The legacy of Ramanujan’s mock theta functions: Harmonic Maass forms in number theory

Ken Ono
Emory University
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Srinivasa Ramanujan (1887-1920)

Srinivasa Ramanujan (1887-1920)
“Death bed letter”

Dear Hardy,

“I am extremely sorry for not writing you a single letter up to now.... I discovered very interesting functions recently which I call “Mock” $\vartheta$-functions. ... they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.”

Ramanujan, January 12, 1920.
Some examples

\[ f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 + q)^2(1 + q^2)^2 \cdots (1 + q^n)^2}, \]

\[ \omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1 - q)^2(1 - q^3)^2 \cdots (1 - q^{2n+1})^2}, \]

\[ \lambda(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (1 - q)(1 - q^3) \cdots (1 - q^{2n-1}) q^n}{(1 + q)(1 + q^2) \cdots (1 + q^{n-1})}. \]
Aftermath of the letter

Although Ramanujan’s secrets died with him, we have:
Aftermath of the letter

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- Works by Atkin, Andrews, Dyson, Selberg, Swinnerton-Dyer, and Watson on these 22 series.
Aftermath of the letter

Although Ramanujan’s secrets died with him, we have:

- Works by Atkin, Andrews, Dyson, Selberg, Swinnerton-Dyer, and Watson on these 22 series.
- Bolster the view that Ramanujan had found something.
G. N. Watson’s 1936 Presidential Address

"Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work... the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance."
G. N. Watson’s 1936 Presidential Address

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As much as any of his earlier work..., the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. ...”
“Ramanujan’s discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end.

As much as any of his earlier work... the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. ...”
Andrews unearths the "Lost Notebook" (1976)

Forgotten in the Trinity College archives.
“Lost Notebook” identities useful for...

- Hypergeometric functions
- Partitions and Additive Number Theory
- Mordell integrals
- Artin $L$-functions
- Mathematical Physics
- Probability theory...
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- Hypergeometric functions
- Partitions and Additive Number Theory
- Mordell integrals
- Artin $L$-functions
- Mathematical Physics
- Probability theory...

“Mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered... This remains a challenge for the future.”

Freeman Dyson, 1987
In his Ph.D. thesis under Zagier ('02), Zwegers investigated:
The legacy of Ramanujan’s mock theta functions: Harmonic Maass forms in number theory

Back to the future

The future is now

In his Ph.D. thesis under Zagier (’02), Zwegers investigated:

- “Lerch-type” series and Mordell integrals.
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- “Lerch-type” series and Mordell integrals.
- Resembling $q$-series of Andrews and Watson on mock thetas.
- Stitched them together give \textit{non-holomorphic Jacobi forms}. 
The legacy of Ramanujan’s mock theta functions: Harmonic Maass forms in number theory
Back to the future

Important Realizations
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- Ramanujan’s 22 examples are **pieces** of Maass forms.
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- Previously thought to be difficult to construct.
Important Realizations

- Ramanujan’s 22 examples are **pieces** of Maass forms.
- Previously thought to be difficult to construct.
- ...giving clues of **general theory** which in turn have **applications**.
Some applications

Partition Theory and $q$-series.
- $q$-series identities ("mock theta conjectures")
- Congruences (Dyson’s ranks)
- Exact formulas

Arithmetic and Modular forms.
- Donaldson invariants
- Eichler-Shimura Theory
- Moonshine for affine Lie superalgebras
- Borcherds-type automorphic products
- $L$-functions and the BSD numbers
Four samples
Four samples

1. *(Maass form congruences)*

**Extend the scope** of Serre, Swinnerton-Dyer, Deligne, Ribet....
Four samples

I. (Maass form congruences)  
**Extend the scope** of Serre, Swinnerton-Dyer, Deligne, Ribet....

II and III. (Exact formulas for Maass forms)  
**Extend and generalize phenomena** obtained previously by Rademacher and Zagier*.
Four samples

I. (Maass form congruences)
Extend the scope of Serre, Swinnerton-Dyer, Deligne, Ribet....

II and III. (Exact formulas for Maass forms)
Extend and generalize phenomena obtained previously by Rademacher and Zagier*.

