesimal shift and \(\alpha^{(T_0, T)}\) is the \((T_0, T)\)-truncated spectral operator.

**Remark:** Note that for \(T_0 = 0\), we recover \(A^{(T)}\) and \(\alpha^{(T)}\).
Definition

The spectral operator $\alpha$ is *quasi-invertible* if its truncation $\alpha^{(T)}$ is invertible for all $T \geq 0$.

Remark: In the definition of "almost invertibility", $T_0$ is allowed to depend on the parameter $c$.

Note: quasi-invertible $\Rightarrow$ almost invertible.
**Definition**

The spectral operator $\alpha$ is *quasi-invertible* if its truncation $\alpha^{(T)}$ is invertible for all $T \geq 0$.

**Definition**

Similarly, $\alpha$ is *almost invertible*, if for some $T_0 \geq 0$, its truncation $\alpha^{(T_0, T)}$ is invertible for all $T \geq T_0$.

Remark: In the definition of “almost invertibility”, $T_0$ is allowed to depend on the parameter $c$.

Note: quasi-invertible $\Rightarrow$ almost invertible.
The spectral operator \( \alpha \) is **quasi-invertible** if its truncation \( \alpha^{(T)} \) is invertible for all \( T \geq 0 \).

Similarly, \( \alpha \) is **almost invertible**, if for some \( T_0 \geq 0 \), its truncation \( \alpha^{(T_0, T)} \) is invertible for all \( T \geq T_0 \).

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**Remark:**

In the definition of “almost invertibility”, $T_0$ is allowed to depend on the parameter $c$.

**Note:**

quasi-invertible $\Rightarrow$ almost invertible.
Theorem

For all $T \geq 0$, $A^{(T)}$ and $a^{(T)}$ are bounded normal operators, with spectra respectively given by

$$\sigma \left( A^{(T)} \right) = \{ c + i\tau : |\tau| \leq T \}$$

and
Theorem

For all $T \geq 0$, $A^T$ and $\alpha^T$ are bounded normal operators, with spectra respectively given by

$$\sigma \left( A^T \right) = \{ c + i\tau : |\tau| \leq T \}$$

and

$$\sigma \left( \alpha^T \right) = \{ \zeta(c + i\tau) : |\tau| \leq T \}.$$
**Theorem**

For all $T \geq 0$, $A^{(T)}$ and $a^{(T)}$ are bounded normal operators, with spectra respectively given by

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**Remarks:**

- Recall that $a^{(T)}$ is invertible if and only if $0 \notin \sigma \left( a^{(T)} \right)$. 
Theorem

For all $T \geq 0$, $A^{(T)}$ and $a^{(T)}$ are bounded normal operators, with spectra respectively given by

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Remarks:

- Recall that $a^{(T)}$ is invertible if and only if $0 \notin \sigma \left( a^{(T)} \right)$.

- More generally, given $0 \leq T_0 \leq T$, the exact counterpart of the above theorem holds for $A^{(T_0, T)}$ and $a^{(T_0, T)}$, except with $|\tau| \leq T$ replaced with $T_0 \leq |\tau| \leq T$. 

Corollary

Assume that $c \geq 0$. Then, the truncated spectral operator $\alpha^{(T)}$ is invertible if and only if $\zeta$ does not vanish on the vertical line segment $\{ s \in \mathbb{C} : \text{Re}(s) = c, |\text{Im}(s)| \leq T \}$. 
Assume that $c \geq 0$. Then, the truncated spectral operator $\alpha^{(T)}$ is invertible if and only if $\zeta$ does not vanish on the vertical line segment $\{s \in \mathbb{C} : \text{Re}(s) = c, |\text{Im}(s)| \leq T\}$.

**Remark:**
Naturally, given $0 \leq T_0 \leq T$, the same result is true for $\alpha^{(T_0, T)}$ provided $|\text{Im}(s)| \leq T$ is replaced with $T_0 \leq |\text{Im}(s)| \leq T$. 

Theorem

Assume that $c \geq 0$. Then,

1. $a$ is quasi-invertible if and only if $\zeta$ does not vanish (i.e., does not have any zeroes) on the vertical line $\Re(s) = c$.

