# Algebra qualifying exam

University of Hawai‘i at Mānoa

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Instructions:

- You have 4 hours to complete this exam.
- There are 6 sections.
- You should answer 12 questions and answer at least one question in each of the 6 sections; the first question in each section is often (maybe not always) the easiest. If you have time, you may answer more than 12 questions—your entire paper will be read and taken into consideration.

## 1. Group theory

**1.** Let $G$ be a group. Recall that the *center of* $G$, denoted $Z(G)$, is the set of elements of $G$ that commute with every other element. Let $p$ be a prime number.

(a) Show that $Z(G)$ is a normal subgroup of $G$.

(b) Show that if $G/Z(G)$ is cyclic, then $G$ is abelian.

(c) Suppose $G$ is a finite group whose order is a power of $p$ (a so-called *$p$-group*). Show that $Z(G)$ is non-trivial.

(d) Show that if $G$ has order $p^2$ for a prime number $p$, then $G$ is abelian.

**2.** Let $G$ be a group. Recall that the *commutator* of two elements $g, h \in G$ is

$$[g,h] := g^{-1}h^{-1}gh.$$

Recall also that the *commutator subgroup of* $G$ is its subgroup $[G,G]$ generated by commutators $[g,h]$ as $g$ and $h$ vary over the elements of $G$.

(a) Show that $[G,G]$ is a normal subgroup of $G$.

(b) Show that $G/[G,G]$ is abelian.

(c) Show that if $H$ is a subgroup of $G$ containing $[G,G]$, then $H$ is normal.

**3.** Let $p$ be a prime number and let $\mathrm{Aff}_1(p)$ be the subgroup of $\mathrm{GL}_2(\mathbf{Z}/p\mathbf{Z})$ whose elements are

$$\mathrm{Aff}_1(p) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in (\mathbf{Z}/p\mathbf{Z})^\times, b \in \mathbf{Z}/p\mathbf{Z} \right\}.$$

Let $K := \mathbf{Z}/p\mathbf{Z}$ and let $H := (\mathbf{Z}/p\mathbf{Z})^\times$ act on $K$ by multiplication. Show that

$$K \rtimes H \cong \mathrm{Aff}_1(p).$$

**4.** Let $X$ and $Y$ be two finite sets and suppose a group $G$ acts on $X$ and on $Y$. Suppose there is a $G$-equivariant function $f : X \to Y$. Show that if $G$ acts transitively on $Y$, then $\#Y$ divides $\#X$.

## 2. Fields and Galois theory

In this section, $\mathbf{Q}$ denotes the field of rational numbers.

**1.** (a) Prove that $\alpha = \sqrt{5} + \sqrt{7}$ is algebraic over $\mathbf{Q}$ by explicitly finding a polynomial $f(x) \in \mathbf{Q}[x]$ of degree 4 having $\alpha$ as a root.

(b) Prove that this $f(x)$ is irreducible.

(c) Show that $\mathbf{Q}(\alpha)/\mathbf{Q}$ is Galois and determine its Galois group. Justify your answer.

(d) Illustrate the Fundamental Theorem of Galois Theory by drawing the lattice of intermediate fields and the corresponding subgroups. You don't have to prove your answer.

**2.** Let $E/F$ be a non-normal field extension with $[E : F] = 4$, and let $K$ be the normal closure of $E/F$.

(a) How big can $[K : F]$ be? Justify your answer.

(b) Give an example of $K \supset E \supset F$ attaining this biggest degree.

(c) Suppose now that $E/F$ is any separable, non-normal degree 4 extension and that $\alpha$ is a primitive element for it, i.e. $E = F(\alpha)$. Let $p(x) \in F[x]$ be the minimal polynomial for $\alpha$ over $F$. Is $p(x)$ solvable by radicals? (Answer definitely yes, definitely no, or maybe; justify your answer.)

**3.** Let $K \supset L \supset F$ be a tower of fields. Prove or disprove each of the following.
(a) If $K/F$ is Galois, then $K/L$ is Galois.
(b) If $K/F$ is Galois, then $L/F$ is Galois.
(c) If $L/F$ and $K/L$ are both Galois, then $K/F$ is Galois.

**4.** (a) Prove that every finite integral domain is a field.

(b) Fix a prime $p$. Sketch the proof that every field with $p^2$ elements is isomorphic.

(c) Exhibit infinitely many pairwise non-isomorphic quadratic extensions of $\mathbf{Q}$, and justify that that they are pairwise non-isomorphic.

