A SURVEY OF EUCLID'S ELEMENTS

SPRING 2012

1. Definitions, Axioms and Postulates

Definition 1.1. **1.** *A* **point** *is that which has no part.*

2. *A* **line** *is breadth-less length.*

3. *The* **extremities** *of a line are points.*

4. *A* **straight line** *is a line which lies evenly with the points on itself.*

8. *A* **plane angle** *is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*

10. *When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is* **right**, *and the straight line standing on the other is called a* **perpendicular** *to that on which it stands.*

15. *A* **circle** *is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.*

21. Rectilineal figures *are those which are contained by straight lines,* **trilateral** *figures being those contained by three, ...*

23. Parallel *straight lines are straight lines which being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

Definition 1.2. (1) *The Euclidean plane* $\mathbb{E}$ *is a set of* **points** *with distinguished subsets called (straight)* **lines**.

(2) **line segment** AB

(3) **half ray** *emanating from an* **initial point**

(4) *Two distinct points* A, B *determine a unique half ray, denoted* $\overrightarrow{AB}$*, which has* A *as initial point and contains* B.

(5) *Line segments* AB *and* $A'B'$ *may or may not be* **congruent**, $AB \equiv A'B'$.

(6) *Two angles* $\angle A$ *and* $\angle A'$ *may or may not be* **congruent**, $\angle A \equiv \angle A'$.

(7) *A set of points is* **collinear** *if the set is contained in some straight line.*

(8) *A triangle* $\Delta\, ABC$ *is any set* $\{A, B, C\}$ *of non–collinear points. The points* A, B, C *are the* **vertices** *of the triangle. The line segments* AB, BC, CA *are called the* **sides** *of the triangle.*

(9) *An* **angle** *is a set of two half–rays* h, k *with common initial point not both contained in the same line. We write* $\angle(h, k) = \{h, k\}$.

(10) *Let* $\Delta\, ABC$ *be a triangle. The angles* $\angle A = \angle(\overrightarrow{AB}, \overrightarrow{AC})$, $\angle B = \angle(\overrightarrow{BC}, \overrightarrow{BA})$, *and* $\angle C = \angle(\overrightarrow{CA}, \overrightarrow{CB})$ *are the* **angles** *of the triangle.*

(11) *Two triangles are* **congruent** *if their vertices can be paired in such a way that all the corresponding sides are congruent and all the corresponding angles are congruent. If the vertices of the one triangles are labeled* A, B, C *and the corresponding vertices of the other are labeled* A', B', C', *then we write* $\Delta\, ABC \equiv \Delta\, A'B'C'$ *and we have* $AB \equiv A'B'$, $BC \equiv B'C'$, $CA \equiv C'A'$, $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$, *and* $\angle C \equiv \angle C'$.

(12) *Let* C *be a point, and* AB *a line segment. The* **circle** *with* **center** C *and* **radius** AB *is the set of all points* P *such that* $PC \equiv AB$.

(13) *Two (different) lines are* **parallel** *if they do not intersect (in the sense of set theory). We also agree, for technical reasons, that a line is parallel to itself.*

1.3. Euclid's Postulates *Let the following be postulated:*

(1) *To draw a straight line from any point to any point.*
(2) *To produce a finite straight line continuously in a straight line.*
(3) *To describe a circle with any center and distance.*
(4) *That all right angles are equal to one another.*
(5) *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

1.4. Euclid's Common Notions or Axioms

(1) *Things which are equal to the same thing are also equal to one another.*
(2) *If equals be added to equals, the wholes are equal.*
(3) *If equals be subtracted from equals, the remainders are equal.*
(4) *Things which coincide with one another are equal to one another.*
(5) *The whole is greater than the part.*

2. Book I. Propositions

After the definitions, postulates, and axioms, the **propositions** follow with proofs.

In the following some propositions are stated in the translation given in *Euclid, The Thirteen Books of THE ELEMENTS*, Translated with introduction and commentary by Sir Thomas L. Hearth, Dover Publications 1956. Most propositions are translated into modern mathematical language and labeled by a decimal number indicating section number and item number.

Propositions 1 to 3 state that certain constructions are possible.

2.1. Proposition 4 *If two triangles have two sides equal to two sides respectively, and have the enclosed angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides sub-tend.*

Proof. Superposition.

