René Descartes and Analytic Geometry

What is Analytic Geometry?

- **Ingredients:** (Euclid's) geometry, the real numbers $\mathbb{R}$
- **The tool:** A coordinate system
- **The result:** Geometry can be treated and transmitted algebraically and numerically.

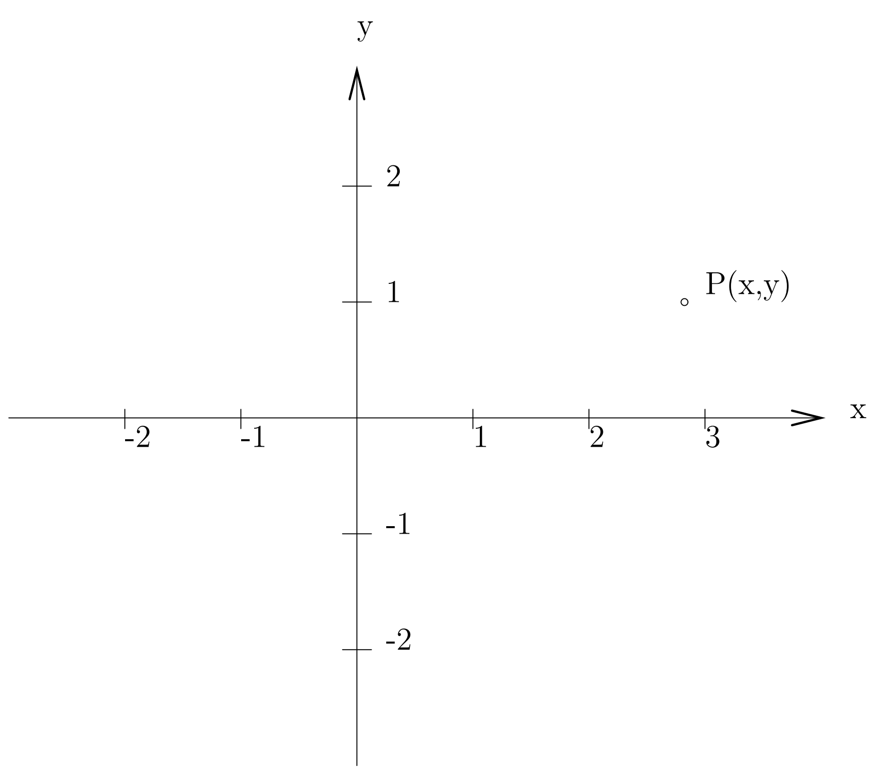


FIGURE 1. Cartesian Coordinates

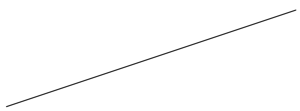
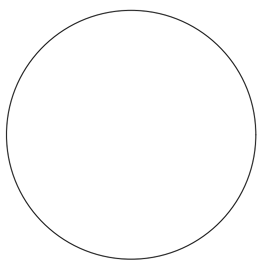
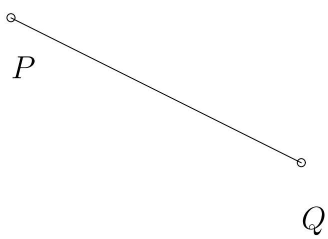
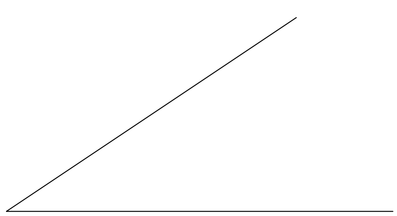
Geometry	**Analytic Geometry**	**Name**
	(x, y)	point
	$Ax + By + C = 0$	straight line
	$(x - h)^2 + (y - k)^2 = r^2$	circle, center (h, k), radius r
	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	distance P to Q
	$m = \frac{y_2 - y_1}{x_2 - x_1}$	slope, inclination versus x-axis

FIGURE 2. Translation table

- Many problems of geometry can be solved by working with lengths of line segments.
- Call the unknown line segments $x, y, z, \ldots$ and the given line segments $a, b, c, \ldots$, find relations between them by computing the same quantity in two ways. Solve the equations.

In what sense does $Ax + By + C = 0$ describe a straight line?