(d) Construct a field with 27 elements.

## 3. Category theory

**1.** Let $\mathrm{Set}_*$ be the category of pointed sets, i.e. the category whose objects are pairs $(X, x_0)$ where $X$ is a set and $x_0$ is an element of $X$ called the *basepoint*, and whose morphisms $(X, x_0) \to (Y, y_0)$ are functions $f : X \to Y$ such that $f(x_0) = y_0$.

(a) Given two pointed sets $(X, x_0)$ and $(Y, y_0)$, show that $(X \times Y, (x_0, y_0))$ gives their product in the category of pointed sets.

(b) Given two pointed sets $(X, x_0)$ and $(Y, y_0)$, define their *wedge sum* $X \vee Y$ to be their disjoint union with the basepoints identified, i.e. the pair $((X \sqcup Y)/\sim, [x_0])$, where $X \sqcup Y$ is the usual disjoint union, $\sim$ is the equivalence relation where $x_0 \sim y_0$ and all other points are only equivalent to themselves, and $(X \sqcup Y)/\sim$ denotes the set of equivalence classes and $[x_0]$ the equivalence class of $x_0$. Show that the wedge sum is the coproduct in the category of pointed sets.

**2.** Given a category $\mathcal{C}$ and a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}$, a *coalgebra of* $\mathcal{F}$ is a pair $(A, \alpha)$ such that $A$ is an object of $\mathcal{C}$ and $\alpha : A \to \mathcal{F}(A)$ is a morphism in $\mathcal{C}$. The object $A$ is called the *carrier* of the coalgebra $(A, \alpha)$. A morphism $(A, \alpha) \to (B, \beta)$ of coalgebras of $\mathcal{F}$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & \mathcal{F}(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(f)} \\ B & \xrightarrow{\beta} & \mathcal{F}(B) \end{array}$$

(a) Show that if $A$ is a carrier of a coalgebra for $\mathcal{F}$, then so is $\mathcal{F}(A)$.

(b) Suppose $(Z, \zeta)$ is a terminal object in the category of coalgebras of $\mathcal{F}$. Show that $\zeta$ is then an isomorphism.

## 4. Ring theory

In this section, all rings and all ring homomorphisms are unital (i.e. all rings have an identity element and all ring homomorphisms map the identity in the domain to the identity in the target). By an ideal in a ring, we mean a two-sided ideal. $\mathbf{C}$ denotes the field of complex numbers, $\mathbf{R}$ denotes the field of real numbers, $\mathbf{Z}$ denotes the ring of integers, and $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$, the field with $p$ elements for $p \in \mathbf{Z}$ prime.

**1.** Let $M$ be the ring of $2 \times 2$ matrices over $\mathbf{R}$, and define $S \subset M$ to be the subset of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

(a) Show $S$ is a subring of $M$ and that in fact $S \cong \mathbf{C}$.

(b) Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$. Prove there is a matrix $X \in M$ such that $X^4 + 13X = A$.

(c) Let $R$ be any (non-zero) ring, and let $h : M \to R$ be a ring homomorphism. Show that $h$ is injective.

**2.** (a) Let $R$ be a (non-zero) commutative ring. Prove that if every proper ideal of $R$ is prime, then $R$ is a field.

(b) Prove that every non-zero prime ideal of a PID is maximal.

(c) Prove that the quotient of a PID by a prime ideal is a PID.

**3.** (a) Let $R = \mathbf{Z}[\sqrt{10}] = \{a + b\sqrt{10} : a, b \in \mathbf{Z}\}$. Let $\mathfrak{p} = \left(2, \sqrt{10}\right)$. Prove that $\mathfrak{p}$ is a prime ideal of $R$.

(b) Prove that 2 is an irreducible element of $R$ but not a prime element of $R$.

(c) Show that $-3 + \sqrt{10}$ is a unit of $R$.

(d) Show that $R$ has infinitely many units.

**4.** Consider two quotient rings: $R_1 = \mathbf{F}_p[x]/\left(x^2 - 2\right)$ and $R_2 = \mathbf{F}_p[x]/\left(x^2 - 3\right)$. Determine whether the rings $R_1$ and $R_2$ are isomorphic for the three cases $p = 2, 5$, and 11.

## 5. Modules and multilinear algebra

In this section, $\mathbf{Z}$ denotes the ring of integers and $\mathbf{Q}$ denotes the field of rational numbers.

