APPLIED MATH QUALIFYING EXAM, JANUARY 2020

Solve all six problems. You have 4 hours. Good luck! You need to demonstrate prociency in each area.

Problem 1. Consider the following system:

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= y - x^3.\end{aligned}$$

(a) Show that origin is a fixed point, and that linear stability analysis does not yield an answer regarding its stability.

(b) Use a Liapunov function of the form $V(x, y) = ax^4 + bx^2 + cxy + dy^2$ to deduce that the origin is unstable.

Problem 2. Consider a planar system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^2$$

with flow $\varphi(t, x)$. Recall that a *trapping region* for this system is a compact, connected set $D \subset \mathbb{R}^2$ such that $\varphi(t, D) \subsetneq D$ for all $t > 0$ (where $\subsetneq$ means *proper* subset).

Now, assume that $D \subset \mathbb{R}^2$ is a closed region whose boundary, ∂D, is a simple, smooth closed curve that is not a periodic trajectory of the flow. For each $x \in \partial D$ let $n(x)$ denote the inward unit normal to ∂D at x, and recall that $\langle f(x), n(x)\rangle$ denotes the scalar product of the vectors $f(x)$ and $n(x)$.

(a) Show that the condition $\langle f(x), n(x)\rangle > 0$ for all $x \in \partial D$ is sufficient for D to be a trapping region.

(b) Show that the condition $\langle f(x), n(x)\rangle \geq 0$ for all $x \in \partial D$ is **not** sufficient for D to be a trapping region.

Problem 3. Consider the following planar system

$$\begin{aligned}\dot{x} &= \mu x + y\\ \dot{y} &= -x - y^3\end{aligned}$$

(a) Show that the origin is a fixed point and perform its linear stability analysis for all values of μ.

(b) Show that the system has a stable limit cycle for $\mu > 0$. What kind of a bifurcation occurs at $\mu = 0$?

Problem 4. Consider a planar diffeomorphism $f(x, y) = \big(f_1(x), f_2(y)\big)$. Suppose that $f_1(x)$ has a 2-cycle $\{x_1^*, x_2^*\}$, and $f_2(y)$ has a fixed point y^*.

(a) Show that f has a 2-cycle $\{(x_1^*, y^*), (x_2^*, y^*)\}$.

(b) If the 2-cycle of $f_1(x)$ is asymptotically stable, how does the stability of y^* affect the stability of the 2-cycle in f?

Problem 5. Suppose that v is a nonzero column vector in $\mathbb{C}^n$ ($n > 1$) and the matrix $A = \dfrac{vv^*}{v^*v}$, where v^* denotes the Hermitian conjugate of v.

(a) What are the eigenvalues of A? Explain.

(b) Is the matrix $I + A$ (I is the $n \times n$ identity matrix) diagonalizable? Explain.

(c) Find the determinant of $I + A$.

(d) What is A^{2020}? Explain.

Problem 6. Let $D \subset \mathbb{R}^3$ be a region with smooth boundary, ∂D. Show that the boundary value problem

$$
\begin{aligned}
\nabla^2 u - \lambda u &= h(x), \qquad && x \in D, \\
\frac{\partial u}{\partial n} + ku &= g(x), \qquad && x \in \partial D,
\end{aligned}
$$

where the functions h and g are smooth, has a unique solution provided $k > 0$ and $\lambda > 0$. Does uniqueness still hold if one of k and λ is zero while the other is strictly positive? Comment on uniqueness when both $k = 0$ and $\lambda = 0$.