IV. (Birch and Swinnerton-Dyer Numbers)
Unify work of Waldspurger and Gross-Zagier on BSD numbers $+\epsilon$. 
Comments
Since I, II, and III are very broad topics, I shall choose partitions to illustrate our results.
Since I, II, and III are very **broad** topics, I shall choose *partitions* to illustrate our results.

Along the way, I will explain some of the essential features (e.g. definitions) of the theory.
Adding and counting

Definition

A *partition* is any nonincreasing sequence of integers summing to $n$.

$$p(n) := \#\{\text{partitions of } n\}.$$
Adding and counting

**Definition**

*A partition* is any nonincreasing sequence of integers summing to \( n \).

\[
p(n) := \#\{\text{partitions of } n\}.
\]

**Example**

The partitions of 4 are:

\[
4, \quad 3 + 1, \quad 2 + 2, \quad 2 + 1 + 1, \quad 1 + 1 + 1 + 1,
\]

and so \( p(4) = 5 \).
Ramanujan’s Congruences

Theorem (Ramanujan)

For every $n$, we have

\[ p(5n + 4) \equiv 0 \pmod{5}, \]
\[ p(7n + 5) \equiv 0 \pmod{7}, \]
\[ p(11n + 6) \equiv 0 \pmod{11}. \]
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Integer Partitions

Ramanujan's Congruences

Theorem (Ramanujan)

For every $n$, we have

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p(5n + 4) \equiv 0 \pmod{5},
\]
\[
p(7n + 5) \equiv 0 \pmod{7},
\]
\[
p(11n + 6) \equiv 0 \pmod{11}.
\]

Remark

Attempting to explain them, Dyson defined the "rank."
Dyson’s Rank

Definition

The rank of a partition is its largest part minus its number of parts.

\[ N(m, n) := \# \{ \text{partitions of } n \text{ with rank } m \}. \]
### Dyson’s Rank

**Definition**

The *rank* of a partition is its largest part minus its number of parts.

\[ N(m, n) := \#\{ \text{partitions of } n \text{ with rank } m \}. \]

**Example**

The ranks of the partitions of 4:

<table>
<thead>
<tr>
<th>Partition</th>
<th>Largest Part</th>
<th># Parts</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>1</td>
<td>3 ≡ 3 (mod 5)</td>
</tr>
<tr>
<td>3 + 1</td>
<td>3</td>
<td>2</td>
<td>1 ≡ 1 (mod 5)</td>
</tr>
<tr>
<td>2 + 2</td>
<td>2</td>
<td>2</td>
<td>0 ≡ 0 (mod 5)</td>
</tr>
<tr>
<td>2 + 1 + 1</td>
<td>2</td>
<td>3</td>
<td>−1 ≡ 4 (mod 5)</td>
</tr>
<tr>
<td>1 + 1 + 1 + 1</td>
<td>1</td>
<td>4</td>
<td>−3 ≡ 2 (mod 5)</td>
</tr>
</tbody>
</table>
Dyson’s Conjecture

Definition

If $0 \leq r, t$, then let

$$N(r, t; n) := \# \{ \text{partitions of } n \text{ with rank } \equiv r \mod t \}.$$
Dyson’s Conjecture

Definition

If $0 \leq r, t$, then let

$$N(r, t; n) := \# \{\text{partitions of } n \text{ with rank } \equiv r \mod t\}.$$ 

Conjecture (Dyson, 1944)

For every $n$ and every $r$, we have

$$N(r, 5; 5n + 4) = p(5n + 4)/5,$$
$$N(r, 7; 7n + 5) = p(7n + 5)/7.$$
A famous theorem

Theorem (Atkin and Swinnerton-Dyer, 1954)

*Dyson's Conjecture is true.*
A famous theorem

**Theorem (Atkin and Swinnerton-Dyer, 1954)**

*Dyson's Conjecture is true.*

**Remark**

*The proof depends on the generating function:*

\[
R(w; q) = \sum_{m,n} N(m, n) w^m q^n := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n},
\]

where

\[
(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).
\]
Revealing specializations

Example

For \( w = 1 \) and \( q := e^{2\pi i z} \), we have the **modular form**

\[
q^{-1} R(1; q^{24}) = \sum_{n=0}^{\infty} p(n) q^{24n-1}.
\]
Revealing specializations

