Finance 651: PDEs and Stochastic Calculus
Midterm Examination
November 9, 2012
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Disclaimer: It is essential to write legibly and show your work. If your work is absent or illegible, and at the same time your answer is not perfectly correct, then no partial credit can be awarded. Completely correct answers which are given without justification may receive little or no credit.

During this exam, you are not permitted to use calculators, notes, or books, nor to collaborate with others.

Improvements to the questions that were written on the white board during the exam are in blue. Solutions are in red.
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The top 6 of 7 answers count.
Problem 1. Suppose we have a stock priced at $S_0 = 4$ at time 0, and at either $S_1(H) = uS_0$ or $S_1(T) = dS_0$ at time 1, where $u = 2$ and $d = 1/2$. Consider a call option $V$ on the stock with strike $K = 7$, which can be exercised at time 1.

(i) Find the price $V_0$ of the option at time 0 if the interest rate is $r = 1/4$.

(ii) Find the price $V_0$ of the option at time 0 if the interest rate is $r = 1/3$.

Solution to (i):

\[
\tilde{E}\left(\frac{1}{1 + r}(S_1 - K)^+\right) = \frac{2}{5} \cdot (8 - 7) = \frac{2}{5}
\]

Solution to (ii): the risk-neutral probabilities are now changed, we have

\[
\tilde{p} = \frac{1 + r - d}{u - d} = \frac{1 + \frac{1}{3} - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{6 + 2 - 3}{12 - 3} = \frac{5}{9}
\]

and so

\[
\tilde{E}\left(\frac{1}{1 + r}(S_1 - K)^+\right) = \frac{1}{1 + \frac{1}{3}} \cdot (8 - 7) \frac{5}{9} = \frac{5}{12}
\]
Problem 2. Toss a coin repeatedly. Assume the probability of head on each toss is \( \frac{1}{2} \), as is the probability of tail. Let \( X_j = 1 \) if the \( j \)th toss results in a head and \( X_j = -1 \) if the \( j \)th toss results in a tail. Consider the stochastic processes \( I_0, I_1, I_2, \ldots \) and \( J_0, J_1, J_2, \ldots \) defined by \( I_0 = J_0 = 0, J_1 = 0, \)

\[
I_n = \sum_{j=1}^{n-1} X_j X_{j+1}, \quad \text{for } n \geq 1 \quad \text{and} \\
J_n = X_{n-1} X_n, \quad \text{for } n \geq 2.
\]

Note that

\[
\sum_{i=1}^{0} (\cdots) = 0.
\]

(i) Determine whether or not \( I_0, I_1, I_2, \ldots \) is a martingale. That is, whether it is an adapted process where the \( I_j \) only depends on the first \( j \) coin tosses, and whether \( \mathbb{E}_n(I_{n+1}) = I_n \).

(ii) Determine whether or not \( J_0, J_1, J_2, \ldots \) is a martingale.

Solution: \( I_n \) and \( J_n \) are both adapted processes, and

\[
\mathbb{E}_n(I_{n+1}) = \mathbb{E}_n(I_n + X_n X_{n+1}) = I_n + X_n \mathbb{E}_n(X_{n+1}) = I_n
\]

but

\[
\mathbb{E}_n(J_{n+1}) = \mathbb{E}_n(X_n X_{n+1}) = X_n \mathbb{E}_n(X_{n+1}) = 0 \neq J_n
\]

So \( I_n \) is a martingale but \( J_n \) is not.
Problem 3. Consider a binomial pricing model, but at each time $n \geq 1$, the “up” factor $u_n(\omega_1 \cdots \omega_n)$ and the “down factor” $d_n(\omega_1 \cdots \omega_n)$ and the interest rate $r_n(\omega_1 \cdots \omega_n)$ are allowed to depend on $n$ and on the first $n$ coin tosses $\omega_1 \omega_2 \cdots \omega_n$. The initial up factor $u_0$, the initial down factor $d_0$, and the initial interest rate $r_0$ are not random. More specifically, the stock prices $S_0$, $S_1$, and $S_2$ at times zero, one, and two, respectively, and the interest rates $r_0$, $r_1$, are as given in Figure 1.

