Algebra qualifying exam

University of Hawaiʻi at Mānoa

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Instructions:

- You have 4 hours to complete this exam.
- There are 6 sections.
- You should answer 12 questions and answer at least one question in each of the 6 sections. If you have time, you may answer more than 12 questions—your entire paper will be read and taken into consideration.
- For every solution, please indicate clearly which problem you are working on. More generally, the more legible your writing is, the better it will be appreciated!

The following notations are used throughout:

- $\mathbb{Z}$ stands for the ring rational integers.
- $\mathbb{Q}$ stands for the field of rational numbers.
- $\mathbb{R}$ stands for the field of real numbers.
- $\mathbb{C}$ stands for the field of complex numbers.
- $\mathbb{F}_{p^n}$ stands for the field which consists of p^n elements, where p is a prime, and $n > 0$ is an integer.

1. Group theory

1. Let a group G act on a set E, both from the left and from the right (these are two different actions, of course). Assume that the number of orbits with respect to the left action is finite, and let $\{e_1, \ldots, e_s\}$ be a complete set of representatives of the orbits. Let $S = \{1, \ldots, s\}$.

Fix an element $g \in G$, and define a map $\varphi_g = \varphi : S \to S$ as follows.

For every $i \in S$, there exist $h \in G$ and (unique) $\varphi(i) \in S$ such that

$$e_i g = h e_{\varphi(i)}.$$

Is the map φ_g necessarily a permutation of S? Prove your answer.

2. Let p be a prime, and let $m \geq 1$ be an integer. Let A be an abelian group of prime power order $|A| = p^m$.

Assume that for every $n \leq m$

$$|\{a \in A \mid p^n a = 0\}| = p^n$$

Does that imply that A is cyclic? Prove the statement or give a counterexample.

3. Denote by $\mathcal{Q}_8$ the quaternion group. Can one present $\mathcal{Q}_8$ as a semi-direct product of two (non-trivial) subgroups?

4. Let G be a non-trivial p-group, that is $|G| = p^e$ with $e \geq 1$. Prove or disprove that G must have a non-trivial center.

2. Fields and Galois theory

1. Let k be a field, and let G be a finite subgroup of the multiplicative group k^*. Is G necessarily cyclic? Justify your answer.

2. Let $[L : K] = n$ be a finite-dimensional separable field extension, and assume that

$$|\mathrm{Aut}(L/K)| = n.$$

Is L necessarily a Galois extension of K? Prove your answer.

3. Let k be an algebraically closed field, and let $a \in k^*$ be a non-zero element. Let p be prime.

How many distinct solutions may the equation $x^p = a$ have?

Show that every possibility in your list really happens, and prove that your list exhausts all possibilities.

4. Prove or give a counter-example that Galois extensions are transitive.

That is, for

$$L \supset K \supset F$$

if $L \supset K$ and $K \supset F$ are both Galois, then so is $L \supset F$.

3. Category theory

1. Consider the category $\mathcal{C}$ with one object and the identity morphism, and the category $\mathcal{D}$ with two objects, a and b, the identity morphisms, and one morphisms each from a to b and b to a that are both isomorphisms. Are the categories $\mathcal{C}$ and $\mathcal{D}$ equivalent and/or isomorphic? Justify your answers.

2. Suppose A is an object of category $\mathcal{C}$. For any object $C \in \mathcal{C}$, we have a set of morphisms $\mathrm{Mor}(C, A)$. If we have a morphism $f : B \to C$, we get a map of sets

$$(*) \quad \mathrm{Mor}(C, A) \to \mathrm{Mor}(B, A)$$

by composition. Suppose you have two objects A and A' in category $\mathcal{C}$, and maps

$$\iota_C : \mathrm{Mor}(C, A) \to \mathrm{Mor}(C, A')$$

for every object $C \in \mathcal{C}$ that commute with the maps $(*)$. Show that there is a unique morphism $g : A \to A'$ such that (as C ranges over the objects of $\mathcal{C}$)

$$\iota_C(u) = g \circ u.$$

3. Let A be a commutative ring with identity.

Let $\mathcal{F}$ be the forgetful functor from the category of A-algebras to the category of sets. Show that the polynomial ring $A[x]$ represents functor $\mathcal{F}$.

4. Ring theory

1. Let R be a commutative ring with $1_R \neq 0_R$. Let $n \in R$ be a nilpotent in R, and let $u \in R$ be a unit. Is the element $u + n \in R$ necessarily a unit? Justify your answer.

2. Let A be a commutative ring (with identity $1_A \in A$), and let $I \subset A$ and $J \subset A$ be ideals.
Prove (or give a counterexample) that

$$IJ = I \cap J.$$

3. Give an example of a U.F.D. which is not a P.I.D. . Prove at least one of the two claims.

4. Give an example of an integral domain R which has an element $x \in R$ such that x is irreducible while not prime. Prove your claims.

5. Modules and multilinear algebra

1. Does an R-module which is not Noetherian exist for any non-zero ring R? Prove your claim.

2. Is it true that, for a $\mathbb{Z}$-module A, the exactness of the sequence

$$0 \longrightarrow X \xrightarrow{f} Y$$

of $\mathbb{Z}$-modules implies the exactness of the sequence

$$0 \longrightarrow A \otimes_{\mathbb{Z}} X \xrightarrow{1\otimes f} A \otimes_{\mathbb{Z}} Y$$

of $\mathbb{Z}$-modules?

Prove that or provide a counterexample with a proof.

3. For a $\mathbb{Z}$-module M, denote by $\mathcal{T}(M)$ the tensor algebra of M.

Describe the addition and multiplication on the set $\mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})$ which make this set into a ring such that we have a ring isomorphism

$$\mathcal{T}(\mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})$$

4. Let R be a commutative ring, and let M and N be Noetherian R-modules. Prove that the direct sum $M \oplus N$ is a Noetherian R-module.

6. Commutative algebra

1. Let R be a commutative ring with $1_R \neq 0_R$, and let M be a finitely generated R-module. Let D be a multiplicatively closed subset in R containing 1_R. Assume that $D^{-1}M = 0$. Does there exist an element $d \in D$ such that $dm = 0$ for every $m \in M$? Prove your answer.

2. Let R be a commutative ring with $1_R \neq 0_R$. Assume that the polynomial ring $R[x]$ is Noetherian. Does that imply that R is Noetherian? Prove your answer.

3. Let S be an integral domain, and let R be a subring of S with $1_R = 1_S$. Suppose that S is integral over R.

Assume that R is a field. Does that imply that S is a field? Prove your answer.

4. Is any U.F.D.integrally closed? Justify your answer.