APPLIED MATHEMATICS QUALIFYING EXAM, AUGUST 2021

Solve all six problems. You have 4 hours. Good luck! You need to demonstrate prociency in each area.

Problem 1. Consider the following system with a real parameter k:

$$\dot{x} = (k - \sqrt{x^2 + y^2})x + y$$
$$\dot{y} = -x + (k - \sqrt{x^2 + y^2})y$$

(a) Show that origin is the only equilibrium point. Investigate and comment on the linear stability of this equilibrium point.

(b) Use the Lyapunov function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ to deduce that the origin is stable for $k \leqslant 0$ and unstable if $k > 0$. Also find the value of k for which the origin is asymptotically stable.

(c) The system can be transformed to the following polar form

$$\dot{r} = (k - r)r$$
$$\dot{\theta} = -1$$

Consider the transformed system with the initial condition $r(0) = r_0$ and $\theta(0) = 0$ to verify that the solution to the system is given by

$$r = \frac{k\, r_0}{r_0 + (k - r_0) \exp(-kt)}$$
$$\theta = -t$$

(d) Use the solution in (c) to prove that for $k > 0$ we can define a Poincaré map $P : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$P(r) = \frac{kr}{r + (k - r) \exp(-2\pi k)}$$

(e) Use Poincaré map, or any other approach, to show that the limit cycle in (c) is stable.

Problem 2. A simple model of pigmentation on an animal body is given by

$$\frac{dx}{dt} = s - rx + \frac{x^2}{1 + x^2},$$

where $x(t)$ measures the concentration of a pigment on the body, $s \geqslant 0$ is a parameter that represents some biochemical signal that promotes the pigmentation, and $r > 0$ is another parameter that represents degradation of the pigment.

(a) In the absence of the biochemical signal, i.e. $s = 0$, show that, in addition to $x = 0$, there are two positive equilibrium points if $r < r_c$, where r_c is to be determined by you.

(b) Denoting the right hand side of the model by $f(x, r, s)$, sketch the graph of f for $s = 0$ and $r = r_0 \in (0, r_c)$. Based on the sketch, describe (informally) how the equilibrium points change when $r = r_0$ and s is varied between 0 and ∞, and when $s = 0$ and r is varied between 0 and ∞.

(c) Show that for $s > 1$ there is only one equilibrium point regardless of the value of r (you may need to consider the sign of $\frac{\partial f}{\partial x}$). Describe (informally) what happens if the system with $s > 1$ is at the equilibrium, but then we set $s = 0$ (note that this behavior depends on the value of r).

(d) Recall that the bifurcation curves in (r, s) space are obtained by equating f and $\frac{\partial f}{\partial x}$ to zero. Find *parametric equations* for these curves (note that you may regard x as a parameter, or you may want to set $x = \tan \alpha$). Describe the bifucations that occur when $r > 0$ is fixed and s is varied between 0 and ∞.

Problem 3. Recall that the Hartman Grobman Theorem says that, under certain assumptions, a nonlinear systems "looks alike" its linearization. More precisely, the statement of the theorem is as follows:

Consider a system $\dot{x} = f(x) \in \mathbb{R}^n$, with $f \in C^1(\mathbb{R}^n)$, and let $\varphi_t(x)$ denote its flow. Assume that x^* be a hyperbolic equilibrium point. Then there exists a neighborhood N of x^* such that φ is topologically conjugate to the flow of the corresponding linearized system, $\dot{x} = Df(x^*)$.

This problem will test your knowledge of some concepts and ideas involved in the proof of the theorem.

(a) Define what an hyperbolic equilibrium point is and sketch an example of a possible phase portraits around a non hyperbolic point.
(b) Describe how the equivalence classes (under linear and hence topological conjugacy) for *planar linear systems* are determined by the corresponding eigenvalues as well as stables, unstable, and center spaces.
(c) Give a formal definition of topological conjugacy, denoting the homeomorphism between the neighborhoods by H.
(d) Restate the Hartman-Grobman theorem using the formal definition of topologically conjugacy. In your statement, use A to denote the linearization of f at x^*, i.e. $A = Df(x^*)$, and use $\psi_t(x) = e^{At}x$ to denote the flow of the linearized system.
(e) The difficulty of the proof lies in the construction of the homeomorphism H. But suppose that H_1 is a unique homeorphism satisfying

$$H_1(x) = (\psi_{-1} \circ H_1 \circ \varphi_1)(x) = e^{-A}(H_1 \circ \varphi_1)(x).$$

Show that the sought homeomorphism H is given by

$$H(x) = \int_0^1 (\psi_{-s} \circ H_1 \circ \varphi_s)(x)\,ds.$$

You do not need to construct $H_1(x)$.

Problem 4. Let us consider the following difference equation:

$$y_{n+1} = F(y_n) = 3y_n - 6\gamma y_n + 2\gamma y_n^2,$$

where $\gamma > 0$ is the bifurcation parameter.

(a) Perform linear stability analysis of the fixed points for all values of γ.
(b) State the general equation which determines the existence of a k-cycle, then show that the existence of a 2-cycle is related to the solutions of the following equation:

$$4\gamma^2 y^2 + (8\gamma - 12\gamma^2)y + 4 - 6\gamma = 0.$$

Use the above equation to justify that the existence of a 2-cycle requires $\gamma \geqslant \frac{2}{3}$.
(c) Show that the 2-cycle is stable for $\frac{2}{3} \leqslant \gamma \leqslant \frac{1}{6}(2 + \sqrt{6})$.

Problem 5. Suppose that u is a smooth solution of the initial-value problem

$$\frac{\partial u}{\partial t} - a\Delta u + b|\nabla u|^2 = 0, \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \tag{1}$$

$$u = g, \quad \text{on} \quad \mathbb{R}^n \times \{0\}, \tag{2}$$

where $a > 0$, $b \in \mathbb{R}$, and $g : \mathbb{R}^n \to \mathbb{R}$ is given.

(a) Show that if $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

$$a\phi'' + b\phi' = 0,$$

then the function $w = \phi(u)$ satisfies

$$\begin{aligned} \frac{\partial w}{\partial t} - a\Delta w &= 0, && \text{in} \quad \mathbb{R}^n \times (0, \infty), \\ w &= \phi(g), && \text{on} \quad \mathbb{R}^n \times \{0\}. \end{aligned}$$

(b) Use part (a) to find an explicit formula for the solution to (1-2) in terms the heat kernel $\frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}$.

Problem 6. Let $m, n \in \mathbb{N}$ and $A, B \in \mathbb{R}^{m \times n}$ with $\operatorname{rank} B = 1$. Show that

$$\operatorname{rank}(A - B) = \operatorname{rank} A - 1$$

if and only if there exist vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ such that $y^T A x \neq 0$ and

$$B = \frac{A x y^T A}{y^T A x}.$$