APPLIED MATH QUALIFYING EXAM, AUGUST 2022

Solve all six problems. You have 4 hours. Good luck! You need to demonstrate proficiency in each area.

Problem 1. Consider the following model of interactions between animal pollinators and plants:

$$\dot{x} = x\left((K - x) + \frac{y}{1+y}\right)$$
$$\dot{y} = -\frac{y}{2} + \frac{xy}{1+y},$$

where $K > 0$.

(a) For all positive values of K, find equilibrium points and determine the values K_0 and K_1 such that the system has two equilibrium points in the positive quadrant when $K \in (K_0, K_1)$.

(b) Perform linear stability analysis of all equilibrium points for $K \in (0, K_1)$.
Note: Do not plug in the expressions for the nontrivial equilibrium points into the Jacobian matrix. Consider the signs of the trace and the determinant of the Jacobian matrix and use the equations that the equilibrium points satisfy to simplify the algebra.

(c) Describe the bifurcation that happens at $K = K_0$.

Problem 2. The Lotka-Volterra model for competing species captures the population oscillations, but is structurally unstable. A structurally stable modification that still captures the oscillations is the Holling-Tanner model. Consider the following specific example of the latter model:

$$\dot{x} = (4 - x)x - \frac{6xy}{1+x}$$
$$\dot{y} = s\left(1 - \frac{y}{x}\right)y,$$

where $s > 0$. Show that there is a stable limit cycle when $s \in \left(0, \frac{1}{2}\right)$.
Hint: Considering a part of the trajectory starting at $(4, 0)$ and a part of the nullcline passing through $(4, 0)$ may help construct a positively invariant region.

Problem 3. Suppose the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, has an equilibrium at the origin. Prove that if there exists a function $V : U \to \mathbb{R}$, where U is a neighborhood of the origin, such that:

(a) there are points arbitrarily close to the origin at which V is positive;

(b) $\dot{V}(x) = \langle \nabla V(x), f(x) \rangle > 0$ for all $x \in U \setminus \{0\}$;

(c) $V(0) = 0$,

then the origin is unstable.

Problem 4. Consider the system $x \mapsto g(x)$, where $g : [0,1] \to [0,1]$ is given by

$$g(x) = \begin{cases} 1 - 3x, & x \in \left[0, \frac{1}{3}\right], \\ -\frac{1}{3} + x, & x \in \left[\frac{1}{3}, 1\right]. \end{cases}$$

Note that 0 is a periodic point with the orbit $\left\{0, 1, \frac{2}{3}, \frac{1}{3}\right\}$. Also note that $g\left(\left[\frac{1}{3}, \frac{2}{3}\right]\right) = \left[0, \frac{1}{3}\right]$ and $g\left(\left[\frac{2}{3}, 1\right]\right) = \left[\frac{1}{3}, \frac{2}{3}\right]$.

(a) Show that any point of the form $x = m3^{-n}$, where $m, n \in \mathbb{N}$, $m \leqslant 3^n$, is eventually periodic, i.e. there exists $k \in \mathbb{N}$ such that the point $g^k(x)$ is periodic.
(b) Suppose that $x \in (0,1)$ is a periodic point. We'll denote by O_x its orbit and by $\min O_x$ the point in the orbit that's closest to 0. Show that for any periodic $x \in (0,1) \setminus \left\{\frac{1}{3}, \frac{2}{3}\right\}$ we have $\min O_x \in \left(0, \frac{1}{4}\right)$, then show that actually $\min O_x \in \left(0, \frac{2}{9}\right)$.
(c) Let $I_0 = \left(0, \frac{1}{3}\right)$, $I_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$, $I_2 = \left(\frac{2}{3}, 1\right)$. If $x \in (0,1) \setminus \left\{\frac{1}{3}, \frac{2}{3}\right\}$ is periodic with period k and $x_0 = \min O_x$, then the itinerary of x is the (symbolic) sequence $I_{n_0} \dots I_{n_{k-1}}$ such that $g^j(x_0) \in I_{n_j}$, $j = 0, \dots, k-1$. Show that if x has itinerary $I_{n_0} \dots I_{n_{k-1}}$ then there cannot be a periodic point with period $2k$ and itinerary $I_{n_0} \dots I_{n_{k-1}} I_{n_0} \dots I_{n_{k-1}}$.

Problem 5. Let $A, B \subset \mathbb{R}^k$ be open bounded domains with smooth boundaries and such that $\bar{A} \cap \bar{B} = \emptyset$, where $\bar{A}, \bar{B}$ denote the corresponding closures. Let $D = \mathbb{R}^k \setminus (\bar{A} \cup \bar{B})$, so that $\partial D = \partial A \cup \partial B$. Suppose that u is the solution to the following problem:

$$\begin{aligned} \Delta u &= 0, \quad x \in D, \\ u(x) &= c_1, \quad x \in \partial A, \\ u(x) &= c_2, \quad x \in \partial B, \\ u(x) &\to 0, \quad \|x\| \to \infty, \\ \int_{\partial A} \frac{\partial u}{\partial n} dS &> 0, \\ \int_{\partial B} \frac{\partial u}{\partial n} dS &= 0, \end{aligned}$$

where c_1 and c_2 are constants, and n denotes the outward normal to the boundary of D (i.e. it is pointing into A or B).

(a) Show that $0 < c_2 < c_1$.
Hint: You may find it useful to use the Hopf lemma which states that if a harmonic function u has a value at a boundary point x that is strictly greater than its values at the interior points of some neighborhood of x, then $\frac{\partial u}{\partial n}(x) > 0$.
(b) Show that $u(x) > 0$ for $x \in D$.

Problem 6. Let $A_1, \dots, A_n$ be $m \times m$ matrices. Prove that the product $A_1 \cdots A_n$ and any cyclic permutation of this product, $A_{r+1} \cdots A_n A_1 \cdots A_r$, $1 \leqslant r < n$, have the same set of eigenvalues.

Hint: Are the eigenvectors of $A_1 A_2$ related to the eigenvectors of $A_2 A_1$?