Example

- For $w = 1$ and $q := e^{2\pi iz}$, we have the **modular form**

\[ q^{-1}R(1; q^{24}) = \sum_{n=0}^{\infty} p(n)q^{24n-1}. \]

- For $w = -1$, we have Ramanujan’s **mock theta**

\[ R(-1; q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1 + q)^2(1 + q^2)^2 \cdots (1 + q^n)^2}. \]
A modular form is any meromorphic function \( f(z) \) on \( \mathbb{H} \) for which

\[
f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z)
\]

for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \text{SL}_2(\mathbb{Z}). \)
Question

*Since the deepest facts about $p(n)$ come from modular form theory, is $R(w; q)$, for roots of unity $w \neq 1$, modular?*
Question

Since the deepest facts about \( p(n) \) come from modular form theory, is \( R(w; q) \), for roots of unity \( w \neq 1 \), modular?

“Theorem” (Bringmann-O)

If \( w \neq 1 \) is a root of unity, then \( R(w; q) \) is the \textbf{holomorphic part} of a \textbf{harmonic Maass form}.
Notation. Throughout, let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$. 
Defining Maass forms

**Notation.** Throughout, let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$.

**Hyperbolic Laplacian.**

$$\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$
Harmonic Maass forms

“Definition”

A harmonic Maass form is any smooth function $f$ on $\mathbb{H}$ satisfying:

1. For all $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \subset \text{SL}_2(\mathbb{Z})$ we have
   $$f\left( \frac{az+b}{cz+d} \right) = (cz+d)^k f(z).$$

2. We have that $\Delta_k f = 0$. 
Harmonic Maass forms

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2. We have that $\Delta_k f = 0$. 

"Definition"
A bit more precisely

Definition

If $0 < a < c$ are integers, then let

$$ S \left( \frac{a}{c} ; z \right) := B(a, c) \int_{-\bar{z}}^{i\infty} \frac{\Theta \left( \frac{a}{c} ; \ell_c \tau \right)}{\sqrt{-i(\tau + z)}} \ d\tau, $$

and define

$$ D \left( \frac{a}{c} ; z \right) = -S \left( \frac{a}{c} ; z \right) + q^{-\ell c^2/24} R \left( \zeta a/c ; q^{\ell c} \right) \Theta \left( \frac{a}{c} ; \ell_c \tau \right). $$
A bit more precisely

**Definition**

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and define $D \left( \frac{a}{c}; z \right)$ by

$$D \left( \frac{a}{c}; z \right) := -S \left( \frac{a}{c}; z \right) + \frac{1}{24} \ell_c R(\zeta^a_c; q^{\ell_c})$$

\[\Theta \text{ — integral} \quad \text{Dyson’s series}\]
The first theorem

Theorem (Bringmann-O)

If $0 < a < c$, then $D\left(\frac{a}{c}; z\right)$ is a weight $1/2$ harmonic Maass form.
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Dyson’s ranks

The first theorem

Theorem (Bringmann-O)

If $0 < a < c$, then $D \left( \frac{a}{c}; z \right)$ is a weight $1/2$ harmonic Maass form.

Remark

By exhibiting Maass forms, this is already interesting.
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Dyson’s ranks

Partition congruences revisited

**Theorem (O, 2000)**

*For primes $Q \geq 5$, there are infinitely many non-nested progressions $An + B$ for which*

$$p(An + B) \equiv 0 \pmod{Q}.$$
Partition congruences revisited

**Theorem (O, 2000)**

*For primes $Q \geq 5$, there are infinitely many non-nested progressions $A_n + B$ for which*

$$p(A_n + B) \equiv 0 \pmod{Q}.$$  

**Examples.** Simplest ones for $17 \leq Q \leq 31$:

\begin{align*}
    p(48037937n + 1122838) &\equiv 0 \pmod{17}, \\
    p(1977147619n + 815655) &\equiv 0 \pmod{19}, \\
    p(14375n + 3474) &\equiv 0 \pmod{23}, \\
    p(348104768909n + 43819835) &\equiv 0 \pmod{29}, \\
    p(4063467631n + 30064597) &\equiv 0 \pmod{31}.
\end{align*}
I. Maass form congruences