2.2. Proposition 4 bis *Let* $\Delta\, ABC$ *and* $\Delta\, A'B'C'$ *be triangles such that* $AB \equiv A'B'$, $AC \equiv A'C'$ *and* $\angle A \equiv \angle A'$. *Then* $\Delta\, ABC \equiv \Delta\, A'B'C'$. (**sas**)

2.3. Proposition 5 *In* $\Delta\, ABC$, *if* $AB \equiv AC$ *then* $\angle B \equiv \angle C$.

2.4. Proposition 6 *In* $\Delta\, ABC$, *if* $\angle B \equiv \angle C$ *then* $AB \equiv AC$.

Proposition 7 is preparatory to Proposition 8.

2.5. Proposition 8 *Let* $\Delta\, ABC$ *and* $\Delta\, A'B'C'$ *be triangles such that* $AB \equiv A'B'$, $BC \equiv B'C'$ *and* $CA \equiv C'A'$. *Then* $\Delta\, ABC \equiv \Delta\, A'B'C'$. (**sss**)

Proposition 9 describes a method for bisecting an angle. Similarly, Proposition 10 tell how to bisect a line segment. Proposition 11 contains a construction of the perpendicular to a line at a point on the line.

2.6. Proposition 12 *To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line.*

2.7. Proposition 12 *There is a compass and straightedge construction for the perpendicular of a given line passing through a point not on the line.*

Construction. Choose a point D in the half plane of the given line l not containing the given point C. Draw the circle with center C and radius CD. It cuts l in points G, E. Let H be the midpoint of GE. Then HC is the desired perpendicular. $\square$

2.8. Proposition 13 *Vertical angles are congruent.*

What follows now are "geometric inequalities". They are proved without the use of Postulate 5.

2.9. Proposition 16 *In $\triangle ABC$, the exterior angle at C is larger than either $\angle A$ or $\angle B$.*

2.10. Proposition 17 *In* $\Delta\, ABC$, $\angle A + \angle B < 2R$.

2.11. Proposition 18 *In* $\Delta\, ABC$, *if* $BC > AC$, *then* $\angle A > \angle B$.

2.12. Proposition 19 *In* $\Delta\, ABC$, *if* $\angle A > \angle B$ *then* $BC > AC$.

It is interesting that the Proposition 18 implies its converse, Proposition 19.

2.13. Proposition 20 (Triangle Inequality) *In any* $\Delta\, ABC$, $AC + BC > AB$.

2.14. Proposition 21 *Let* $\Delta\, ABC$ *be given, and let* C' *be a point in the interior of* $\Delta\, ABC$. *Then* $AC + BC > AC' + BC'$ *and* $\angle C' > \angle C$.

2.15. Proposition 22 *If* a, b, *and* c *are line segments such that* $a + b > c$ *then there is a triangle* $\Delta\, ABC$ *such that* $AB \equiv c$, $BC \equiv a$, *and* $CA \equiv b$.

2.16. Proposition 25 *If two triangles have two sides equal to two sides respectively, but have the base greater than the base, they will also have the one of the angles contained by the equal straight lines greater than the other.*

2.17. Proposition 25 *Let* $\Delta\, ABC$ *and* $\Delta\, A'B'C'$ *be triangles such that* $AB \equiv A'B'$ *and* $AC \equiv A'C'$ *but* $BC > B'C'$ *then* $\angle A > \angle A'$.

2.18. Proposition 26 *Let* $\Delta\, ABC$ *and* $\Delta\, A'B'C'$ *be triangles such that* $AB \equiv A'B'$, $\angle A \equiv \angle A'$ *and* $\angle B \equiv \angle B'$ *then* $\Delta\, ABC \equiv \Delta\, A'B'C'$. **(asa)**

The following congruence theorem does not appear in the Elements.

2.19. Proposition *Let* $\Delta\, ABC$ *and* $\Delta\, A'B'C'$ *be triangles such that* $\angle A \equiv \angle A'$, $AB \equiv A'B'$, $BC \equiv B'C'$ *and* $BC > AB$ *then* $\Delta\, ABC \equiv \Delta\, A'B'C'$. **(asS)**

2.20. Proposition 27 *If a line cuts a pair of lines such that the alternating angles are congruent then the lines of the pair are parallel.*

2.21. Proposition 28 *If a line cuts a pair of lines such that corresponding angles are congruent, then the lines of the pair are parallel.*

Now, for the first time, Postulate 5 will be used.