Example. A point (a, b) is on the straight line ℓ given by $3x - 5y + 3 = 0$ if and only if $3a - 5b + 3 = 0$, i.e.,

$$\ell = \{(x, y) \mid 3x - 5y + 3 = 0\}$$

Example. A point (a, b) is on the circle $\mathcal{C}$ given by $(x-5)^2 + (y+3)^2 = 25$ if and only if $(a-5)^2 + (b+3)^2 = 25$, i.e.,

$$\mathcal{C} = \{(x, y) \mid (x-5)^2 + (y+3)^2 = 25\}$$

René Descartes (1596-1650)

- Lesser nobility, but inheritence from mother rendered Descartes financially independent.
- Educated at a Jesuit college entering at the age of eight years, leaving at age 16.
- Being in poor health he was granted permission to remain in bed until 11 o'clock in the morning, a custom he maintained until the year of his death.
- Law degree from Poitiers in 1616.
- Enlisted as gentlemen soldier. After two years in Holland he travelled through all of Europe. In a hot room revelation of "the foundation of a marvellous science" and calling to pursue it.
- Settled down in Holland.
- First major treatise on physics, Le Monde, ou Traité de la Lumière. News that Galileo Galilei was condemned to house arrest. Decided not to risk publication. World starts with original vortices and develops according to laws of mechanics.

- Wrote **Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences.** Three appendices were La Dioptrique, Les Météores, and La Géométrie. He popularizes the analytic approach to geometry.
- In 1649 Queen Christina of Sweden persuaded Descartes to come to Stockholm as her personal tutor. The Queen worked early in the morning. After only a few months in the cold northern climate, Descartes died of pneumonia.
- Descartes constitutes "the turning point between medieval and modern mathematics".

Discours de la Méthode (Discussion of Method)

- Clear influence of Euclid: looking for basic principles, logical deduction.
- Doubt! Disregard acquired knowledge, start from the beginning.
- “I think therefore I am”.
- God exists because there must be perfection.
- Search for basic principles and derive (explain) phenomena.
- Verify the truth of the basic principles by experimentally checking the consequences.

From Discours de la Méthode

I thought the following four [rules] would be enough, provided that I made a firm and constant resolution not to fail even once in the observance of them.

- **Never accept anything as true if there is not evident knowledge of its being so, that is, carefully avoid precipitancy and prejudice**;
- **Divide each problem into as many parts as is feasible, and as is requisite for its better solution**;
- **Direct your thoughts in an orderly way; beginning with the simplest objects, those most apt to be known, and ascending little by little, in steps as it were, to the knowledge of the most complex**;
- **to make throughout such complete enumerations and such general surveys that might make sure of leaving nothing out**.

These long chains of perfectly simple and easy reasonings by means of which geometers are accustomed to carry out their most difficult demonstrations had led me to fancy that everything that can fall under human knowledge forms a similar sequence.

It is not enough to have a good mind. The main thing is to use it well.

If you would be a real seeker after truth, you must at least once in your life doubt, as far as possible, all things.

Optics

Descartes explains refraction, reflection, function of the eye, lenses, how to construct lenses.

Meterology

Descartes explains the rainbow, develops a theory of the weather.

“Despite its many faults, the subject of meteorology was set on course after publication of Les Météores particularly through the work of Boyle, Hooke and Halley.”

On geometry

- Replace the geometric algebra of the Greeks by numerical algebra. Think of line segments as numbers. Then, e.g., $a^2 + b$ makes sense.

See pages 177 and 178.

- Pappus: Appolonius says that the problem of the locus related to three or four lines was not entirely solved by Euclid, and that neither he himself or anyone else was able to solve it completely.

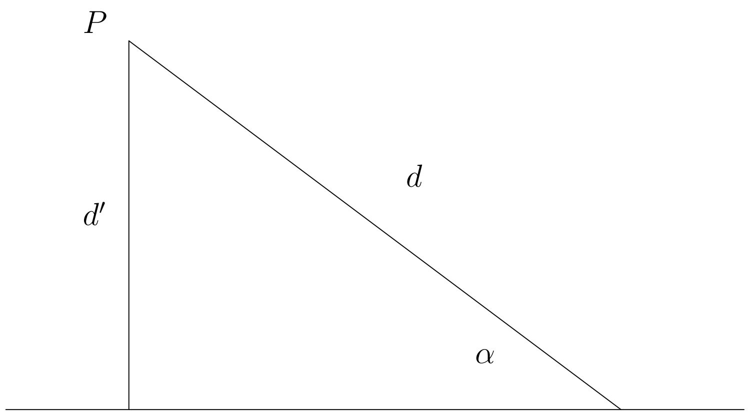
  Descartes solved this 1850 year old problem.

