Algebra qualifying exam

University of Hawai‘i at Mānoa

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Instructions:

- You have 4 hours to complete this exam.
- There are 6 sections.
- You should answer 12 questions and answer at least one question in each of the 6 sections. If you have time, you may answer more than 12 questions—your entire paper will be read and taken into consideration.
- For every solution, please indicate clearly which problem you are working on. More generally,

  the more legible your writing is, the better it will be appreciated!

The following notations are used throughout:

- $\mathbb{Z}$ stands for the ring rational integers.
- $\mathbb{Q}$ stands for the field of rational numbers.
- $\mathbb{R}$ stands for the field of real numbers.
- $\mathbb{C}$ stands for the field of complex numbers.
- $\mathbb{F}_{p^n}$ stands for the field which consists of p^n elements, where p is a prime, and $n > 0$ is an integer.

1. Group theory

1. Let G be a finite group and suppose that the order of G is odd. May there exist a non-identity element $x \in G$ such that $x = ax^{-1}a^{-1}$ with some element $a \in G$ (i.e. x and x^{-1} are conjugate)? Justify your answer.

2. Let G be the group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with a, b, c elements in the finite field $\mathbb{F}_2$. Clearly $|G| = 8$. Assume (without proof) that

- G is nonabelian;
- There are only two non-isomorphic nonabelian groups of order 8: the dihedral group D_8 and the quaternion group Q_8.

Which of the two is G isomorphic to?

3. How many non-isomorphic finite groups of order 15 are there? Justify your answer.

4. Let G be a finite group of order $363 = 3 \times 11^2$. Then G is necessarily a semidirect product. True or False? Justify your answer.

2. Fields and Galois theory

1. Let E be the splitting filed of $x^4 + 1$ over $\mathbb{Q}$.

(a) Find $Gal(E/\mathbb{Q})$, the Galois group of E over $\mathbb{Q}$.

(b) Find all subfields of E.

(c) Find the automorphisms of E that have fixed fields $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(i)$.

(d) Is there an automorphism of E whose fixed field is $\mathbb{Q}$?

2. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{5})$.

(a) What is the order of the group $Gal(E/\mathbb{Q})$?

(b) What is the order of $Gal(\mathbb{Q}(\sqrt{10})/\mathbb{Q})$?

3. (a) Find the smallest field that contains exactly six subfields.

(b) Write out the addition and multiplication tables for $\mathbb{F}_4$.

4. Let F be a finite extension of $\mathbb{Q}$. Prove that there are only a finite number of roots of unity in F.

3. Category theory

1. Call $\mathbf{1}$ the category which has exactly one object and one map (the identity on that object). For any category $\mathcal{A}$, there is precisely one functor $\mathcal{A} \to \mathbf{1}$.

Denote by **Ring** the category of rings with (multiplicative) identity, where morphisms are ring homomorphisms preserving the identities.

Does the functor $\mathbf{Ring} \to \mathbf{1}$ have a left adjoint? Justify your answer.

Hint. The left adjoint to $\mathbf{Ring} \to \mathbf{1}$ should be a functor $\mathbf{1} \to \mathbf{Ring}$. Since $\mathbf{1}$ has only one object, any such functor, if exists, is fully characterized by a certain ring which is the image in **Ring** of that object. The adjunction may be reformulated as a universal property of this ring. That allows one to either prove or rule out the existence of the ring in question, therefore the left adjoint functor.

2. Recall that any group can be thought of as a category with just one object (the morphisms from the object to itself are the elements of the group). Then a group homomorphism $H \to G$ can be thought of as a functor between the one-object categories.

Consider groups $\mathbb{Z}$ of integers with respect to addition and a group G this way.

Group homomorphisms $\mathbb{Z} \to G$ are in one-to-one correspondence with elements of G because such a homomorphism is fully determined by the image of $1 \in \mathbb{Z}$, and for every $g \in G$ there is a (unique) homomorphism $\mathbb{Z} \to G$ such that $1 \mapsto g$.

As we think about the group homomorphisms $\mathbb{Z} \to G$ as functors between the one-object categories, elements of G can now be interpreted as such functors. Natural isomorphism between functors thus determines an equivalence relation on G. Describe the equivalence classes in purely group-theoretic terms (with a proof).

4. Ring theory

1. Let R be a commutative ring with identity. Let $I \subset R$ and $J \subset R$ be ideals. Let

$$K = \{ij \mid i \in I, \ \ j \in J\} \subset R.$$

Is it true that K is an ideal in R? Justify your answer.

2. Is there a UFD which is not noetherian? Justify your answer.

3. Find a ring with exactly 3 prime ideals. Justify your answer.

4. Let R be a simple ring (with identity, not necessarily commutative). Is R necessarily a division ring?

Recall that R is said to be simple when its only two-sided ideals are $\{0\}$ and R.

Recall that R is said to be a division ring if every non-zero element in R is a two-sided unit.

5. Modules and multilinear algebra

1. True or false? Justify.

- $\mathbb{Q}$ is an injective $\mathbb{Z}$-module
- $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.
- $\mathbb{Z}/6\mathbb{Z}$ is a projective $\mathbb{Z}$-module.
- $\mathbb{Z}/12\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/20\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z}$

2. Let M be an R-module. Let $\mathcal{T}(M)$ denote the tensor algebra.

- Suppose $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$. Simplify $\mathcal{T}(M)$.
- Now suppose $R = \mathbb{Z}$ and $M = \mathbb{Z}/n\mathbb{Z}$. Simplify $\mathcal{T}(M)$.

3. Let V be an n-dimensional vector space over $\mathbb{C}$. Consider the twist map $\tau : V \otimes V \to V \otimes V$ defined by sending the pure tensor $s \otimes t$ to $t \otimes s$, and then extending linearly. What are the eigenspaces of τ? Write down the characteristic polynomial as well as the minimal polynomial of τ.

4. Give an example of

(a) An exact sequence of $\mathbb{C}[x, y]$-modules that does not split.
(b) A perfect pairing between $\wedge^k V$ and $\wedge^{n-k} V$ for V an n-dimensional space.

6. Commutative algebra

1. Let R be a commutative ring with 1. Prove the converse to Hilbert's basis theorem: if the polynomial ring $R[x]$ is Noetherian, then R is Noetherian.

2. Let R be a commutative ring with 1. Prove the following two statements are equivalent.

(i) R is a local ring with unique maximal ideal M,

(ii) if M is the set of elements of R that are not units, then M is an ideal.

3. Let $I = (x^2, xy) \subseteq \mathbb{R}[x, y]$.

(a) Show that the following is a primary decomposition of the ideal I

$$I = (x) \cap (x^2, y).$$

(b) What are the associated primes of I?