2.22. Proposition 29 *If* a, b *are a pair of parallel lines then the corresponding angles at a transversal are congruent.*

2.23. Proposition 30 *If a is parallel to b, and b is parallel to c, then a is parallel to c.*

The famous next theorem contains the important fact that the angle sum of a triangle is 180°.

2.24. Proposition 32 *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.*

Propositions 33 to 36 deal with parallelograms.

Exercise 2.25. *Recall that a* **parallelogram** *is a quadrilateral with opposite sides parallel.*

(1) *(***Proposition 33***) Let* $\square ABCD$ *be a quadrilateral with sides* AB*,* BC*,* CD*,* DA *such that* AB *is opposite* CD*, and* BC *is opposite* DA*. Suppose that* $AD \equiv BC$ *and* $\overleftrightarrow{AD}$ *is parallel to* $\overleftrightarrow{BC}$*. Prove that* $AB \equiv CD$ *and* $\overleftrightarrow{AB}$ *is parallel to* $\overleftrightarrow{DC}$*.*

(2) *(***Proposition 34***) Let* $\square ABCD$ *be the parallelogram with sides* AB*,* BC*,* CD*,* DA *such that* AB *is opposite* CD*, and* BC *is opposite* DA*. Prove that* $AB \equiv CD$*,* $BC \equiv AD$ *and that the diagonals* BD *and* AC *bisect one another.*

The following proposition deals with area for the first time. When two plane figures are called "equal" in Euclid, it means in modern terms that they have equal areas. The concept of area is treated as a known, unquestioned concept, which is not satisfactory nowadays. It is interesting, however, to observe which properties of area are used in the proofs.

2.26. Proposition 35 *Parallelograms which are on the same base and in the same parallels are equal to one another.*

Proof. Let $ABCD$, $EBCF$ be parallelograms on the same base BC and in the same parallels AF, BC; I say that $ABCD$ is equal to the parallelogram $EBCF$. For, since $ABCD$ is a parallelogram, AD is equal to BC. For the same reason EF is equal to BC, so that AD is also equal to EF [C.N. 1]; and DE is common; therefore the whole AE is equal to the whole DF [C.N. 2]. But AB is also equal to DC [I. 34]; therefore the two sides EA, AB are equal to the two sides FD, DC respectively, and therefore the angle FDC is equal to the angle EAB, the exterior to the interior [I. 29]; therefore the base EB is equal to the base FC, and the triangle EAB will be equal to the triangle FDC [I, 4]. Let DGE be subtracted from each; therefore the trapezium $ABGD$ which remains is equal to the trapezium $EGCF$ which remains [C.N. 3]. Let the triangle GBC be added to each; therefore the whole parallelogram $ABCD$ is equal to the whole parallelogram $EBCF$ [C.N. 2]. $\square$

2.27. Proposition 38 *Triangles which are on equal bases and in the same parallels are equal to one another.*

2.28. Proposition 47 (Theorem of Pythagoras) *In right–angled triangles the square on the side sub-tending the right angle is equal to the squares on the sides containing the right angle.*

The following is the converse of the Pythagorean Theorem.

2.29. Proposition 48 *If in a triangle the square on one of the sides be equal to the squares on the remaining two sides of the triangle , the angle contained by the remaining two sides of the triangle is right.*

3. Book II

Book II contains a number of propositions on area which is the way to deal with products in Euclidean mathematics. Some propositions amount to algebraic identities which are very simple in today's algebraic language; some propositions use the Pythagorean Theorem to solve quadratic equations. An example is Proposition 14.

3.1. Proposition 14 *To construct a square equal to a given rectilineal figure.*

3.2. Corollary *For any positive real number a, construct $\sqrt{a}$.*

The following proposition is much more tricky.

3.3. Proposition 11 *Let AB be a given line segment. Find a point $C \in AB$ such that the square over AC has the same area as the rectangle with sides AB and CB.*

Proof. Let $r = |AB|$ (the length of AB), and let $a = |AC|$. The problem amounts to constructing

$$a = \frac{r}{2}(-1 + \sqrt{5}). \quad \square$$

4. Book III

This part of the Elements deals with circles and their properties.

Here is a sampling of definitions from Book III.

Definition 4.1.
2. *A straight line is said to* **touch a circle** *which, meeting the circle and being produced, does not cut the circle.*
6. *A* **segment of a circle** *is the figure contained by a straight line and a circumference of a circle.*

8. *An* **angle in a segment** *is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the* **base of the segment**, *is contained by the straight lines so joined.*

Recall that a straight line cuts a circle in at most two points.