Find the risk-neutral probabilities of the four possible paths through the graph:

\[ \tilde{P}(HH) \]
\[ \tilde{P}(HT) \]
\[ \tilde{P}(TH) \]
\[ \tilde{P}(TT) \]

such that the time-zero price of an option that pays off $V_2$ at time two is given by the risk-neutral pricing formula

\[ V_0 = \tilde{E} \left[ \frac{V_2}{(1 + r_0)(1 + r_1)} \right] . \]

Hint: start by finding the conditional probability of $\omega_1 \omega_2 = HH$ given $\omega_1 = H$, and then work your way through the tree.
Space for work on Problem 3. The risk neutral probability $\tilde{p} = \tilde{P}(HH \mid H)$ is such that
\[
8 = \left( \frac{1}{1 + (1/4)} \right) (12\tilde{p} + 8\tilde{q}) = \frac{4}{5} (12\tilde{p} + 8 - 8\tilde{p})
\]
so $\tilde{p} = 1/2$.

Proceeding similarly find
\[
\tilde{P}(HT \mid H) = 1/2, \\
\tilde{P}(TH \mid T) = \tilde{p} = 1/6
\]
using
\[
\frac{1}{1 + \frac{1}{2}} (8\tilde{p} + 2(1 - \tilde{p})) = 2
\]
and $\tilde{P}(TT \mid T) = 5/6$ and finally
\[
\tilde{P}(H) = \tilde{p} = 1/2 \text{ where } 4 = \frac{1}{1 + \frac{1}{4}} (8\tilde{p} + 2(1 - \tilde{p}))
\]
and $\tilde{P}(T) = 1/2$. Thus $\tilde{P}(HH) = 1/4 = \tilde{P}(HT)$ and $\tilde{P}(TH) = \frac{11}{26} = \frac{1}{12}$, and $\tilde{P}(TT) = \frac{5}{26} = \frac{5}{12}$. 
Problem 4. In this problem we price a call option on an average stock price over 2 periods. The intrinsic value of the option is the average stock price so far minus the strike $K$ (or 0 if that is negative):

$$G_n = \left( \frac{1}{n+1} \sum_{i=0}^{n} S_i - K \right)^+$$

where $K = 4$, $u = 2$, $d = 1/2$, $\bar{p} = \bar{q} = 1/2$, and $r = 1/4$.

(i) Fill in the two missing parts labeled “???” in Figure 2. Justify your answers!

$S_2(TH) = uS_2(T) = 2 \cdot 2 = 4$ and $S_2(TT) = dS_2(T) = \frac{1}{2} \cdot 2 = 1$. 

Figure 2: Intrinsic values.
Problem 4 continued.

(ii) Fill in the two missing parts labeled “???” in Figure 3. Justify your answers!

\[ V_1(H) = \frac{2}{5} \left( \frac{28}{3} - 4 + \frac{16}{3} - 4 \right) = \frac{8}{3} \]

(since \( \frac{8}{3} > 2 \), we do not exercise the option).

\[ V_0 = \frac{2}{5} \cdot \frac{8}{3} = \frac{16}{15} \]

(since \( \frac{16}{15} > 0 \), we do not exercise the option).
Problem 5. Suppose the interest rate $r = 0$ and we have a stock priced at either $S_1(H) = 6$ or $S_1(T) = 3$ at time 1. At time 0 the stock is priced at $S_0 = 4$. Now consider a derivative security that pays off either $V_1(H) = 9$ or $V_1(T) = 2$ at time 1. Find the price $V_0$ of the security at time 0.

Solution: We can use either of the following approaches.

(i) Answer using risk-neutral pricing.

(ii) Answer without using risk-neutral pricing, but instead using either a hedging argument or an argument about $V$ being a linear function of $S$.

(i) The up and down factors are $u = 6/4$ and $d = 3/4$. The risk-neutral probability of heads is

$$\tilde{p} = \frac{1 + r - d}{u - d} = \frac{1 - \frac{3}{4}}{\frac{6}{4} - \frac{3}{4}} = \frac{4 - 3}{6 - 3} = \frac{1}{3}.$$ 

So

$$V_0 = \tilde{E}(V_1) = \frac{1}{3} \cdot 9 + \frac{2}{3} \cdot 2 = 4 + \frac{1}{3}.$$ 