2. $a$ is almost invertible if and only if all but (at most) finitely many zeroes of $\zeta$ are off the vertical line $\Re(s) = c$. 
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Corollary

1. \( \alpha \) is quasi-invertible for all \( c \in \left( \frac{1}{2}, 1 \right) \) if and only if the Riemann hypothesis is true.
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1. \( \alpha \) is quasi-invertible for all \( c \in (\frac{1}{2}, 1) \) if and only if the Riemann hypothesis is true.

2. \( \alpha \) is almost invertible for all \( c \in (\frac{1}{2}, 1) \) if and only if the Riemann hypothesis (RH) is “almost true” (i.e., on every vertical line \( \text{Re}(s) = c, \ c > \frac{1}{2} \), there are at most finitely many exceptions to RH).
Remark:

According to our previous results, we have (for the spectral operator $a$):

1. invertible $\Rightarrow$ quasi-invertible $\Rightarrow$ almost invertible.
Remark:

According to our previous results, we have (for the spectral operator $\alpha$):

1. invertible $\Rightarrow$ quasi-invertible $\Rightarrow$ almost invertible.
2. For $c = \frac{1}{2}$, $\alpha$ is not almost (and hence, not quasi- etc.) invertible. (This follows from Hardy’s theorem according to which $\zeta$ has infinitely many zeroes on the critical line $\text{Re}(s) = \frac{1}{2}$.)
Remark:

According to our previous results, we have (for the spectral operator $\alpha$):

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3. For $c > 1$, $\alpha$ is quasi- (and hence, almost) invertible. In fact, we will next see that $\alpha$ is also invertible for $c > 1$. 
Recall that, by the Spectral Mapping Theorem (SMT) for unbounded normal operators, \( \sigma(\alpha) \) (the spectrum of \( \alpha \)) is equal to the closure of the range of \( \zeta \) on the vertical line \( \sigma(A) = \{ \text{Re}(s) = c \} \). Hence, \( \sigma(\alpha) \) is equal to \( \{ \zeta(c + i\tau) : \tau \in \mathbb{R} \} \) union its limit points (in \( \mathbb{C} \)).

Also, by definition of \( \sigma(\alpha) \), \( \alpha \) is invertible if and only if \( 0 \notin \sigma(\alpha) \).
Theorem

For $c > 1$, $\sigma(a)$ is bounded and $0 \not\in \sigma(a)$. Hence, $a$ is invertible.

2 (Universality) For $c \in \left(\frac{1}{2}, 1\right)$, $\sigma(a) = C$. This follows from the Bohr–Landau Theorem and, more generally, from the universality of $\zeta$ in the right critical strip $\frac{1}{2} < \Re(s) < 1$.

Hence, $a$ is not invertible (because $0 \in \sigma(a)$).

3 For $c \in \left(0, \frac{1}{2}\right)$, $\sigma(a)$ is unbounded and conditionally (i.e., under RH), $\sigma(a) \neq C$ and, in particular, $0 \not\in \sigma(a)$, so that $a$ is invertible.

Remark: The last statement in the third part of the theorem follows from the non-universality of $\zeta$ in the left critical strip $0 < \Re(s) < \frac{1}{2}$; see [KaSt].
Theorem

1. For $c > 1$, $\sigma(\alpha)$ is bounded and $0 \notin \sigma(\alpha)$. Hence, $\alpha$ is invertible.
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2. (Universality) For $c \in (\frac{1}{2}, 1)$, $\sigma(\alpha) = \mathbb{C}$. (This follows from the Bohr–Landau Theorem and, more generally, from the universality of $\zeta$ in the right critical strip $\frac{1}{2} < \Re(s) < 1$.) Hence, $\alpha$ is not invertible (because $0 \in \sigma(\alpha)$).
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1. For $c > 1$, $\sigma(\alpha)$ is bounded and $0 \notin \sigma(\alpha)$. Hence, $\alpha$ is invertible.

2. (Universality) For $c \in (\frac{1}{2}, 1)$, $\sigma(\alpha) = \mathbb{C}$. (This follows from the Bohr–Landau Theorem and, more generally, from the universality of $\zeta$ in the right critical strip $\frac{1}{2} < \text{Re}(s) < 1$.) Hence, $\alpha$ is not invertible (because $0 \in \sigma(\alpha)$).

