Finance 651: PDEs and Stochastic Calculus
Midterm Examination
November 9, 2012

↑ Student name ↑

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Disclaimer: It is essential to write legibly and *show your work*. If your work is absent or illegible, and at the same time your answer is not perfectly correct, then no partial credit can be awarded. Completely correct answers which are given without justification may receive little or no credit.

During this exam, you are not permitted to use calculators, notes, or books, nor to collaborate with others.

Improvements to the questions that were written on the white board during the exam are in blue.

Score: _______

Problem	Score/5
1	
2	
3	
4	
5	
6	
7	
Total/30	

The top 6 of 7 answers count.

Score: _______

Problem 1. Suppose we have a stock priced at $S_0 = 4$ at time 0, and at either $S_1(H) = uS_0$ or $S_1(T) = dS_0$ at time 1, where $u = 2$ and $d = 1/2$. Consider a call option V on the stock with strike $K = 7$, which can be exercised at time 1.

(i) Find the price V_0 of the option at time 0 if the interest rate is $r = 1/4$.

(ii) Find the price V_0 of the option at time 0 if the interest rate is $r = 1/3$.

Score: _______

Problem 2. Toss a coin repeatedly. Assume the probability of head on each toss is $\frac{1}{2}$, as is the probability of tail. Let $X_j = 1$ if the jth toss results in a head and $X_j = -1$ if the jth toss results in a tail. Consider the stochastic processes $I_0, I_1, I_2, \ldots$ and $J_0, J_1, J_2, \ldots$ defined by $I_0 = J_0 = 0$, $J_1 = 0$,

$$I_n = \sum_{j=1}^{n-1} X_j X_{j+1}, \text{ for } n \geq 1 \text{ and}$$

$$J_n = X_{n-1} X_n, \quad \text{for } n \geq 2.$$

Note that

$$\sum_{i=1}^{0} (\cdots) = 0.$$

(i) Determine whether or not $I_0, I_1, I_2, \ldots$ is a martingale. That is, whether it is an adapted process where the I_j only depends on the first j coin tosses, and whether $\mathbb{E}_n(I_{n+1}) = I_n$.

(ii) Determine whether or not $J_0, J_1, J_2, \ldots$ is a martingale.

Score: _______

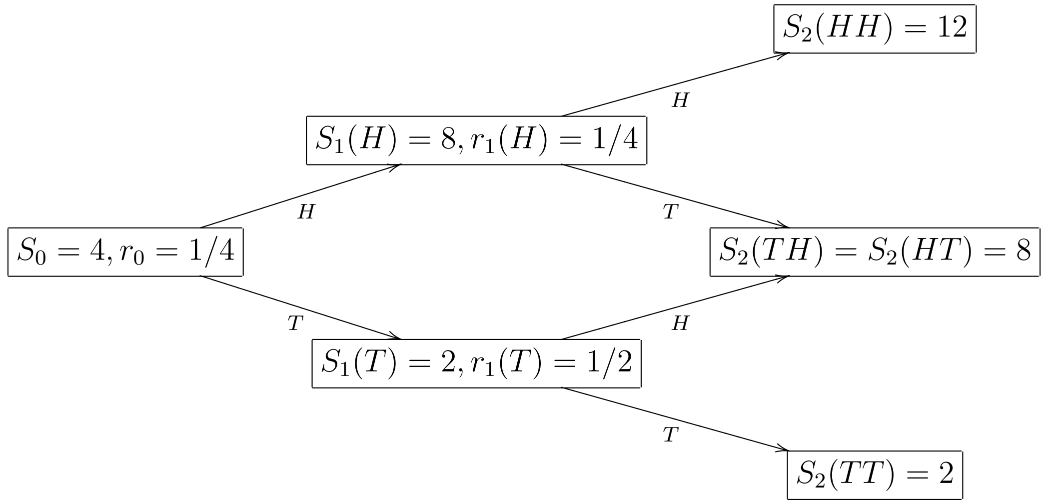

Figure 1: A stochastic volatility, random interest rate model.