(ii) $V$ is 9 or 2 according as $S$ is 6 or 3; so let us find $a$ and $b$ with

$$V_1 = aS_1 + b$$

in both cases, giving

$$9 = 6a + b$$

$$2 = 3a + b$$

Subtracting the second equation from the first we find that

$$7 = 3a$$

so $a = 7/3$ and then $b = -5$. Then we should also have

$$V_0 = aS_0 + b = \frac{7}{3} \cdot 4 - 5 = 4 + \frac{1}{3}.$$ 

To construe this as a hedging argument, let us observe that $7/3$ shares of stock, together with a debt of 5, is worth the same as the option. So starting with an initial capital of $4 + \frac{1}{3}$ we can borrow 5, then buy $7/3$ shares of stock, and we will at time 1 have exactly the value of the option.
Problem 6. A 2-period interest rate swap is a contract that makes payments $S_1$, $S_2$ at times 1, 2, respectively, where $S_n = K - R_{n-1}$, $n = 1, 2$. The fixed rate $K$ is constant. The 2-period swap rate $SR_2$ is the value of $K$ that makes the time-zero no-arbitrage price of the interest rate swap equal to zero. Assume the risk-neutral probabilities are $\tilde{P}(H) = \tilde{P}(T) = 1/2$.

(i) Find $SR_2$ for the interest rate process with $R_0 = 1/4$ and $R_1(H) = 1/4$, $R_1(T) = 1/2$, as in Problem 3; but do not look at Problem 3 too much when doing this problem.

(ii) Determine whether your answer is larger than $1/3$.

Solution: The value of the payments at time zero is

$$0 = \frac{1}{1 + R_0}(K - R_0) + \tilde{E} \frac{1}{(1 + R_0)(1 + R_1)}(K - R_1)$$

$$= \frac{4}{5} \left( K - \frac{1}{4} \right) + \frac{14}{25} \left( \frac{1}{(1 + R_1(H))(K - R_1(H))} + \frac{1}{(1 + R_1(T))(K - R_1(T))} \right)$$

$$= \frac{4}{5} \left( K - \frac{1}{4} \right) + \frac{14}{25} \left( \frac{1}{(1 + \frac{1}{4})}(K - \frac{1}{4}) + \frac{1}{(1 + \frac{1}{2})}(K - \frac{1}{2}) \right)$$

Multiplying both sides by 5 and simplifying the fractions,

$$0 = 4 \left( K - \frac{1}{4} \right) + 2 \left( \frac{4}{5} (K - \frac{1}{4}) + \frac{2}{3} (K - \frac{1}{2}) \right)$$

Multiplying by 15,

$$0 = 15 \cdot 4 \left( K - \frac{1}{4} \right) + 2 \left( 12(K - \frac{1}{4}) + 10(K - \frac{1}{2}) \right)$$

Simplifying,

$$104K - 15 - 6 - 10 = 0$$

$$K = \frac{31}{104} < \frac{1}{3}$$

since $93 < 104$. 

Score: _______
Problem 7. Let $\tau_2$ be the first hitting time of 2 for a symmetric random walk $M_n = \sum_{i=1}^{n} X_i$. (Symmetric means that $P(X_i = 1) = P(X_i = -1) = 1/2$.)

(i) Calculate $P(\tau_2 \leq 5)$.

(ii) Suppose we think of the times $n$ for the random walk $M_n$ as trading dates, and consider an option which pays $1 at the random time $\tau_2$ if $\tau_2 \leq 5$. (If $\tau_2 > 5$ then the option expires worthless and pays zero.) Suppose the interest rate is 0 and the risk-neutral probabilities are $\tilde{p} = \tilde{q} = 1/2$. Find the price $V_0$ at time 0 of the option.

(iii) Now suppose the interest rate in force between the steps of the random walk is $r = 1/4$. Keep assuming that $\tilde{p} = \tilde{q} = 1/2$. Find $V_0$, and determine whether $V_0 < 1/2$.

Solution:

(i) Since one cannot be at an odd position at an even time, this is the same as $P(\tau_2 \leq 4)$. If we start with T then it must be THHH. If we start with H then it can be either HH, or we can start with HT, and then it must be HTHH. So the probability is $\frac{1}{4} + \frac{2}{16} = \frac{3}{8}$.

(ii) Since there is no interest, the answer is again $3/8$.

(iii) Now

$$V_0 = \left(\frac{1}{2}\right)^2 \left(\frac{4}{5}\right)^2 + 2 \left(\frac{1}{2}\right)^4 \left(\frac{4}{5}\right)^4 = \frac{4}{25} + \frac{32}{625} = \frac{132}{625} < 1/2.$$