Exercise 4.2. (Proposition 10) *Show that two circles intersect in at most two points.*

Definition 4.3. *Let $\mathcal{C}$ be a circle with center Z, and let A, B be points on the circle, i.e., $A, B \in \mathcal{C}$.*

(1) *The line segment AB is a* **chord** *of $\mathcal{C}$.*
(2) *A straight line which intersects the circle in two points is called a* **secant** *of the circle.*
(3) *A straight line which intersects the circle in exactly one point is said to* **touch** *the circle, and to be* **tangent** *to the circle.*
(4) *An* **arc** *of a circle is the intersection of the circle with a half–plane of a secant.*
(5) *The* **central angle** *over the chord AB is the angle $\angle AZB$.*

(6) *An* **inscribed angle** *is an angle* $\angle ACB$ *where* C *is some point on the circle.*

(7) *Two circles which intersect in exactly one point are said to* **touch** *one another.*

Exercise 4.4. (Proposition 1) *Given three (distinct) points of a circle, construct the center by compass and ruler alone.*

4.5. Proposition 10 *A circle does not cut a circle at more than two points.*

4.6. Proposition 16 *The straight line drawn at right angles to the diameter of a circle from its extremities will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semi-circle is greater, and the remaining angle less, than any acute rectilineal angle.*

4.7. Proposition 18 *The tangent at a point* A *of a circle is perpendicular to the radius vector through* A*.*

4.8. Proposition 20 *Let AB be a chord of a circle $\mathcal{C}$ with center Z. Then the central angle over the chord AB is twice the size of any inscribed angle $\angle ACB$ when C and Z are on the same side of $\overleftrightarrow{AB}$.*

4.9. Proposition 21 *In a circle the inscribed angles over the same chord AB and on the same side of $\overleftrightarrow{AB}$ are congruent.*

4.10. Proposition 36 *Let $\mathcal{C}$ be a circle and let P be a point of the exterior of $\mathcal{C}$. Let $T \in \mathcal{C}$ such that $\overleftrightarrow{PT}$ is tangent to $\mathcal{C}$ and let some other line through P intersect $\mathcal{C}$ in the points A and B. Then*

$$PA \cdot PB = PT^2.$$

Exercise 4.11. *Let $\mathcal{C}$ be a circle with center Z and let P be a point of the exterior of $\mathcal{C}$. Let $T \in \mathcal{C}$ such that $\overleftrightarrow{PT}$ is tangent to $\mathcal{C}$ and let $\overleftrightarrow{PZ}$ intersect $\mathcal{C}$ in the points A and B. Prove that*

$$PA \cdot PB = PT^2.$$

5. Book IV

This book deals with connections between circles and triangles essentially. Here are some sample theorems.

5.1. Proposition 4 *In a given triangle to inscribe a circle.*

Proposition 5.2. *The angle bisector is the locus of all points equidistant from the legs of the angle.*

5.3. Proposition 5 *About a given triangle to circumscribe a circle.*

5.4. Proposition 10 *To construct an isosceles triangle having each of the angles at the base double of the remaining one.*

5.5. Proposition 11 *In a given circle, inscribe a regular pentagon.*

6. Book V

This book contains the theory of proportions and the algebra of line segments. Already the definitions are hard to understand and the propositions are complicated, especially when compared with the elegant algebraic language which is available to us today. However, this Book throws considerable light on the Greek substitute for real number.

Here are some sample definitions.

Definition 6.1. (1) *A magnitude is a* **part** *of a magnitude, the less of the greater, when it measures the greater.*

(2) *The greater is a* **multitude** *of the less when it is measured by the less.*

(3) *A* **ratio** *is a sort of relation in respect of size between two magnitudes of the same kind.*

(4) *Magnitudes are said to* **have a ratio** *to one another which are capable when multiplied, of exceeding one another.*

(5) *Magnitudes are said to be* **in the same ratio**, *the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and the fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.*

(6) *Let magnitudes which have the same ratio be called* **proportional**.

