Problem 1. Consider the following system:
\[ \dot{x} = y, \quad \dot{y} = y - x^3. \]
(a) Show that origin is a fixed point, and that linear stability analysis does not yield an answer regarding its stability.
(b) Use a Lyapunov function of the form
\[ V(x, y) = ax^4 + bx^2 + cxy + dy^2 \]
to deduce that the origin is unstable.

Problem 2. Consider a planar system
\[ \dot{x} = f(x), \quad x \in \mathbb{R}^2 \]
with flow \( \varphi(t, x) \). Recall that a trapping region for this system is a compact, connected set \( D \subset \mathbb{R}^2 \) such that \( \varphi(t, D) \subsetneq D \) for all \( t > 0 \) (where \( \subsetneq \) means proper subset).
Now, assume that \( D \subset \mathbb{R}^2 \) is a closed region whose boundary, \( \partial D \), is a simple, smooth closed curve that is not a periodic trajectory of the flow. For each \( x \in \partial D \) let \( n(x) \) denote the inward unit normal to \( \partial D \) at \( x \), and recall that \( \langle f(x), n(x) \rangle \) denotes the scalar product of the vectors \( f(x) \) and \( n(x) \).
(a) Show that the condition \( \langle f(x), n(x) \rangle > 0 \) for all \( x \in \partial D \) is sufficient for \( D \) to be a trapping region.
(b) Show that the condition \( \langle f(x), n(x) \rangle \geq 0 \) for all \( x \in \partial D \) is not sufficient for \( D \) to be a trapping region.

Problem 3. Consider the following planar system
\[ \dot{x} = \mu x + y, \quad \dot{y} = -x - y^3. \]
(a) Show that the origin is a fixed point and perform its linear stability analysis for all values of \( \mu \).
(b) Show that the system has a stable limit cycle for \( \mu > 0 \). What kind of a bifurcation occurs at \( \mu = 0 \)?

Problem 4. Consider a planar diffeomorphism \( f(x, y) = (f_1(x), f_2(y)) \). Suppose that \( f_1(x) \) has a 2-cycle \( \{x_1^*, x_2^*\} \), and \( f_2(y) \) has a fixed point \( y^* \).
(a) Show that \( f \) has a 2-cycle \( \{(x_1^*, y^*), (x_2^*, y^*)\} \).
(b) If the 2-cycle of \( f_1(x) \) is asymptotically stable, how does the stability of \( y^* \) affect the stability of the 2-cycle in \( f \)?
Problem 5. Suppose that $v$ is a nonzero column vector in $\mathbb{C}^n$ ($n > 1$) and the matrix $A = \frac{vv^*}{v^*v}$, where $v^*$ denotes the Hermitian conjugate of $v$.

(a) What are the eigenvalues of $A$? Explain.
(b) Is the matrix $I + A$ ($I$ is the $n \times n$ identity matrix) diagonalizable? Explain.
(c) Find the determinant of $I + A$.
(d) What is $A^{2020}$? Explain.

Problem 6. Let $D \subset \mathbb{R}^3$ be a region with smooth boundary, $\partial D$. Show that the boundary value problem

$$\nabla^2 u - \lambda u = h(x), \quad x \in D,$$
$$\frac{\partial u}{\partial n} + ku = g(x), \quad x \in \partial D,$$

where the functions $h$ and $g$ are smooth, has a unique solution provided $k > 0$ and $\lambda > 0$. Does uniqueness still hold if one of $k$ and $\lambda$ is zero while the other is strictly positive? Comment on uniqueness when both $k = 0$ and $\lambda = 0$. 