  See pages 183, 184

- **The Problem:** Given straight lines $a_1, a_2, \ldots$ and angles $\alpha_1, \alpha_2, \ldots$. Given a point P, let $d_1, d_2, \ldots$ be the lengths of the line segments PQ_i where Q_i is on a_i and PQ_i makes the angle α_i with a_i. Find the locus of all points P such that certain products of the d_i have a given ratio with the product of the rest of the d_i.

Observations.

(1) Without loss of generality $d_i = \mathrm{dist}(P, a_i)$ because $d' = d\sin(\alpha)$.



(1) In a Cartesian coordinate system the distance of a point $P = P(x, y)$ from a line $a:\ Ax + By + C = 0$ is given by the formula

$$\mathrm{dist}(P, a) = \frac{|Ax+By+C|}{\sqrt{A^2+B^2}}.$$

(1) Solutions:

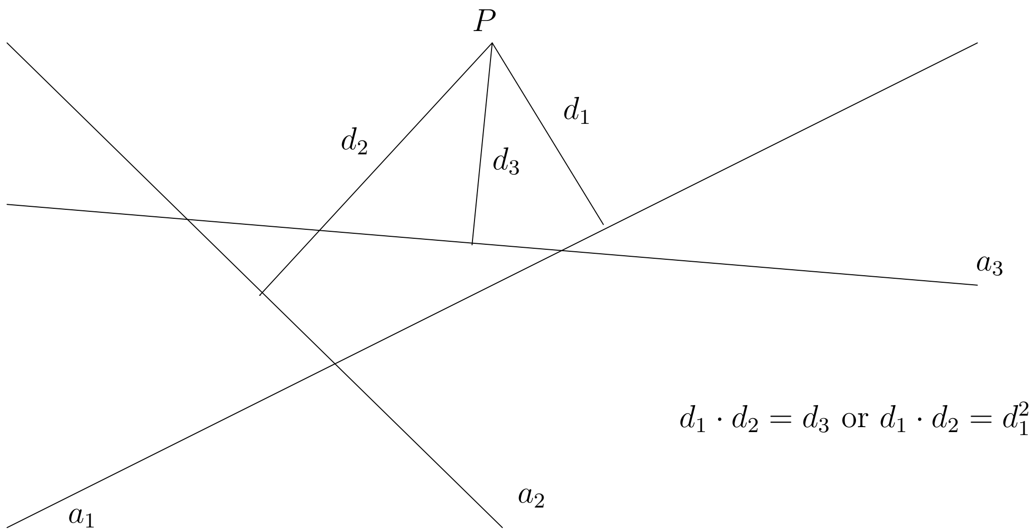


FIGURE 3. A problem of Appolonius and Pappus

(a) Two lines:

$$|A_1x + B_1y + C_1| = f|A_2x + B_2y + C_2|, \quad f \in \mathbb{R}.$$

The locus is a pair of straight lines.

(b) Three lines:

$$|A_1x+B_1y+C_1||A_2x+B_2y+C_2| = f|A_3x+B_3y+C_3|.$$

The locus is a conic section.

(c) Four lines:

$$\begin{aligned} &|A_1x + B_1y + C_1||A_2x + B_2y + C_2| \\ = \ & f|A_3x + B_3y + C_3||A_4x + B_4y + C_4|. \end{aligned}$$

The locus is a conic section.

