Algebra qualifying exam

University of Hawaiʻi at Mānoa

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Instructions:

- You have 4 hours to complete this exam.
- There are 6 sections.
- You should answer 12 questions and answer at least one question in each of the 6 sections. If you have time, you may answer more than 12 questions—your entire paper will be read and taken into consideration.
- For every solution, please indicate clearly which problem you are working on. More generally,

  the more legible your writing is, the better it will be appreciated!

The following notations are used throughout:

- $\mathbb{Z}$ stands for the ring rational integers.
- $\mathbb{Q}$ stands for the field of rational numbers.
- $\mathbb{R}$ stands for the field of real numbers.
- $\mathbb{C}$ stands for the field of complex numbers.
- $\mathbb{F}_{p^n}$ stands for the field which consists of p^n elements, where p is a prime, and $n > 0$ is an integer.

1. Group theory

1. Let G be a (not necessarily finite) abelian group. Let $c \in G$ be an element of finite order $|c|$ such that for any element $a \in G$ of finite order $|a|$ we have that $|a| \leq |c|$.

Is it true that for every element $a \in G$ of finite order $|a|$ we have that $|a|$ divides $|c|$? Justify your answer.

2. Prove or give a counterexample to the statement "a semidirect product of two abelian groups is an abelian group".

3. Prove or give a counterexample to the statement "an infinite simple group has no subgroups of finite index".

4. Let G be a group. Assume that its order $|G| = 217 = 7 \cdot 31$. Prove that G is abelian.

2. Fields and Galois theory

1. Does there exist a Galois extension K of $\mathbb{Q}$ such that the Galois group $\mathrm{Gal}_{\mathbb{Q}}K \simeq \mathbb{Z}/5\mathbb{Z}$ is cyclic of order 5?

2. Let F be a field of characteristic $\mathrm{char}F = p > 0$. Let $K = F(u)$ be an algebraic extension. Assume that for all $n \geq 0$ we have $u^{p^n} \notin F$. Is it true that there exist $n \geq 0$ such that $F(u^{p^n})$ is a separable extension of F? Prove your answer.

3. Let p be a prime, and let $\mathbb{F}_p$ be the finite field out of p elements. Let $f \in \mathbb{F}[x]$ be an irreducible polynomial of degree $\deg f = n$. Is it true that f divides $\left(x^{p^n} - x\right)$ in $\mathbb{F}[x]$? Justify your answer.

4. Let $K \supset F$ be a field extension. Assume that

$$[K : F] = 2$$

Is K necessarily a normal extension of F? Prove your answer.

3. Category theory

1. For a group G, we denote by $[G, G]$ its commutator subgroup.

Recall that $[G, G]$ is defined as the subgroup generated by commutators $[x, y] = x^{-1}y^{-1}xy$ of all elements $x, y \in G$ of the group G; the subgroup $[G, G]$ is normal in G, and there is an exact sequence

$$1 \longrightarrow [G, G] \xrightarrow{\varphi} G \xrightarrow{\psi} G/[G, G] \longrightarrow 1$$

with **"natural" ("canonical")** maps φ and ψ.

The word "natural" ("canonical") translates into a certain claim about the maps φ and ψ. Formulate this claim explicitly for the map φ. Furthermore, write down a statement which one has to prove in order to support this claim. Reduce it to the commutativity of certain diagrams. (Do not do the actual proof: it is lengthy and trivial.)

2. Let $\mathcal{F}$ be the forgetful functor from the category of groups to the category of sets. What is left-adjoint functor to $\mathcal{F}$?

4. Ring theory

1. A prime ideal in a ring A is called minimal if it does not contain properly any other prime ideals.

Let k be a field. Find all minimal prime ideals in the ring $k[x, y]/(xy)$.

You have to prove both that the ideals you present are prime, and that there are no more minimal prime ideals in this ring.

2. Let K be a field, and let $A \supset K$ be an integral domain which contains K, and is a finite-dimensional vector space over K. Prove that A must be a field.

An example of this situation is $K = \mathbb{R}$ and $A = \mathbb{C}$ (as a two-dimensional vector space over $\mathbb{R}$), or, more generally, A is a finite algebraic extension of K. In all these examples, A is a field by construction.

3. Consider the ring $R = \mathbb{Z}[x]$. Find a prime ideal $I \subset R$ such that R/I has 4 elements.

4. Give an example (with a justification) of an integral domain R and an element $a \in R$ which is irreducible while not prime.

5. Modules and multilinear algebra

1. Let R be an integral domain. Let M be a free R-module, and let N be an R-submodule of M. Can we conclude that N is a free R-module? Prove it or give a counterexample.

2. Is any flat $\mathbb{Z}$-module projective? Prove it or provide a counterexample (with a justification).

3. Let M and N be $\mathbb{Z}$-modules.

Let $M' \subset M$ be a $\mathbb{Z}$-submodule in M and let $N' \subset N$ be a $\mathbb{Z}$-submodule in N.

Let $x \in M'$ and $y \in N'$.

Can it happen that $x \otimes y = 0$ in $M \otimes_{\mathbb{Z}} N$ while $x \otimes y \neq 0$ in $M' \otimes_{\mathbb{Z}} N'$?

Justify your answer.

4. Recall that a module M is called irreducible (simple) if $M \neq 0$ and if 0 and M are the only submodules of M.

Let R be a ring with 1 and let M be an irreducible left R-module. Prove that $\mathrm{End}_R(M)$ is a division ring.

6. Commutative algebra

1. Consider the subring of $\mathbb{C}$

$$S = \{a + 2bi | a, b \in \mathbb{Z}\}$$

(here, as usual $i^2 = -1$).

Find the integral closure of S. Justify your answer.

2. Let R be a commutative ring with identity, and let $\mathfrak{a} \subset R$ and $\mathfrak{b} \subset R$ be two distinct maximal ideals.

Is it true that for any positive integers $m > 0$ and $n > 0$ we have that

$$\mathfrak{a}^n + \mathfrak{b}^m = R \ \ ?$$

Prove your answer.

3. Let R be a commutative ring with $1 \neq 0$, and let M be an R-module. Assume that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subset R$.

Can we conclude that $M = 0$? Prove it or give a counterexample.

4. Let R be a commutative local ring with $1 \neq 0$, and $I \subset R$ be the maximal ideal. Assume that I is a finitely generated ideal.

Prove that $I = I^2$ implies that A is a field.