**Theorem (Bringmann-O)**

There are progressions \( An + B \) s.t. for all \( 0 \leq r < t \)

\[
N(r, t; An + B) \equiv 0 \pmod{Q}.
\]
I. Maass form congruences

Theorem (Bringmann-O)

There are progressions $An + B$ s.t. for all $0 \leq r < t$

$$N(r, t; An + B) \equiv 0 \pmod{Q}.$$ 

Remark

This is a Dyson-style proof of

$$p(An + B) \equiv 0 \pmod{Q}.$$
“Proof” of the second theorem
“Proof” of the second theorem

- The Fourier expansions in $q := e^{2\pi i z}$ are special:

$$D \left( \frac{a}{c} ; z \right) = q^{-\frac{\ell_c}{24}} R(\zeta^a_c ; q^\ell_c) + \sum_{n \in \mathbb{Z}} B(a, c, n) \gamma(c, y; n) q^{-\tilde{\ell}_c n^2}.$$
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Proofs of the two theorems

“Proof” of the second theorem

- The Fourier expansions in $q := e^{2\pi i z}$ are special:

$$D \left( \frac{a}{c}; z \right) = q^{-\frac{\ell_c}{24}} R(\zeta^a_c; q^\ell_c) + \sum_{n \in \mathbb{Z}} B(a, c, n) \gamma(c, y; n) q^{-\tilde{\ell}_cn^2}.$$

- The “bad” coefficients are so sparse that the proof becomes

  Shimura correspondence + $\ell$-adic Galois repns + $\epsilon$. 
II. Exact formulas of Rademacher-type

Problem (Rademacher)

Define $\alpha(n)$ by

$$f(q) = R(-1; q)$$

$$= \sum_{n=0}^{\infty} \alpha(n)q^n = 1 + q - 2q^2 + \cdots + 487q^{47} + 9473q^{89} - \cdots.$$
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Define $\alpha(n)$ by

$$f(q) = R(-1; q) = \sum_{n=0}^{\infty} \alpha(n)q^n = 1 + q - 2q^2 + \cdots + 487q^{47} + 9473q^{89} - \cdots.$$ 

Find an exact formula for

$$\alpha(n) = N(0, 2; n) - N(1, 2; n)$$

\begin{align*}
\uparrow & \quad \uparrow \\
\text{even rank} & \quad \text{odd rank},
\end{align*}
Rademacher-type exact formula

**Conjecture (Andrews-Dragonette, 1966)**

If $n$ is a positive integer, then

$$
\alpha(n) = \pi (24n - 1)^{-\frac{1}{4}} \\
\times \sum_{k=1}^{\infty} \frac{(-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor} A_{2k} \left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot l_{\frac{1}{2}} \left(\frac{\pi \sqrt{24n - 1}}{12k}\right),
$$

where $A_k(n)$ is a “Kloosterman-type sum”.

\[\alpha(n) = \pi (24n - 1)^{-\frac{1}{4}} \times \sum_{k=1}^{\infty} \frac{(-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor} A_{2k} \left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot l_{\frac{1}{2}} \left(\frac{\pi \sqrt{24n - 1}}{12k}\right),\]
II. Exact formulas for Maass forms

**Theorem (Bringmann-O, 2006)**

*The Andrews-Dragonette Conjecture is true.*
II. Exact formulas for Maass forms

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Idea of the Proof.
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Idea of the Proof.

- By the first theorem, the holomorphic part of the Maass form $D\left(\frac{1}{2}; z\right)$ is $q^{-1}R(-1; q^{24})$. 
II. Exact formulas for Maass forms

Theorem (Bringmann-O, 2006)

The Andrews-Dragonette Conjecture is true.

Idea of the Proof.

- By the first theorem, the holomorphic part of the Maass form $D\left(\frac{1}{2}; z\right)$ is $q^{-1} R(-1; q^{24})$.