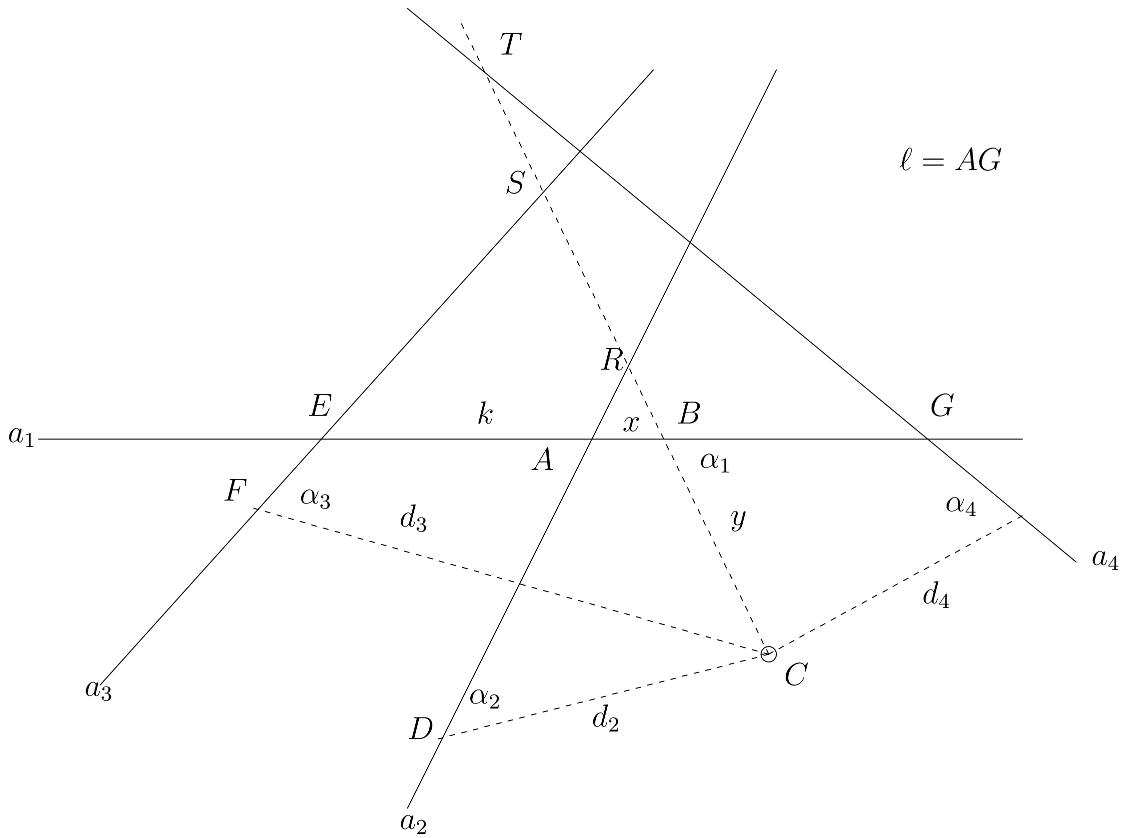
(d) Five lines:

$$|A_1x + B_1y + C_1||A_2x + B_2y + C_2| =$$
$$f|A_3x + B_3y + C_3||A_4x + B_4y + C_4|$$
$$\cdot\ |A_5x + B_5y + C_5|, \quad f \in \mathbb{R}.$$

Here without loss of generality $B_5 = 0$, then, given x we have a quadratic equation in y that can be solved by compass and ruler alone.

(e) Six or more lines: The equations can have high degrees, out of reach of ancient geometers.

(2) Descartes' solution.



Set $x = AB$ and $y = CB$. Angles in the configuration are largely known.

Using essentially the Law of Sines, he computes that

$$\begin{aligned} CB &= y \\ CD &= r_2 y + r_1 x \\ CF &= r_4 y + r_3 r_4 k + r_3 r_4 x \\ CH &= r_6 y + r_5 r_6 \ell + r_5 r_6 x, \end{aligned}$$

where $k = AB$, $\ell = AG$, and the r_i are fractions of sines of known angles. His conclusions: Same as above.

(3) Descartes essentially created relations $F(x, y) = 0$ and functions $y = f(x)$.

General Curves

- Curves can be described algebraically with respect to lines of reference.

