

# APPLIED MATH QUALIFYING EXAM, JANUARY 2023

Solve all six problems. You have 4 hours. Good luck! You need to demonstrate proficiency in each area.

**Problem 1.** Consider the following planar system:

$$\begin{aligned}\dot{x} &= y^3 - 4x \\ \dot{y} &= y^3 - y - 3x\end{aligned}$$

- (a) Find all equilibrium points and perform linear stability analysis.
- (b) Show that the line  $y = x$  is an invariant set.
- (c) Show that for any trajectory  $(x(t), y(t))$  we have  $|x(t) - y(t)| \rightarrow 0$  as  $t \rightarrow \infty$ .

**Problem 2.** The following system for certain chemical reactions, developed by the Belgian scientists Prigogine and Lefever, is known as the Brusselator:

$$\begin{aligned}\dot{x} &= a - (b + 1)x + x^2y, \\ \dot{y} &= bx - x^2y.\end{aligned}$$

Here,  $x$  and  $y$  are the (non-negative) concentrations of the two chemicals,  $a$  and  $b$  are positive parameters. Does the Brusselator admit the possibility of Hopf bifurcation?

Note: You'll need to keep one of the parameters fixed and see what happens when the other one varies.

**Problem 3.** Let  $\varphi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the flow of a continuous dynamical system, and let  $A \subset \mathbb{R}^n$  be an attracting set, i.e.  $A$  is a closed invariant set and there is a neighborhood  $U \supset A$  such that  $\forall t \geq 0, \varphi(t, U) \subset U$  and  $\bigcap_{t \geq 0} \varphi(t, U) = A$ . Suppose that  $\bar{x} \in A$  is a hyperbolic equilibrium point of a saddle-type, and let  $W^s(\bar{x})$  and  $W^u(\bar{x})$  denote the stable and unstable manifolds of  $\bar{x}$ . Must the following be true?

- (a)  $W^s(\bar{x}) \subset A$
- (b)  $W^u(\bar{x}) \subset A$

Justify your answer.

**Problem 4.** We shall need the following definition: given a discrete dynamical system  $x \mapsto g(x)$ , a point  $x$  is called *eventually periodic* if  $\exists n \in \mathbb{N}$  such that the point  $y = g^n(x)$  is periodic (in other words,  $g^n(x) = g^{n+k}(x)$  for some positive integer  $k > 0$ ).

Now, consider the map  $g : [0, 1] \rightarrow [0, 1]$  given by

$$g(x) = \begin{cases} 2x, & x \in \left[0, \frac{1}{2}\right] \\ c, & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

where  $c \in (0, \frac{1}{2})$ . Show that *every* point  $x \in [0, 1]$  of the discrete dynamical system  $x \mapsto g(x)$  is *eventually periodic*.

**Problem 5.** Let  $c$  and  $L$  be positive numbers, and let  $u_0 \in C^\infty([0, L])$  be a nonnegative function that vanishes at  $x = 0$  and  $x = L$  but is not identically zero. Suppose that  $u(x, t)$  is a smooth, nonnegative solution to the initial-value problem

$$\begin{aligned} u_t &= u_{xx} + c^2 u + u^2, & x &\in (0, L), t > 0, \\ u(0, t) &= u(L, t) = 0, & t &> 0, \\ u(x, 0) &= u_0(x), & x &\in [0, L]. \end{aligned}$$

Let  $\phi(x) = \frac{\pi}{2L} \sin\left(\frac{\pi x}{L}\right)$  and  $E(t) = \int_0^L u(x, t) \phi(x) dx$ . Note that  $\int_0^L \phi(x) dx = 1$ .

- (a) Show that if  $cL \geq \pi$ , then  $E'(t) \geq E(t)^2$  for all  $t$ . (Hint: As an intermediate step, apply the Cauchy-Schwarz inequality to  $\int_0^L (u\sqrt{\phi}) \sqrt{\phi} dx$ .)
- (b) Deduce that the solution  $u(x, t)$  does not exist beyond time  $t = 1/E(0)$ .

**Problem 6.** Let  $A \in \mathbb{C}^{n \times n}$  be a normal matrix, and let  $S$  be a  $k$ -dimensional subspace of  $\mathbb{C}^n$ . Let  $\gamma \in \mathbb{C}$  and  $\varepsilon > 0$  be given. Suppose that  $\|Ax - \gamma x\|_2 < \varepsilon$  for every unit vector  $x \in S$ . Show that  $A$  has at least  $k$  eigenvalues (counting multiplicities) in a disk of radius  $\varepsilon$  centered at  $\gamma$ .