APPLIED MATH QUALIFYING EXAM, AUGUST 2020

Solve all six problems. You have 4 hours. You need to demonstrate proficiency in each area. Good luck!

Problem 1. Consider a general predator-prey system:

$$\begin{aligned}\dot{x} &= xg(x) - yp(x)\\ \dot{y} &= y\left(-d + q(x)\right)\end{aligned}$$

where $d > 0$ and the functions $g(x), p(x), q(x)$ are smooth for $x \geq 0$. Let $K > 0$, and assume that

$$g(x) > 0 \text{ for } x < K,\ g(x) < 0 \text{ for } x > K, \text{ and } g(K) = 0.$$

Assume also that

$$p(0) = 0,\ p(x) > 0 \text{ for } x > 0,$$

and

$$q(0) = 0,\ q(K) > d,\ \frac{d}{dx}q(x) > 0 \text{ for } x > 0,$$

i.e. $q(x)$ is monotonically increasing.

(a) Show that x- and y-nullclines are given by the equations $y = \dfrac{xg(x)}{p(x)}$ and $x = \bar{x}$, respectively, where $\bar{x}$ is the unique point such that $q(\bar{x}) = d$. Deduce that there is a single equilibrium point, $\mathbf{w} = (\bar{x}, \bar{y})$, in the interior of the first quadrant, with $\bar{y} = \dfrac{\bar{x}g(\bar{x})}{p(\bar{x})}$.

(b) Show that $\mathbf{w}$ is locally asymptotically stable if and only if $\dfrac{d}{dx}\left(\dfrac{xg(x)}{p(x)}\right)$ is negative at $x = \bar{x}$.
Hint: Recall that the product and the sum of the eigenvalues of the Jacobian at $\mathbf{w}$ *are equal to the determinant and the trace, respectively. Then consider the signs of the determinant and the trace.*

Problem 2. Consider the following predator-prey system:

$$\begin{aligned}\dot{x} &= rx\left(1 - \frac{x}{K}\right) - y\frac{cx}{a+x}\\ \dot{y} &= y\left(-d + \frac{bx}{a+x}\right)\end{aligned}$$

where all parameters are positive with $b > d$ and $K > \dfrac{ad}{b-d}$. Note that this system is a particular case of the system from Problem 1.

(a) Use the results of Problem 1 (even if you haven't solved it) to show that there is a single equilibrium point $\mathbf{w} = (\bar{x}, \bar{y})$ with $\bar{x} = \dfrac{ad}{b-d}$ in the interior of the first quadrant, and that it is locally asymptotically stable if and only if $K < a + 2\bar{x}$.

(b) Show that the system has a limit cycle if $K > a + 2\bar{x}$.

Problem 3. Consider the following planar system:

$$\dot{x} = (1 + a^2)x + (2 - 6a)y + f(x, y)$$
$$\dot{y} = -x - 2y + g(x, y)$$

where a is a parameter. The functions $f(x, y)$, $g(x, y)$ are smooth and can be expanded into a Taylor series around the origin starting with quadratic terms (that is, the values of the functions and their first derivatives at the origin are all zero).

(a) For which values of a is the origin locally asymptotically stable?

(b) For which values of a is it possible to have a Hopf bifircation?

Problem 4. Recall that a homoclinic orbit of a discrete dynamical system $x \mapsto f(x)$ is a bi-inifinite sequence

$$\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$$

such that $x_{n+1} = f(x_n)$, $n \in \mathbb{Z}$, and

$$\lim_{n\to-\infty} x_n = \lim_{n\to\infty} x_n = p$$

for some p in the domain of f. It is known that presence of a homoclinic orbit implies chaos.

Consider the system $x \mapsto f(x)$ for $f(x) = 1 - 2|x|$.

(a) Find equilibrium points and perform linear stability analysis for each of them.

(b) Show that the Lyapunov exponent is positive, which suggests chaos.

(c) Show that this system has a homoclinic orbit, and hence is chaotic.
Hint: Often, a homoclinic orbit can be constructed by "iterating back" an unstable equilibrium point. This gives the part of the orbit for $n \to -\infty$. The other part then becomes a constant sequence which simply repeats the unstable equilibrium point.

Problem 5. Suppose that a collection of particles with a unit total mass diffuse according to the equation

$$u_t = au_{xx}, \qquad -\infty < x < \infty, t > 0,$$

where $u(x, t)$ represents the density of the particles, $a > 0$. Assume that all solutions and their derivatives are smooth with a faster than polynomial decay at infinity (i.e. $\lim_{x\to\pm\infty} |x|^m u(x, t) = 0$ and $\lim_{x\to\pm\infty} |x|^m u_x(x, t) = 0$ for any $m \geq 1$).

(a) Given that $u(x, t) = \dfrac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ is the fundamental solutions of the above diffusion equation for $a = 1$, rescale time to find the fundamental solution for any $a > 0$.

(b) Suppose that the initial density profile is given by $u(x, 0) = f(x)$ for some function $f(x)$ (again, assumed to be smooth and rapidly decaying at infinity). Show that the first moment of the density $u(x, t)$ (which represents the expected position of a particle) is constant in t:

$$\int_{-\infty}^{\infty} su(s, t)ds = \int_{-\infty}^{\infty} sf(s)ds.$$

What is the value of the first moment if $f(x)$ is even?

(c) Now consider the second moment (which represents the "average" spread of the particles around their expected position):

$$M_2(t) = \int_{-\infty}^{\infty} s^2 u(s,t) ds.$$

Show that $M_2(t) = 2at + M_2(0)$, where $M_2(0)$ is the second moment at time $t = 0$.

Problem 6. Let A be a symmetric $n \times n$ matrix with non-negative elements. Prove that for any nonzero $x \in \mathbb{R}^n$ with non-negative elements the following inequality holds:

$$\left(\frac{\langle x, Ax \rangle}{\langle x, x \rangle} \right)^m \leq \frac{\langle x, A^m x \rangle}{\langle x, x \rangle}, \quad m \in \mathbb{Z}_+,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner (dot) product.

Hint: Use induction.