Solve all six problems. You have 4 hours. Good luck!
You need to demonstrate proficiency in each area

Problem 1: Consider the dynamics of a particle governed by the differential equation
\[ \ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0 \]
with parameter \( a > 0 \).
(1) Find and classify all the fixed points.
(2) Show that the system has a circular limit cycle, and find its amplitude and period.
(3) Determine the stability of the limit cycle.

Problem 2: Let \( \varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) be a flow. Recall that a closed invariant set \( A \subset \mathbb{R}^n \) is called an attracting set for \( \varphi \) if there is a neighborhood \( U \supset A \) such that \( \forall t \geq 0, \varphi(t,U) \subset U \) and \( \cap_{t>0}\varphi(t,U) = A \). Now, consider a set \( K \) which is a union of a homoclinic orbit and a hyperbolic equilibrium point, as shown in the figure. Can \( K \) be an attracting set?

Problem 3: Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a vector field, \( G \subset \mathbb{R}^2 \) an invariant open set. Show that if there is a function \( B : \mathbb{R}^2 \to \mathbb{R} \), \( B(x) > 0 \) for \( x \in G \), such that \( \text{div}(B(x)f(x)) = 0 \) for \( x \in G \), then there is a first integral for \( f \) in \( G \), i.e. a function \( H : G \to \mathbb{R} \) that is constant along the trajectories: \( \dot{H} = \nabla H(x) \cdot f(x) = 0, x \in G \).

Hint: Recall that a differential equation \( P(x,y)dx + Q(x,y)dy = 0 \) is called exact if \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \), in which case there is a function \( F(x,y) \) such that \( \frac{\partial F}{\partial x} = P \), \( \frac{\partial F}{\partial y} = Q \).
**Problem 4:** Recall that a discrete dynamical system $x \mapsto g(x)$, $g : [a, b] \to [a, b]$, is called chaotic if

1. $g$ is transitive on $[a, b]$, i.e. for any two subintervals $U_1, U_2 \subset [a, b]$ there is $n > 0$ such that $g^n(U_1) \cap g^n(U_2) \neq \emptyset$.
2. $g$ has sensitive dependence on the initial conditions, i.e. there is $c > 0$ such that for all $x_0 \in [a, b]$ and any interval $U \ni x_0$, there is $y_0 \in U$ and $n > 0$ such that $|g^n(x_0) - g^n(y_0)| \geq c$.
3. Periodic points of $g$ are dense in $[a, b]$.

Consider the tent map $g : [0, 1] \to [0, 1]$ given by

$$g(x) = \begin{cases} 
2x, & x \in \left[0, \frac{1}{2}\right] \\
2x - 1, & x \in \left(\frac{1}{2}, 1\right]
\end{cases}$$

Prove that the dynamical system $x \mapsto g(x)$ is chaotic.

*Hint: Find an explicit form of intervals which $g^n$ maps onto $(0, 1]$, show that the length of any such interval decreases to zero as $n \to \infty$ and deduce the properties needed for chaos.*

**Problem 5:** The spherical mean of a function $u$ is defined as

$$U = \frac{1}{\alpha(n)n^{n-1}} \int_{\partial B(x, r)} u dS(y),$$

and satisfies

$$\frac{\partial U}{\partial r} = \frac{1}{n \alpha(n) r^{n-1}} \int_{B(x, r)} \Delta u dy.$$ 

Here $\alpha(n)$ is the volume of an $n$ dimensional sphere. You do not need to prove this second relation.

1. Suppose that $u = u(x)$ is harmonic. Use the formula above to prove the mean value formula for Laplace’s equation.
2. Suppose instead that $u = u(x, t)$ is a solution of the wave equation $u_{tt} = \Delta u$, show that

\begin{equation} (r^{n-1}U_r)_r = \frac{1}{\alpha(n)} \int_{\partial B(x, r)} u_{tt} dS. \end{equation}

3. Use equation (1) to derive a PDE for function $U(r, t)$.

**Problem 6:** Let $A$ be an $n \times n$ matrix with integer entries, and assume that $A$ is invertible.

1. Prove that if $A^{-1}$ has integer entries then $|\det A| = 1$.
2. Prove that if $|\det A| = 1$ then $A^{-1}$ has integer entries. *Hint: Cramer’s rule*