FIN 651: PDEs and Stochastic Calculus
Final Exam
December 14, 2012

Solutions
↑ Student name ↑

Instructor: Bjørn Kjos-Hanssen

Disclaimer: It is essential to write legibly and *show your work*. If your work is absent or illegible, and at the same time your answer is not perfectly correct, then no partial credit can be awarded. Completely correct answers which are given without justification may receive little or no credit.

During this exam, you are not permitted to use calculators, notes, or books, nor to collaborate with others.

Score: _______

Problem	Score/4
1	
2	
3	
4	
5	
6	
Total/20	
$\times 5/2$ =Total/50	

The top 5 of 6 questions count.

Score: _______

Problem 1. *Calculate the variance of the Ito integral* $\int_0^T W_t dW_t$.

Hint #1: you may use without proof the *Ito isometry*, which says that

$$\mathbb{E}\left(\left(\int_0^T \Delta_t dW_t\right)^2\right) = \mathbb{E}\int_0^T \Delta_t^2 dt.$$

Hint #2 (alternative to Hint #1): You may start by using Ito's formula to calculate $d(W^2)$.

Solution: $\Delta = W$ and $\int \mathbb{E}(W_s^2)ds = \int s ds = T^2/2$.
Alternatively,

$$d(W^2) = 2WdW + \frac{1}{2}2dt$$

so

$$W_T^2 = \int 2WdW + T$$

so

$$\int WdW = \frac{1}{2}(W_T^2 - T)$$

and

$$\mathrm{Var}\int WdW = \mathrm{Var}\frac{1}{2}(W_T^2 - T) = \frac{1}{4}\mathrm{Var}(W_T^2) = \frac{1}{4}(\mathbb{E}(W_T^4) - (E(W_T^2))^2)$$

using the fourth moment of $N(0, \sigma^2)$ is $3\sigma^4$,

$$= \frac{1}{4}(3T^2 - T^2) = T^2/2$$

Score: _______

Problem 2. (a) Solve the stochastic differential equation

$$dX_t = t\,dt - dW_t$$

(in other words, find X_t) by integrating both sides from 0 to T.

(b) Consider the stochastic differential equation

$$dY_t = t\,dt + dW_t$$

What can be said about X_t and Y_t, assuming $X_0 = Y_0$ and the same Brownian motion W_t is used to define both X_t and Y_t? Justify.

(i) $X_t = Y_t$ almost surely?

(ii) X_t and Y_t are not equal almost surely, but they have the same probability distribution?

(iii) X_t and Y_t are not equal almost surely, and they also do not have the same probability distribution?

(c) Draw (sketch) what a typical random path of the solution Y_t may look like in a (t, Y_t) coordinate system.

Score: _______

Space for work on Problem 2. (a) $X_t = X_0 + \frac{1}{2}t^2 - (W_t - W_0)$ and $W_0 = 0$. (b) (ii) not equal almost surely, but have the same distribution since W_t and $-W_t$ have the same distribution. (c) Sketch $\frac{1}{2}t^2$ and some random motion around it.

Score: _______

Problem 3. The Black-Scholes-Merton partial differential equation for the price $c(t,x)$ of a derivative security at time t when the current stock price is $x = S_t$ is

$$\frac{\partial c}{\partial t} + rx\frac{\partial c}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2} = rc$$

(a) Show that

$$c(t,x) = ax + be^{rt}$$

is a solution for any constants a and b.

(b) Describe (in words, in finance terms) the derivative security with price process given by the function in (a), in other words $c(t,S_t) = aS_t + be^{rt}$.

Solution: The option pays the value of a shares of the stock, plus b invested in a money market account with continuously compounded interest rate r.

Score: _______

Problem 4. *Evaluate the stochastic integral*

$$\int_0^T W_t^{1/2} dW_t.$$

- Explanation: Find an equivalent expression that does not involve stochastic (Ito, "dW") integrals, but only regular (but possibly random) integrals and functions.
- Hint: use Ito's formula to expand $d(W_t^{3/2})$.