- Construct the “right” Poincaré series

$$P(z) = \frac{2}{\sqrt{\pi}} \sum_{M \in \Gamma_{\infty} \setminus \Gamma_0(2)} \chi(M)^{-1} (cz + d)^{-\frac{1}{2}} \phi(Mz),$$

where $\phi$ is a Whittaker function.
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Exact formulas for Maass forms

Idea of the proof

- The “right” one has a Fourier expansion

\[ P(24z) = \text{Nonholomorphic function} + \sum_{n=1}^{\infty} \beta(n)q^{24n-1}, \]

where the \( \beta(n) \)'s equal the expressions in the conjecture.
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Exact formulas for Maass forms

Idea of the proof

- The “right” one has a Fourier expansion

\[ P(24z) = \text{Nonholomorphic function} + \sum_{n=1}^{\infty} \beta(n)q^{24n-1}, \]

where the \( \beta(n) \)'s equal the expressions in the conjecture.

- Somehow prove that \( D \left( \frac{1}{2}; z \right) - P(24z) \) is 0.
II. Exact formulas for Maass forms

**Theorem (Bringmann-O)**

We have formulas for all harmonic Maass forms with weight $\leq 1/2$. 

Remark: Gives the theorem of Rademacher-Zuckerman for non-positive weight modular forms as a special case.
II. Exact formulas for Maass forms

Theorem (Bringmann-O)

We have formulas for all harmonic Maass forms with weight $\leq 1/2$.

Remark

Gives the theorem of Rademacher-Zuckerman for non-positive weight modular forms as a special case.
Relation to classical modular forms

\[ S_k(\Gamma) := \text{weight } k \text{ cusp forms on } \Gamma, \]
\[ H_{2-k}(\Gamma) := \text{weight } 2 - k \text{ harmonic Maass forms on } \Gamma. \]
Relation to classical modular forms

\[ S_k(\Gamma) := \text{weight } k \text{ cusp forms on } \Gamma, \]

\[ H_{2-k}(\Gamma) := \text{weight } 2 - k \text{ harmonic Maass forms on } \Gamma. \]

**Lemma**

If \( w \in \frac{1}{2} \mathbb{Z} \) and \( \xi_w := 2iy^w \overline{\frac{\partial}{\partial z}} \), then

\[ \xi_{2-k} : H_{2-k}(\Gamma) \longrightarrow S_k(\Gamma). \]

Moreover, this map is surjective.
Harmonic Maass forms have two parts \((q := e^{2\pi i z})\)

**Fundamental Lemma**

If \(f \in H_{2-k}\) and \(\Gamma(a, x)\) is the incomplete \(\Gamma\)-function, then

\[
f(z) = \sum_{n \gg -\infty} c_f^+(n)q^n + \sum_{n < 0} c_f^-(n)\Gamma(k - 1, 4\pi|n|y)q^n.
\]

\[
\uparrow \quad \uparrow
\]

Holomorphic part \(f^+\) \quad Nonholomorphic part \(f^-\)
Relation with classical modular forms

**Fundamental Lemma**

If $\xi_w := 2iy^w \frac{\partial}{\partial z}$, then

$$\xi : H_{2-k} \longrightarrow S_k$$

satisfies

$$\xi(f) = \xi(f^- + f^+) = \xi(f^-).$$
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Fourier expansions and $\xi$

Relation with classical modular forms

**Fundamental Lemma**

If $\xi_w := 2iy^w \frac{\partial}{\partial z}$, then

$$\xi : H_{2-k} \longrightarrow S_k$$

satisfies

$$\xi(f) = \xi(f^- + f^+) = \xi(f^-).$$

**Question**

*What does the holomorphic part $f^+$ unearth?*
Source of rich information

Question

Can one use **holomorphic** parts of Maass forms to **unearth** hidden information related to $S_k(\Gamma)$?
Question

Can one use holomorphic parts of Maass forms to unearth hidden information related to $S_k(\Gamma)$?

Answer. Yes, and we discuss two applications:
Source of rich information

**Question**

*Can one use holomorphic parts of Maass forms to unearth hidden information related to $S_k(\Gamma)$?*

**Answer.** Yes, and we discuss two applications:

- III. Exact formulas for Maass forms of Zagier-type.
- BSD numbers
Rademacher’s “exact formula”

**Theorem (Rademacher (1943))**

*If* $n$ *is a positive integer, then*

$$p(n) = \text{CRAZY convergent infinite sum.}$$
III. Exact formulas of Zagier-type

Theorem (Bruinier-O)

Let $F$ be the Maass form for which $\xi_{-2}(F)$ is in $S_4(6)$.