Problem 3. Consider a binomial pricing model, but at each time $n \geq 1$, the "up" factor $u_n(\omega_1 \cdots \omega_n)$ and the "down factor" $d_n(\omega_1 \cdots \omega_n)$ and the interest rate $r_n(\omega_1 \cdots \omega_n)$ are allowed to depend on n and on the first n coin tosses $\omega_1\omega_2 \cdots \omega_n$. The initial up factor u_0, the initial down factor d_0, and the initial interest rate r_0 are not random. More specifically, the stock prices S_0, S_1, and S_2 at times zero, one, and two, respectively, and the interest rates r_0, r_1, are as given in Figure 1.

Find the risk-neutral probabilities of the four possible paths through the graph:

$\tilde{\mathbb{P}}(HH)$

$\tilde{\mathbb{P}}(HT)$

$\tilde{\mathbb{P}}(TH)$

$\tilde{\mathbb{P}}(TT)$

such that the time-zero price of an option that pays off V_2 at time two is given by the risk-neutral pricing formula

$$V_0 = \tilde{\mathbb{E}}\left[\frac{V_2}{(1+r_0)(1+r_1)}\right].$$

Hint: start by finding the conditional probability of $\omega_1\omega_2 = HH$ given $\omega_1 = H$, and then work your way through the tree.

Score: _______

Space for work on Problem 3.

Score: ________

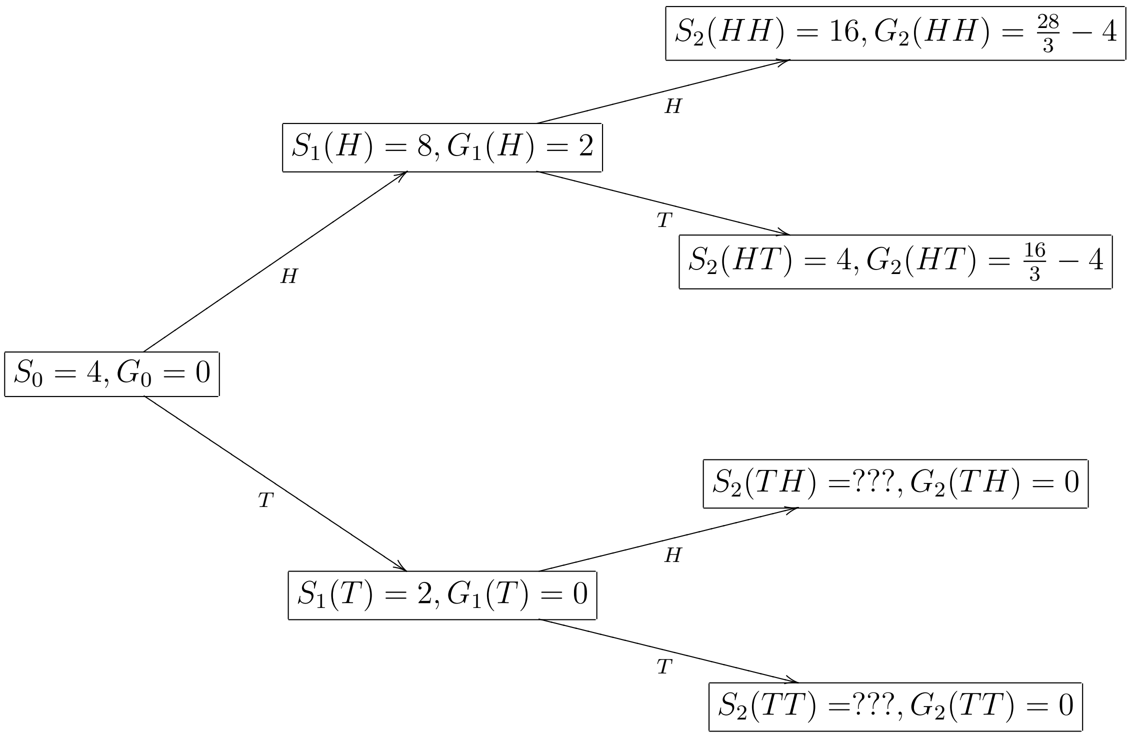

Figure 2: Intrinsic values.