Solution:

$$d(W_t^{3/2}) = \frac{3}{2}W_t^{1/2}dW_t + \frac{1}{2}\frac{3}{2}\left(\frac{1}{2}W^{-1/2}\right)dt$$

$$W_t^{3/2} = \int \frac{3}{2}W_t^{1/2}dW_t + \frac{3}{8}\int W^{-1/2}dt$$

$$\frac{2}{3}W_t^{3/2} = \int W_t^{1/2}dW_t + \frac{1}{4}\int W^{-1/2}dt$$

$$\int_0^T W_t^{1/2}dW_t = \frac{2}{3}W_T^{3/2} - \frac{1}{4}\int_0^T W^{-1/2}dt$$

Score: _______

Problem 5. Recall that a stochastic process X_t is a *Markov process* if for each Borel measurable function f, there is a function g such that if $s < t$ then

$$\mathbb{E}_s(f(X_t)) = g(X_s).$$

In other words, our estimate at time s of the distribution of X_t is determined by the value X_s and does not depend on X_u, $u < s$. Here $\mathbb{E}_s(\cdot)$ is what the textbook calls $\mathbb{E}(\cdot \mid \mathcal{F}_s)$.

(a) Simplify the expression:

$$\mathbb{E}_1(W_2)$$

where W_t, $t \geq 0$, is Brownian motion. $\mathbb{E}_1(W_2) = W_1$ since Brownian motion is a martingale and $1 < 2$.

(b) Calculate $\mathbb{E}_s(X_t)$ and $\mathbb{E}_s(Y_t)$ for the following processes.

$$X_t = W_t^2 - t$$

$$Y_t = \int_0^t W_s ds$$

(c) Can you rule out that one or more of them are Markov processes based on your answer to (b)?

$$\begin{aligned}\mathbb{E}_s[X_t] &= \mathbb{E}_s[W_t^2] - t = \mathbb{E}_s[((W_t - W_s) + W_s)^2] - t \\ &= \mathbb{E}_s[(W_t - W_s)^2 + 2(W_t - W_s)W_s + W_s^2] - t \\ &= t - s + 0 + W_s^2 - t = W_s^2 - s = X_s\end{aligned}$$

so we cannot rule out that X_t is a Markov process (and by the way, we just showed that X_t is a martingale). And

$$\mathbb{E}_s\left[Y_t\right] = \mathbb{E}_s\left[\int_0^t W_u du\right]$$

$$= \mathbb{E}_s\left[\int_s^t W_u du + \int_0^s W_u du\right]$$

$$= \int_s^t (\mathbb{E}_s W_u) du + \int_0^s W_u du$$

$$= \int_s^t W_s du + \int_0^s W_u du$$

$$= (t-s)W_s + \int_0^s W_u du = (t-s)W_s + Y_s$$

which is not determined by Y_s, so Y_t is not a Markov process. Note: if you are unable to do the formal calculation in part (b), you can try to guess or give an informal reasoning for what the answer should be. Also in part (c) you can try to explain informally whether the answers in (b) depend only on X_s, Y_s or whether they depend also on X_u, $u < s$ or Y_u, $u < s$.

Score: _______

Space for work on Problem 5.

Score: ______

Problem 6. Let W be any normal random variable with mean 0 and variance 1. Calculate the moment generating function

$$M_W(u) = \mathbb{E}(e^{uW}).$$

Hint: This will involve "completing the square".

Solution:

$$\begin{aligned}
\mathbb{E}(e^{uW}) &= \int_{-\infty}^{\infty} e^{ux}\left(\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(ux-\frac{1}{2}x^2\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\left[x^2-2ux\right]\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\left[x^2-2ux+u^2-u^2\right]\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\left[(x-u)^2-u^2\right]\right)dx \\
&= e^{u^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\left[(x-u)^2\right]\right)dx \\
&= e^{u^2/2}
\end{aligned}$$