$p(n) = \frac{1}{24} n - \frac{1}{12} \cdot \left( P(\alpha n, 1) + P(\alpha n, 2) + \cdots + P(\alpha n, h_n) \right).$
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Exact formulas as Galois traces

III. Exact formulas of Zagier-type

Theorem (Bruinier-O)

Let $F$ be the Maass form for which $\xi_{-2}(F)$ is in $S_4(6)$. Then

$$P(z) := -\left( \frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y} \right) F(z)$$

has the property that

...
III. Exact formulas of Zagier-type

Theorem (Bruinier-O)

Let $F$ be the Maass form for which $\xi_{-2}(F)$ is in $S_4(6)$. Then

$$\mathbb{P}(z) := - \left( \frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y} \right) F(z)$$

has the property that its sum over disc $-24n + 1$ CM points is

$$p(n) = \frac{1}{24n - 1} \cdot (\mathbb{P}(\alpha_{n,1}) + \mathbb{P}(\alpha_{n,2}) + \cdots + \mathbb{P}(\alpha_{n,h_n})).$$
III. Exact formulas of Zagier-type

Theorem (Bruinier-O)

Let $F$ be the Maass form for which $\xi_{-2}(F)$ is in $S_4(6)$. Then

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has the property that its sum over disc $-24n + 1$ CM points is

$$p(n) = \frac{1}{24n - 1} \cdot (\mathbb{P}(\alpha_{n,1}) + \mathbb{P}(\alpha_{n,2}) + \cdots + \mathbb{P}(\alpha_{n,h_n})) .$$

Moreover, each $(24n - 1)\mathbb{P}(\alpha_{n,m})$ is an algebraic integer.
Hard proof that \( p(1) = 1. \)

If \( \beta := 161529092 + 18648492\sqrt{69}, \) then

\[
\frac{1}{23} \cdot \mathbb{P} \left( \frac{-1 + \sqrt{-23}}{12} \right) = \frac{1}{3} + \frac{\beta^{2/3} + 127972}{6\beta^{1/3}},
\]
\[
\frac{1}{23} \cdot \mathbb{P} \left( \frac{-13 + \sqrt{-23}}{24} \right) = \frac{1}{3} - \frac{\beta^{2/3} + 127972}{12\beta^{1/3}} + \frac{\beta^{2/3} - 127972}{4\sqrt{-3}\beta^{1/3}},
\]
\[
\frac{1}{23} \cdot \mathbb{P} \left( \frac{-25 + \sqrt{-23}}{36} \right) = \frac{1}{3} - \frac{\beta^{2/3} + 127972}{12\beta^{1/3}} - \frac{\beta^{2/3} - 127972}{4\sqrt{-3}\beta^{1/3}},
\]
Hard proof that $p(1) = 1$.

If $\beta := 161529092 + 18648492\sqrt{69}$, then

$$
\frac{1}{23} \cdot \mathbb{P} \left( \frac{-1 + \sqrt{-23}}{12} \right) = \frac{1}{3} + \frac{\beta^{2/3} + 127972}{6\beta^{1/3}},
$$
$$
\frac{1}{23} \cdot \mathbb{P} \left( \frac{-13 + \sqrt{-23}}{24} \right) = \frac{1}{3} - \frac{\beta^{2/3} + 127972}{12\beta^{1/3}} + \frac{\beta^{2/3} - 127972}{4\sqrt{-3}\beta^{1/3}},
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$$

and we see directly that

$$
p(1) = 1 = \frac{1}{23} \left( \mathbb{P}(\alpha_1) + \mathbb{P}(\alpha_2) + \mathbb{P}(\alpha_3) \right).
$$
First few minimal polynomials