**1.** Let $M$ be a $\mathbf{Z}$-module.

(a) Show that every element of $M \otimes_{\mathbf{Z}} \mathbf{Q}$ is of the form $m \otimes \dfrac{1}{d}$ for some integer $d \geq 1$ and some $m \in M$.

(b) Show that if $M$ is a torsion $\mathbf{Z}$-module, then $M \otimes_{\mathbf{Z}} \mathbf{Q} = 0$.

(c) Let $M = \mathbf{Z} \oplus (\mathbf{Z}/10\mathbf{Z}) \oplus \mathbf{Q}/\mathbf{Z}$. What is the dimension of $M \otimes_{\mathbf{Z}} \mathbf{Q}$ as a $\mathbf{Q}$-vector space?

(d) Let $R$ be a commutative ring with identity and suppose $M$ and $N$ are two $R$-modules. Show that $M \otimes_R N = 0$ if and only if for every $R$-module $A$, every $R$-bilinear map $\psi : M \times N \to A$ is 0.

**2.** Suppose all three columns and the bottom two rows of the following commutative diagram are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \\
0 \arrow[r] & A'' \arrow[r] & B'' &
\end{tikzcd}$$

Show that the top row is then exact.

**3.** Let $F$ be a field.

(a) Let $V$ be a finite-dimensional vector space over $F$, let $V^*$ denotes its dual, and let $\mathrm{End}_F(V)$ denote the $F$-vector space of linear transformations $V \to V$. For $v \in V$ and $f \in V^*$, let $T_{v,f} \in \mathrm{End}_F(V)$ be given by

$$T_{v,f}(w) = f(w)v.$$

Show that the map $\psi : V \times V^* \to \mathrm{End}_F(V)$ sending $(v, f)$ to $T_{v,f}$ is an $F$-bilinear map.

(b) Show that the map $\psi$ induces an isomorphism $\widetilde{\psi} : V \otimes_F V^* \xrightarrow{\sim} \mathrm{End}_F(V)$.

(c) Composing $\widetilde{\psi}$ with the trace map $\mathrm{tr} : \mathrm{End}_F(V) \to F$ yields a linear map

$$\widetilde{\varphi} = \mathrm{tr} \circ \widetilde{\psi} : V \otimes_F V^* \to F$$

and hence a bilinear map $\varphi : V \times V^* \to F$. Give a formula for $\varphi(v, f)$.

(d) Does $\varphi$ induce an isomorphism from $V$ to $V^*$?

## 6. Commutative algebra

In this section, $\mathbf{Q}$ denotes the field of rational numbers and $\mathbf{Z}$ denotes the ring of integers. All rings in this section are commutative with identity, and subrings have the same identity as their ambient ring.

**1.** (a) Define what it means for a ring to be Noetherian.
(b) Give an example of a Noetherian ring.
(c) Give an example of a ring that is not Noetherian.
(d) Prove or disprove: A subring of a Noetherian ring is Noetherian.

**2.** Let $R$ be an integral domain with field of fractions $K$, and let $R_{\mathfrak{m}}$ denote the localization of $R$ at a maximal ideal $\mathfrak{m}$. Prove that

$$R = \bigcap_{\mathfrak{m}} R_{\mathfrak{m}},$$

where the intersection is taken over all maximal ideals $\mathfrak{m} \triangleleft R$.

**3.** Recall that for an $R$-module $M$, the annihilator of $M$ is defined as

$$\operatorname{Ann}(M) = \{x \in R \mid xm = 0 \text{ for all } m \in M\}.$$

Note that $\operatorname{Ann}(M) \triangleleft R$ is an ideal.
(a) Let $R$ be a ring, $S$ a multiplicative subset of $R$, and $M$ an $R$-module. Prove that

$$S^{-1}M = 0 \text{ if } \operatorname{Ann}(M) \cap S \neq \emptyset.$$

(b) Prove the converse of part (a) if $M$ is finitely generated.
(c) Let $N$ be another $R$-module. Define a natural homomorphism

$$\sigma : S^{-1} \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N),$$

and prove that $\sigma$ is injective if $M$ is finitely generated as an $R$-module.
(d) Let $R = \mathbf{Z}$, $S = \mathbf{Z} \smallsetminus \{0\}$, and $M = N = \mathbf{Q}/\mathbf{Z}$. Prove that the map $\sigma$ from part (c) is not injective in this case.