APPLIED MATH QUALIFYING EXAM, JANUARY 2022

Solve all six problems. You have 4 hours. Good luck! You need to demonstrate proficiency in each area.

Problem 1. In epidemiological modeling the SEIR model is a compartmental model that has been widely used in the current COVID-19 pandemic to predict the spread of the disease.

The SEIR diagram below shows how individuals move through each compartment in the model, we have S=Susceptible, E=Exposed, I=Infectious, R=Recovered:

$$S \xrightarrow{\beta I/N} E \xrightarrow{\sigma} I \xrightarrow{\gamma} R$$

where the parameters $\beta, N, \sigma, \gamma \geqslant 0$ have following meaning:

- β is the transmission rate, the average rate at which an infected individual can infect a susceptible,
- N is total population,
- $1/\sigma$ is the latency period,
- $1/\gamma$ is the symptomatic period.

The equations describing the SEIR model are given by:

$$\frac{dS}{dt} = -\beta SI/N \tag{1}$$

$$\frac{dE}{dt} = \beta SI/N - \sigma E \tag{2}$$

$$\frac{dI}{dt} = \sigma E - \gamma I \tag{3}$$

$$\frac{dR}{dt} = \gamma I. \tag{4}$$

We know now that for COVID-19 individuals can stay asymptomatic throughout the duration of the infection and still spread to other individuals. We also know that immunity has a finite duration. The new model, SE(R)IRS, can be visualized as

$$\begin{tikzcd} S \arrow[r, "\beta(I+\alpha E)/N"] & E \arrow[r, "\sigma"] \arrow[rr, bend left, "\delta"] & I \arrow[r, "\gamma"] & R \arrow[lll, bend left, "\omega"] \end{tikzcd}$$

where $\alpha \in [0, 1]$ accounts for the reduction of transmission by asymptomatic individuals, and $1/\delta$ is the duration of the course for the asymptomatic. The parameter ω is the inverse of the immunity period.

(a) Modify the equations (1)-(4) of the SEIR model to account for the modification of the SE(R)IRS model.

(b) Show that the resulting system has two equilibria, one disease free and one endemic equilibrium (which is an equilibrium with a nonzero number of infected individuals).

(c) Let $\mathfrak{R}_0 = \left(\frac{\alpha\gamma+\sigma}{\delta+\sigma}\right)\left(\frac{\beta}{\gamma}\right)$ be the basic reproductive number. Prove the following:

If $\mathfrak{R}_0 < 1$ then the disease-free equilibrium is locally asymptotically stable and the endemic equilibrium is irrelevant. If $\mathfrak{R}_0 > 1$ then the endemic equilibrium is locally asymptotically stable and the disease-free equilibrium is unstable.

Hint. For the endemic equilibrium you can choose appropriate values for the parameters and prove it in that specific case.

(d) Explain what the theorem means from a practical view point for the pandemic.

Problem 2. A model that is used to analyze a class of experimental systems known as chemical oscillator is given by

$$\dot{x_1} = \alpha - x_1 - \frac{4\,x_1\,x_2}{1+x_1^2}$$
$$\dot{x_2} = \beta\,x_1\left(1 - \frac{x_2}{1+x_1^2}\right)$$

where x_1 and x_2 are dimensionless concentrations of certain chemicals and α, β are positive parameters.

(a) Show that the system has a periodic orbit when $\beta < \frac{3\alpha}{5} - \frac{25}{\alpha}$

(b) Fixing some $\alpha \geqslant 6.5$ and treating β as a bifurcation parameter, describe the bifurcation at $\beta = \frac{3\alpha}{5} - \frac{25}{\alpha}$.

Problem 3. Let $\varphi(t,x)$ be the flow of a system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, with f smooth. Recall that a closed set $\Lambda \subset \mathbb{R}^n$ is called an attracting set for this system if there is a neighborhood $U \supset \Lambda$ (called a fundamental neighborhood) such that

(a) for every neighborhood V of Λ we have $\varphi(t,U) \subset V$ when t is large enough;

(b) $\bigcap_{t \geqslant T} \varphi(t,U) = \Lambda$ for some T.

Show that if U is open and the closure of $\varphi(t,U)$ is compact and contained in U for all sufficiently large t, then

$$\Lambda = \bigcap_{t \geqslant 0} \varphi(t,U)$$

is a compact attracting set with fundamental neighborhood U.

Problem 4. Consider the dimensionless Nicholson–Bailey host-parasitoid model

$$h_{n+1} = R_0 h_n e^{-p_n},$$
$$p_{n+1} = h_n(1 - e^{-p_n}),$$

where $R_0 > 0$.

(a) Calculate the fixed point inside the positive quadrant and the corresponding value of the Jacobian.

(b) Show that this fixed point is unstable.
Hint: First, show that the stability requirement can be stated as $g(R_0) > 1$, where g is some function that you need to determine. Then show $g(R_0) \leqslant 1$.

Problem 5. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Suppose that u is a smooth function satisfying

$$-\Delta u = f, \qquad \text{in } \Omega$$
$$\frac{\partial u}{\partial n} = g, \qquad \text{on } \partial\Omega,$$

where $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are given functions satisfying $\int_\Omega f dx + \int_{\partial\Omega} g dS = 0$. Show that for every smooth function v we have

$$E(u) \leqslant E(v),$$

where

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx - \int_{\partial\Omega} g v dS.$$

Is the inequality strict for $v \neq u$?

Problem 6. Suppose $R : \mathbb{C}^n \to \mathbb{C}^m$ and $S : \mathbb{C}^m \to \mathbb{C}^k$ are both linear and $\mathrm{Range}(R) = \mathrm{Null}(S)$. Show that the map $RR^* + S^*S$ is invertible, where R^* and S^* denote adjoint maps to R and S, respectively.