<table>
<thead>
<tr>
<th>n</th>
<th>( x^{h_n} - (24n - 1)p(n)x^{h_n-1} + \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x^3 - 23 \cdot 1x^2 + \frac{3592}{23}x - 419 )</td>
</tr>
<tr>
<td>2</td>
<td>( x^5 - 47 \cdot 2x^4 + \frac{169659}{47}x^3 - 65838x^2 + \frac{1092873176}{47^2}x + \frac{1454023}{47} )</td>
</tr>
<tr>
<td>3</td>
<td>( x^7 - 71 \cdot 3x^6 + \frac{1312544}{71}x^5 - 723721x^4 + \frac{44648582886}{71^2}x^3 + \frac{9188934683}{71}x^2 + \frac{166629520876208}{71^3}x + \frac{2791651635293}{71^2} )</td>
</tr>
<tr>
<td>4</td>
<td>( x^8 - 95 \cdot 5x^7 + \frac{9032603}{95}x^6 - 9455070x^5 + \frac{3949512899743}{95^2}x^4 - \frac{97215753021}{19}x^3 + \frac{9776785708507683}{95^3}x^2 - \frac{53144327916296}{19^2}x - \frac{134884469547631}{5^4 \cdot 19} )</td>
</tr>
</tbody>
</table>
Zagier’s famous Berkeley lectures
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- Zagier’s Berkeley lectures give **modular** generating functions for such algebraic traces when

  \[ \xi_{2-k}(F) = 0. \]
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- Using different ideas (i.e. theta integrals with Kudla-Millson kernels), we obtain the general framework.
Zagier’s Berkeley lectures give modular generating fcns for such algebraic traces when

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Generating fcns are holomorphic parts $f^+$ of Maass forms.
III. Exact formulas of Zagier-type
The legacy of Ramanujan’s mock theta functions: Harmonic Maass forms in number theory
Exact formulas as Galois traces

III. Exact formulas of Zagier-type

Theorem (Bruinier-O)

The generating function for the CM Galois traces of a good \( \mathbb{Q} \)-rational Maass form with Laplacian eigenvalue \( \lambda = -2 \) is the holomorphic part \( f^+ \) of a weight \(-1/2\) harmonic Maass form.

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IV: BSD Numbers

The rational points of an elliptic curve over a number field form a finitely generated abelian group.
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*Determine the integers which are areas of rational right triangles.*

Example

1. The number 6 is congruent since it is the area of (3, 4, 5).

2. The number 157 is congruent, since it is the area of

\[
\left( \frac{411340519227716149383203}{21666555693714761309610}, \frac{680 \cdots 540}{411 \cdots 203}, \frac{224 \cdots 041}{891 \cdots 830} \right).
\]
A Classical Diophantine Criterion

Theorem (Easy)

An integer $D$ is congruent if and only if the elliptic curve

$$E_D : \quad Dy^2 = x^3 - x$$

has positive rank.
Quadratic twists

Definition

Let $E/\mathbb{Q}$ be the elliptic curve

$$E : y^2 = x^3 + Ax + B.$$
Quadratic twists

Definition

Let $E / \mathbb{Q}$ be the elliptic curve

$$E : y^2 = x^3 + Ax + B.$$ 

If $\Delta$ is a fundamental discriminant, then the $\Delta$-quadratic twist of $E$ is

$$E(\Delta) : \Delta y^2 = x^3 + Ax + B.$$
Birch and Swinnerton-Dyer Conjecture

Conjecture

If $E/\mathbb{Q}$ is an elliptic curve and $L(E, s)$ is its $L$-function, then

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A good question. How does one compute $\text{ord}_{s=1}(L(E, s))$?
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If $\text{ord}_{s=1}(L(E, s)) \leq 1$, then

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Question

How does one compute \( L(E, 1) \) and \( L'(E, 1) \)?
The legacy of Ramanujan’s mock theta functions: Harmonic Maass forms in number theory
Algebraicity of Maass forms and L-functions
Formulas for L-values and derivatives

Formulas for $L(E(\Delta), 1)$

Theorem (Shimura-Kohnen/Zagier-Waldspurger)

There is a modular form

$$g(z) = \sum_{n=1}^{\infty} b_E(n) q^n$$

such that if $\Delta < 0$ and $\left( \frac{\Delta}{p} \right) = 1$, then

$$L(E(\Delta), 1) = \alpha_E(\Delta) \cdot b_E(|\Delta|)^2.$$
The Gross-Zagier Theorem

Question

What about derivatives?
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**The Gross-Zagier Theorem**