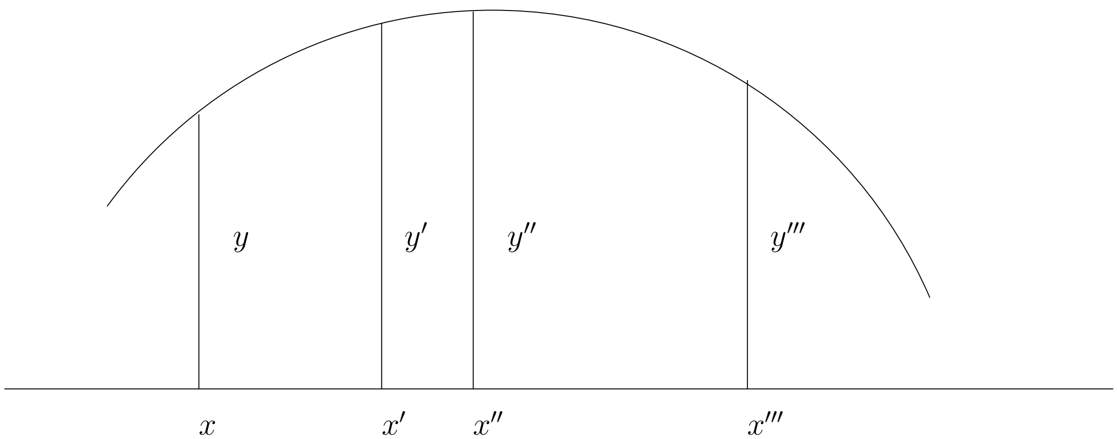
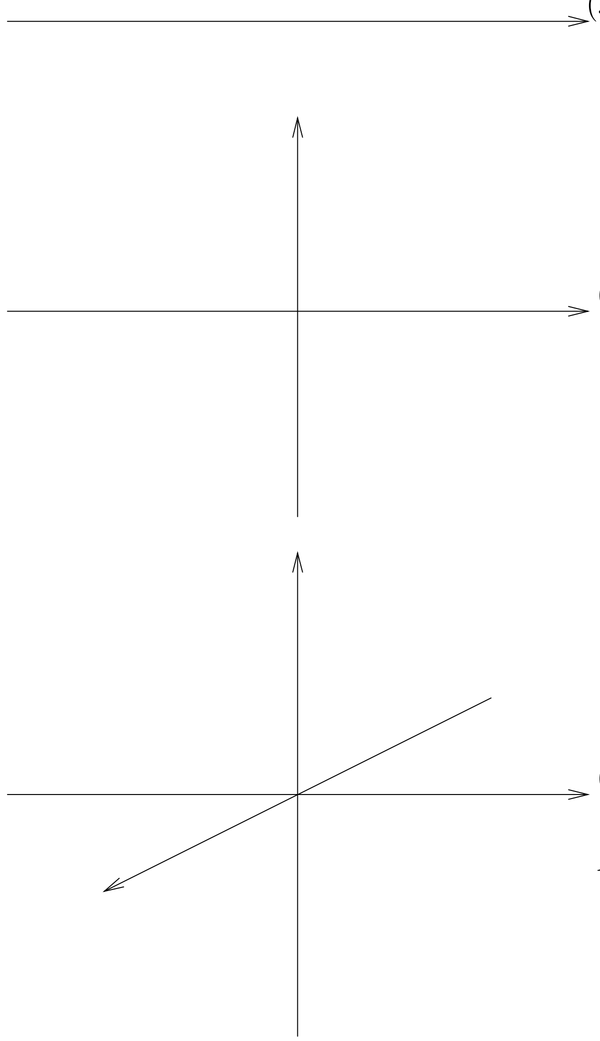


FIGURE 4. General curves according to Descartes

- Displays mechanisms that result in curves whose algebraic descriptions are more and more complicated.
- Essentially explains the graph of a function f:
  graph of $f = \{(x, f(x)) \mid x \in \mathbb{R}\}$.
- Graphs of a relation F:
  graph of $F = \{(x, y) \mid F(x, y) = 0\}$
- Develops a method for finding the tangent to a curve.

The question of dimension



(x), $\quad Ax = B$, $\quad Ax^2 + Bx + C = 0$

(x, y), $\quad Ax + By + C = 0$, $\quad Ax^2 + By^2 + C = 0$

(x, y, z),

$$Ax + By + Cz + D = 0, \quad Ax^2 + By^2 + Cz^2 + D = 0$$

Figure 5

Visualization fails but algebraically we can go on

- (x, y),
  $Ax + By + C = 0$,
  $Ax^2 + By^2 + C = 0$
- (x, y, z),
  $Ax + By + Cz + D = 0$,
  $Ax^2 + By^2 + Cz^2 + D = 0$
- (x, y, z, t),
  $Ax + By + Cz + Dt + E = 0$,
  $Ax^2 + By^2 + Cz^2 + Dt^2 + E = 0$
- (x, y, z, t, u),
  $Ax + By + Cz + Dt + Eu + F = 0$,
  $Ax^2 + By^2 + Cz^2 + Dt^2 + Eu^2 + F = 0$
- $(x_1, x_2, \ldots, x_n)$,
  $A_1x_1 + A_2x_2 + \cdots + A_nx_n + B = 0$,
  $A_1x_1^2 + A_2x_2^2 + \cdots + A_nx_n^2 + B = 0$