Problem 4. In this problem we price a call option on an average stock price over 2 periods. The intrinsic value of the option is the average stock price so far minus the strike K (or 0 if that is negative):

$$G_n = \left(\frac{1}{n+1} \sum_{i=0}^{n} S_i - K \right)^+$$

where $K = 4$, $u = 2$, $d = 1/2$, $\tilde{p} = \tilde{q} = 1/2$, and $r = 1/4$.

(i) Fill in the two missing parts labeled "???" in Figure 2. Justify your answers!

Score: _______

$$
\begin{tikzcd}[column sep=large]
 & & \boxed{V_2(HH) = G_2(HH) = \frac{28}{3} - 4} \\
 & \boxed{V_1(H) = ???, G_1(H) = 2} \arrow[ur, "H"] \arrow[dr, "T"'] & \\
 & & \boxed{V_2(HT) = G_2(HT) = \frac{16}{3} - 4} \\
\boxed{V_0 = ???, G_0 = 0} \arrow[uur, "H"] \arrow[ddr, "T"'] & & \\
 & & \boxed{V_2(TH) = G_2(TH) = 0} \\
 & \boxed{V_1(T) = G_1(T) = 0} \arrow[ur, "H"] \arrow[dr, "T"'] & \\
 & & \boxed{V_2(TT) = G_2(TT) = 0}
\end{tikzcd}
$$

Figure 3: Option prices.

Problem 4 continued.

(ii) Fill in the two missing parts labeled "???" in Figure 3. Justify your answers!

Score: _______

Problem 5. Suppose the interest rate $r = 0$ and we have a stock priced at either $S_1(H) = 6$ or $S_1(T) = 3$ at time 1. At time 0 the stock is priced at $S_0 = 4$. Now consider a derivative security that pays off either $V_1(H) = 9$ or $V_1(T) = 2$ at time 1. Find the price V_0 of the security at time 0.

Score: _______

Problem 6. A 2-period interest rate swap is a contract that makes payments S_1, S_2 at times 1, 2, respectively, where $S_n = K - R_{n-1}$, $n = 1, 2$. The fixed rate K is constant. The 2-period swap rate SR_2 is the value of K that makes the time-zero no-arbitrage price of the interest rate swap equal to zero. Assume the risk-neutral probabilities are $\tilde{\mathbb{P}}(H) = \tilde{\mathbb{P}}(T) = 1/2$.

(i) Find SR_2 for the interest rate process with $R_0 = 1/4$ and $R_1(H) = 1/4$, $R_1(T) = 1/2$, as in Problem 3; but do not look at Problem 3 too much when doing this problem.

(ii) Determine whether your answer is larger than $1/3$.

Score: _______

Problem 7. Let τ_2 be the first hitting time of 2 for a symmetric random walk $M_n = \sum_{i=1}^{n} X_i$. (Symmetric means that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$.)

(i) Calculate $\mathbb{P}(\tau_2 \leq 5)$.

(ii) Suppose we think of the times n for the random walk M_n as trading dates, and consider an option which pays \$1 at the random time τ_2 if $\tau_2 \leq 5$. (If $\tau_2 > 5$ then the option expires worthless and pays zero.) Suppose the interest rate is 0 and the risk-neutral probabilities are $\tilde{p} = \tilde{q} = 1/2$. Find the price V_0 at time 0 of the option.

(iii) Now suppose the interest rate in force between the steps of the random walk is $r = 1/4$. Keep assuming that $\tilde{p} = \tilde{q} = 1/2$. Find V_0, and determine whether $V_0 < 1/2$.