There are 11 more definitions at the start of the book. Definition 6.1.3 says that a certain relationship between the sizes of magnitudes may or may not exist; if it exists it is called "ratio". If a and b are magnitudes "of the same kind", then $a : b = a/b$ is their ratio, so some real number by our comprehension. The next Definition (4) says when such a relationship exists: For any integral multiple ma there is an integral multiple nb such that $nb > ma$ and conversely. This definition says that the ratio $a : b$ can be approximated to any degree of precision by rational numbers. Definition (5), due to Eudoxos of Knidos (408?–355?), then says when two ratios $a : a'$ and $b : b'$ are equal in terms of rationals: $a/a' = b/b'$ if and only if for every rational m/n, we have

$$a/a' \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} m/n \quad \Leftrightarrow \quad b/b' \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} m/n.$$

This is a valid criterion for the equality of the real numbers a/a' and b/b'.

Here are some sample theorems which are translated into modern algebraic formulas. They should be interpreted geometrically in order to reflect the Greek original. Also note that ma where m is a positive integer and a a magnitude (line segment, area, volume), means "m copies of a added together", and does not mean a product. This is analogous to the definition of powers. In the following $m, n, p, \ldots$ stand for positive integers while $a, b, c, \ldots$ stand for magnitudes.

6.2. Proposition 1 $ma + mb + mc + \cdots = m(a + b + c + \ldots)$.

6.3. Proposition 2 $ma + na + pa + \cdots = (m + n + p + \ldots)a$.

6.4. Proposition 3 $n(ma) = (nm)a$.

6.5. Proposition 4 *If* $a : b = c : d$ *then* $ma : nb = mc : nd$.

6.6. Proposition 5 $(ma) - (nb) = (m - n)b$.

There are 25 propositions of this nature altogether.

7. Book VI

The results of this book which deals with **similarity** contains very useful and important results.

Definition 7.1. Similar rectilineal figures *are such as have their angles severally equal and the sides about the equal angles proportional.*

We specialize and rephrase this definition to triangles. Note the analogy to "congruent".

Definition 7.2. *Two triangles are* **similar** *if they can be labelled* $\Delta\, ABC$ *and* $\Delta\, A'B'C'$ *in such a way that* $\angle A \equiv \angle A'$, $\angle B \equiv \angle B'$, $\angle C \equiv \angle C'$, $AB : A'B' = BC : B'C' = CA : C'A'$.

Interesting is the following definition.

Definition 7.3. *A straight line is said to have been* **cut in the extreme and mean ratio** *when, as the whole line is to the greater segment, so is the greater to the less.*

The first proposition says, in modern terms, that the area of parallelograms and triangles is proportional to the product of base and height. The proof requires the definition of "equal ratios".

7.4. Proposition 1 *Triangles and parallelograms which are under the same height are to one another as their bases.*

The following result is basic.

7.5. Proposition 2 *If a straight line be drawn parallel to one of the sides of a triangle, it will cut the sides of the triangle proportionally; and if the sides of the triangle be cut proportionally, the line joining the points of section will be parallel to the remaining side of the triangle.*

7.6. Proposition 2 *Let $\angle CAB$ be cut by a transversal parallel to BC in the points B', C' where the notation is chosen so that $B' \in \overrightarrow{AB}$ and $C' \in \overrightarrow{AC}$. Then*

$$AB : BB' = AC : CC' \quad \textit{if and only if} \quad BC \parallel B'C'.$$

Proof. (Euclid) For let $B'C'$ be drawn parallel to BC, one of the sides of the triangle ABC; I say that, as BB' is to $B'A$, so is CC' to $C'A$. For let BC', CB' be joined. Therefore the triangle $BB'C'$ is equal to the triangle $CB'C'$, for they are on the same base $B'C'$ and in the same parallels $B'C'$, BC [I. 38]. And the triangle $AB'C'$ is another area. But equals have the same ratio to the same; therefore as the triangle $BB'C'$ is to the triangle $AB'C'$, so is the triangle $CB'C'$ to the triangle $AB'C'$. etc. □

7.7. Proposition 3 *In* $\Delta\, ABC$ *if the bisector of* $\angle A$ *meets* BC *in the point* D*, then* $\frac{CD}{DB} = \frac{AC}{AB}$.

The next four proposition are "similarity theorems" in analogy to the "congruence theorems". Recall our definition of similar triangles.