**Question**

*What about derivatives?*

**Theorem (Gross and Zagier)**

*If $\Delta > 0$ and $\left( \frac{\Delta}{p} \right) = 1$, then for suitable $d < 0$ the global Neron-Tate height on $J_0(p)(H)$ of $y_{\Delta,r}(-n, h)$ is*

$$\beta_E(\Delta, d) \cdot L(E(d), 1) \cdot L'(E(\Delta), 1).$$
Natural Question

Question

Find an extension of the Kohnen-Waldspurger theorem giving both

\[ L(E(\Delta), 1) \quad \text{and} \quad L'(E(\Delta), 1). \]
The legacy of Ramanujan’s mock theta functions: Harmonic Maass forms in number theory

Algebraicity of Maass forms and \( L \)-functions

Results on \( L \)-values and derivatives

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$$f_g = f_g^+ + f_g^-$$

$\cap$ $\cap$ $\cap$

$H_{1/2}^+(4p)$ $\rightarrow$ $S_{3/2}^+(4p)$ $\rightarrow$ $S_2(p)$

$\uparrow$ $\uparrow$ $\uparrow$

$\xi$ Kohnen-Shimura $\uparrow$

$E/\mathbb{Q}$

$G$ Modularity
The following are true:

1. If \( \Delta < 0 \) and \( (\Delta_p) = 1 \), then
   \[ L(E(\Delta), 1) = \tilde{\alpha} E(\Delta) \cdot c^{-g(\Delta)^2}. \]

2. If \( \Delta > 0 \) and \( (\Delta_p) = 1 \), then
   \[ L'(E(\Delta), 1) = 0 \iff c + g(\Delta) \text{ is algebraic}. \]
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2. If $\Delta > 0$ and $(\Delta_p) = 1$, then
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Example for $E: \ y^2 = x^3 + 10x^2 - 20x + 8.$

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$c_g^+(-\Delta)$</th>
<th>$L'(E(\Delta), 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>1.0267149116...</td>
<td>1.4792994920...</td>
</tr>
<tr>
<td>$-4$</td>
<td>1.2205364009...</td>
<td>1.8129978972...</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$-136$</td>
<td>$-4.8392675993...$</td>
<td>5.7382407649...</td>
</tr>
<tr>
<td>$-139$</td>
<td>$-6$</td>
<td>0</td>
</tr>
<tr>
<td>$-151$</td>
<td>$-0.8313568817...$</td>
<td>6.6975085515...</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$-815$</td>
<td>121.1944103120...</td>
<td>4.7492583693...</td>
</tr>
<tr>
<td>$-823$</td>
<td>312</td>
<td>0</td>
</tr>
</tbody>
</table>
Overview of the proof
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- The equivalence of $L'(E(\Delta), 1) = 0$ and the algebraicity of $c_g^+(\Delta)$ involves a **detailed study** of Heegner divisors.
Overview of the proof

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- The equivalence of $L'(E(\Delta), 1) = 0$ and the algebraicity of $c^+_g(\Delta)$ involves a detailed study of Heegner divisors.

- Algebraicity is dictated by the vanishing of Heegner divisors, and Gross-Zagier gives the connection to

$$L'(E(\Delta), 1).$$
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“Detailed study” of Heegner divisors

**Theorem (Bruinier-O)**

We have that $\eta_{\Delta}(z, f_g) := -\frac{1}{2} \partial \Phi_\Delta(z, f_g)$ is a differential on $X_0(p)$ with Heegner divisor. Moreover, we have

$$\eta_{\Delta}(z, f_g) = \left( \rho_{f_g, \ell} - \text{sgn}(\Delta) \sqrt{\Delta} \sum_{n \geq 1} \sum_{d \mid n} \frac{n}{d} \left( \frac{\Delta}{d} \right) c_g^+ \left( \frac{|\Delta|n^2}{4Nd^2} \right) e(nz) \right) \cdot 2\pi i dz.$$
Ramanujan $\Rightarrow$ harmonic Maass forms.

Building general theory has applications, such as:

- (Maass form congruences)
  - Extend the scope of Serre, Swinnerton-Dyer, Deligne, Ribet, etc.

- (Exact formulas for Maass forms)
  - Extend and generalize phenomena obtained previously by Rademacher and Zagier $^\ast$.

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