3. For $c \in (0, \frac{1}{2})$, $\sigma(\alpha)$ is unbounded and conditionally (i.e., under RH), $\sigma(\alpha) \neq \mathbb{C}$ and, in particular, $0 \notin \sigma(\alpha)$, so that $\alpha$ is invertible.

Remark: The last statement in the third part of the theorem follows from the non-universality of $\zeta$ in the left critical strip $0 < \text{Re}(s) < \frac{1}{2}$; see [KaSt].
Theorem

1. For \( c > 1 \), \( \sigma(\alpha) \) is bounded and \( 0 \notin \sigma(\alpha) \). Hence, \( \alpha \) is invertible.

2. (Universality) For \( c \in (\frac{1}{2}, 1) \), \( \sigma(\alpha) = \mathbb{C} \). (This follows from the Bohr–Landau Theorem and, more generally, from the universality of \( \zeta \) in the right critical strip \( \frac{1}{2} < \text{Re}(s) < 1 \).) Hence, \( \alpha \) is not invertible (because \( 0 \in \sigma(\alpha) \)).

3. For \( c \in (0, \frac{1}{2}) \), \( \sigma(\alpha) \) is unbounded and conditionally (i.e., under RH), \( \sigma(\alpha) \neq \mathbb{C} \) and, in particular, \( 0 \notin \sigma(\alpha) \), so that \( \alpha \) is invertible.

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The last statement in the third part of the theorem follows from the non-universality of \( \zeta \) in the left critical strip \( 0 < \text{Re}(s) < \frac{1}{2} \); see [KaSt].
Open Problems

1. What is $\sigma(a)$ when $c \in (0, 1/2)$? (It is a very complicated, closed, unbounded, and (conditionally) strict subset of $C$.)

2. Is RH needed for $\sigma(a) \neq C$ to be true? (Compare [KaSt].)

3. Does $0/\in \sigma(a)$, when $c \in (0, 1/2)$? Unconditionally (or else under the Lindelöf hypothesis), we conjecture that $0/\in \sigma(a)$ and hence, that $a$ is invertible for $0 < c < 1/2$.

4. Conjecturally, for $c = 1/2$, we have that $\sigma(a) = C$. Moreover, $a$ is clearly not invertible for $c = 1/2$ since $\zeta$ has zeroes on the critical line and hence, we know that $0/\in \sigma(a)$. 

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Conditionally (i.e., under RH), the spectral operator $a$ is quasi-invertible for all $c \neq \frac{1}{2}$ ($c \in (0, 1)$), and (unconditionally) it is not quasi-invertible (not even almost invertible) for $c = \frac{1}{2}$.

Remark:
1. Recall that $a$ is quasi-invertible for all $c \neq \frac{1}{2} \iff$ RH.
2. Furthermore, $a$ is not almost invertible for $c = \frac{1}{2}$ because $\zeta$ has infinitely many zeroes on the critical line $\text{Re}(s) = \frac{1}{2}$.
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**Theorem**

The spectrum $\sigma(\alpha)$ is non-compact (and hence, unbounded), but (conditionally) not all of $\mathbb{C}$ for $0 < c < \frac{1}{2}$. It is all of $\mathbb{C}$ for $\frac{1}{2} < c < 1$, and compact (and thus, bounded) for $c > 1$. 
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**Theorem**

Unconditionally, the spectral operator $\alpha$ is unbounded for $c < 1$, and is bounded for $c > 1$. 
Universality of the Riemann Zeta Function $\zeta = \zeta(s)$

The “universality” of $\zeta$ roughly means that any non-vanishing holomorphic function in \( \{ \frac{1}{2} < \text{Re}(s) < 1 \} \) can be approximated arbitrarily closely by imaginary translates of $\zeta$. 
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More precisely, we have the following well-known remarkable result (Extended Voronin Theorem).