7.8. Proposition 4 ($\sim$aa) *Let* $\Delta\, ABC$ *and* $\Delta\, A'B'C'$ *be triangles such that* $\angle A \equiv \angle A'$ *and* $\angle B \equiv \angle B'$*. Then* $\Delta\, ABC \sim \Delta\, A'B'C'$*.*

7.9. Proposition 5 ($\sim$sss) *Let* $\Delta\, ABC$ *and* $\Delta\, A'B'C'$ *be triangles. If* $AB : A'B' = BC : B'C' = CA : C'A'$ *then* $\Delta\, ABC \sim \Delta\, A'B'C'$*.*

7.10. Proposition 6 ($\sim$sas) *Let* $\Delta\, ABC$ *and* $\Delta\, A'B'C'$ *be such that* $\angle A \equiv \angle A'$ *and* $AB : A'B' = AC : A'C'$ *then* $\Delta\, ABC \sim \Delta\, A'B'C'$*.*

The following proposition is not quite Euclid's Proposition 7, but a little stronger.

Proposition 7.11. ($\sim$asS) *Let $\Delta\, ABC$ and $\Delta\, A'B'C'$ be triangles. If $\angle A \equiv \angle A'$, $AB : A'B' = CB : C'B'$ and $CB > AB$, then $\Delta\, ABC \sim \Delta\, A'B'C'$.*

7.12. Proposition 8 *Let $\Delta\, ABC$ be a right triangle with $\angle C \equiv R$. Let the foot of the perpendicular from C to AB be H. Then $\Delta\, ABC \sim \Delta\, ACH \sim \Delta\, CBH$.*

7.13. Proposition 9 *By compass and ruler alone, a given line segment can be divided into a prescribed number of congruent line segments.*

Exercise 7.14. (Proposition 11) *Given line segments a, b, by compass and ruler alone, construct a line segment x such that $a : b = x : a$.*

An important result for Greek mathematics is the construction of the so-called forth proportional.

Exercise 7.15. (Proposition 12) *Let a, b, c be given line segments. By compass and ruler alone, find a line segment x such that $a : b = c : x$.*

7.16. Proposition 19 *If $\Delta ABC \sim \Delta A'B'C'$ then*

$$\text{area}\,\Delta\, ABC : \text{area}\,\Delta\, A'B'C' = (AB : A'B')^2.$$

Proposition 7.17. *Let $\mathcal{C}$ be a circle, and P a point in the* **exterior** *of the circle. Suppose that a line through P intersects the circle in the points A and B, and that another line through P intersects the circle in the points A' and B'. Then*

$$PA \cdot PB = PA' \cdot PB'.$$

Proposition 7.18. *Let $\mathcal{C}$ be a circle, and P a point in the* **interior** *of the circle. Suppose that a line through P intersects the circle in the points A and B, and that another line through P intersects the circle in the points A' and B'. Then*

$$PA \cdot PB = PA' \cdot PB'.$$

8. Book VII, VIII, IX

These books deal with **natural numbers** which are defined as a "multitude composed of units". Ratios of numbers are what are rational numbers for us. A good deal of important and standard number theory is contained in these books.

9. Book X

"Book X does not make easy reading" (B. van der Waerden, Science Awakening, p. 172.) It deals via geometry and geometric algebra with what we call today rational and irrational numbers. In fact, 13 different kinds of irrationalities are distinguished.

Definition 9.1. (1) *Those magnitudes are said to be* **commensurable** *which are measured by the same measure, and those* **incommensurable** *which cannot have any common measure.*

(2) *Straight lines are* **commensurable in square** *when the squares on them are measured by the same area, otherwise they are* **incommensurable in square***.*

(3) *Line segments are* **rational** *if they are commensurate with a fixed line segment (or unit), otherwise* **irrational***.*

The book contains 115 propositions none of which is recognizable at first sight.

There is general agreement that the difficulty and the limitations of geometric algebra contributed to the decay of Greek mathematics (Van der Waerden, Science Awakening, p.265.) Author like Archimedes and Apollonius were too difficult to read. However, Van der Waerden disputes that it was a lack of understanding of irrationality which drove the Greek mathematicians into the dead-end street of geometric algebra. Rather it was the discovery of irrationality, e.g. the diagonal of a square is incommensurable with the side of the square, and a strict, logical concept of number which was the root cause.

10. Books XI, XII, and XIII

Book XI deals with solid geometry and theorems on volumes, in geometric language, of course. Book XII uses the **method of exhaustion** to discuss the area of curved figures, e.g. the circle. Finally, Book XIII contains a discussion of the five Platonic solids (regular polyhedra).