**Theorem**

*Let $K$ be any compact subset of $\{\frac{1}{2} < \text{Re}(s) < 1\}$, with connected complement in $\mathbb{C}$. Let $g : K \rightarrow \mathbb{C}$ be a non-vanishing continuous function that is holomorphic in the interior of $K$ (which may be empty). Then, given any $\epsilon > 0$, there exists $\tau \geq 0$ (depending only on $\epsilon$) such that*
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$$J_{sc}(\tau) := \max_{s \in K} |g(s) - \zeta(s + i\tau)| \leq \epsilon.$$
In fact, the set of such $\tau$’s has a positive lower density and, in particular, is infinite. More precisely, we have

$$\liminf_{\rho \to +\infty} \frac{1}{\rho} \text{vol}_1 (\{\tau \in [0, \rho] : J_{sc}(\tau) \leq \epsilon\}) > 0.$$
Remarks:

- Voronin’s Original Universality Theorem (1975) corresponds to the choice of $K := \overline{D\left(\frac{3}{4}, r\right)}$, the closed disk of center $\frac{3}{4}$ and radius $r$, with $0 < r < \frac{1}{4}$ arbitrary.
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- The compact set \( K \) is allowed to have empty interior, in which case \( f \) is only required to be continuous (and without zeroes) in \( K \). In particular, if \( K \) is a line segment (on the real axis), then any continuous curve can be approximated by imaginary translates of \( \zeta \). Thus \( \zeta \) encodes all types of complex behaviors: it is chaotic.
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- If we assume the Riemann hypothesis, then $\zeta(s)$ does not have any zeroes in $\frac{1}{2} < \text{Re}(s) < 1$. Hence, applying the Universality Theorem to $g(s) := \zeta(s)$ and upon some elementary manipulations, one sees that scaled copies of $\zeta$ can be found within itself at all scales. In other words, conditionally, the Riemann zeta function is both fractal and chaotic.
The “universality” of the spectral operator $\alpha = \zeta(\partial)$ roughly means that any non-vanishing holomorphic function of $\partial$ in $\{ \frac{1}{2} < \text{Re}(s) < 1 \}$ can be approximated arbitrarily closely by imaginary translates of $\zeta(\partial)$. 
The “universality” of the spectral operator $\alpha = \zeta(\partial)$ roughly means that any non-vanishing holomorphic function of $\partial$ in $\{\frac{1}{2} < \Re(s) < 1\}$ can be approximated arbitrarily closely by imaginary translates of $\zeta(\partial)$.

More precisely, we have the following operator-theoretic generalization of the Extended Voronin Universality Theorem, expressed in terms of the imaginary translates of the $T$-truncated spectral operators $\alpha^{(T)} = \zeta(\partial^{(T)})$, where $\partial^{(T)} = \partial^{(T)}_c$ is the $T$-truncated infinitesimal shift (with parameter $c$).
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Let $K$ be a compact subset of $\{\frac{1}{2} < \text{Re}(s) < 1\}$ of the following form. Assume, for simplicity, that $K = \mathcal{K} \times [-T_0, T_0]$, for some $T_0 \geq 0$, where $\mathcal{K}$ is a compact subset of $(\frac{1}{2}, 1)$.
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Let $g : K \to \mathbb{C}$ be a non-vanishing continuous function that is holomorphic in the interior of $K$ (which may be empty). Then, given any $\epsilon > 0$, there exists $\tau \geq 0$ (depending only on $\epsilon$) such that
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\[
\mathcal{H}_{op}(\tau) := \sup_{c \in \mathcal{K}, 0 \leq T \leq T_0} \left\| g \left( \partial_c^{(T)} \right) - \zeta \left( \partial_c^{(T)} + i\tau \right) \right\| \leq \epsilon,
\]

where $\partial^{(T)} = \partial_c^{(T)}$ is the $T$-truncated infinitesimal shift (with parameter $c$) and $\| . \|$ is the norm in $\mathcal{B}(\mathbb{H}_c)$ (the space of bounded linear operators on $\mathbb{H}_c$).
In fact, the set of all such $\tau$'s has a positive lower density and, in particular, is infinite. More precisely, we have

$$\liminf_{\rho \to +\infty} \frac{1}{\rho} \text{vol}_1 \left( \{ \tau \in [0, \rho] : \mathcal{H}_{op}(\tau) \leq \epsilon \} \right) > 0.$$
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Remark:
A remarkable feature of the above generalization is the uniformity in the parameter $c \in \mathcal{K}$ and in $T \in [0, T_0]$ of the stated approximation of $g\left(\partial_c(T)\right)$. 
We will next state a further generalization of the *operator-theoretic Extended Voronin Universality Theorem*. For pedagogical reasons, we will choose assumptions (on the compact set $K$) that simplify its formulation. (The appropriate definitions and possible extensions will be given just after the statement of the theorem.)
Theorem

Let $K$ be any compact, vertically convex, subset of \( \{ \frac{1}{2} < \text{Re}(s) < 1 \} \), with connected complement in $\mathbb{C}$. Assume, for simplicity, that $K$ is symmetric with respect to the real axis. Denote by $K$ the projection of $K$ onto the real axis, and for $c \in K$, let

$$T(c) := \sup \{ T \geq 0 : [c - iT, c + iT] \subset K \}.$$

(By construction, $K$ is a compact subset of $(\frac{1}{2}, 1)$ and $0 \leq T(c) < \infty$, for $c \in K$.) Assume further that $c \mapsto T(c)$ is continuous on $K$. 
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J_{op}(\tau) := \sup_{c \in K, 0 \leq T \leq T(c)} \| g \left( \partial_c(T) \right) - \zeta \left( \partial_c(T) + iT \right) \| \leq \epsilon,
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where $\partial^{(T)} = \partial^{(T)}_c$ is the $T$-truncated infinitesimal shift (with parameter $c$) and $\|\cdot\|$ denotes the usual norm in $\mathcal{B}(\mathbb{H}_c)$ (the space of bounded linear operators on $\mathbb{H}_c$).

In fact, the set of such $\tau'$s has a positive lower density and, in particular, is infinite. More precisely, we have

$$\liminf_{\rho \to +\infty} \frac{1}{\rho} \text{vol}_1(\{\tau \in [0, \rho] : \mathcal{J}_{op}(\tau) \leq \epsilon\}) > 0.$$
Remarks:

To say that $K$ is vertically convex means that if $c^{-iT'}$ and $c^{+iT}$ belong to $K$ for some $c \in K$ and $T' \leq 0 \leq T$, then the entire line segment $[c^{-iT'}, c^{+iT}]$ is contained in $K$. Instead of assuming that $K$ is symmetric with respect to the real axis, it would suffice to suppose that $c^{+iT} \in K$ (for some $c \in K$ and $T > 0$) implies that $c^{-iT} \in K$, and vice versa.

As in the scalar case (and taking $K$ to be a line segment), we see that any continuous curve $(\partial (T))c$ can be approximated by imaginary translates of $a(T) = \zeta(\partial (T))c$. Hence, roughly speaking, the spectral operator $a$ (or its $T$-truncations $a(T)$) can emulate any type of complex behavior: it is chaotic.
Remarks:

- To say that $K$ is *vertically convex* means that if $c - iT'$ and $c + iT$ belong to $K$ for some $c \in K$ and $T' \leq 0 \leq T$, then the entire line segment $[c - iT', c + iT]$ is contained in $K$. 
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- As in the scalar case (and taking $K$ to be a line segment), we see that any continuous curve (of $\partial_c^{(T)}$) can be approximated by imaginary translates of $a^{(T)} = \zeta \left( \partial_c^{(T)} \right)$. Hence, roughly speaking, *the spectral operator $a$ (or its $T$-truncations $a^{(T)}$) can emulate any type of complex behavior: it is chaotic.*
Conditionally (i.e., under RH), and applying the above operator-theoretic Universality Theorem to \( g(s) := \zeta(s) \), we see that, roughly speaking, arbitrarily small scaled copies of the spectral operator are encoded within \( \alpha \) itself. In other words, \( \alpha \) (or its \( T \)-truncation) is both chaotic and fractal.
Future Research Directions

1. Study of the global spectral operator $\tilde{a} = \xi(\partial)$, where $\xi(s) = \pi - s^2 \Gamma(s^2) \zeta(s)$ is the global Riemann zeta function.

2. Study of the Euler products representation of $a$; see [HerLa3]. Adelic representation of $\tilde{a}$.

3. Extension to other L-functions (e.g., Dirichlet L-functions, zeta functions of number fields, etc.), as well as to members of the Selberg class.

4. Spectral operator and universality (both for $\zeta$ and other L-functions).
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