AN AXIOM SYSTEM FOR THE EUCLIDEAN PLANE

SPRING 2008

1. History

In spite of its erudition and acumen, by today's standards, there are many defects in Euclid's axiomatic development of geometry. Yet this was not a major preoccupation until the 20th century. Instead, it was the Fifth Postulate on parallels which captured the imagination of the mathematical public. This axiom appeared too complicated by comparison with the others and not an "obvious truth". Euclid himself nourished this impression by putting off its use. Mathematicians tried to prove the Parallel Axiom for hundreds of years. There were claims of success but every time the proofs collapsed under close scrutiny.

Finally, it dawned on some people,

- Karl–Friedrich Gauss (1777–1855),
- Janos Bolyai (1802–1860), and
- Nikolai Ivanovich Lobachevski (1792–1856),

to be precise, that the Fifth Postulate could not be proved, that one could postulate the existence of more than one parallel to a line through a point, and "create a strange new universe" (Letter of Janos Bolyai to his father Wolfgang Bolyai.) This was the discovery of Non–Euclidean geometry and a breakthrough shattering many ingrained philosophical and mathematical ideas.

2. Axiomatic Mathematics

In ancient Greece philosophy and mathematics were closely interwoven, and the Greek intellectuals questioned one another and insisted on certainty and rigor. The novel idea carried out by Euclid with astonishing elegance and completeness was to begin with simple unquestioned truths and derive logically other truths. Whoever would accept the original truths or postulates, would have to accept equally their logical consequences. It was clear that any such theory would have to start somewhere with postulates; otherwise an unending chain of questions and answers would result and nothing would ever be sure. Similarly, there must be simple basic objects in terms of which other objects can be defined.

In the Elements these are prominently the points and lines, but Euclid's definitions of point and line suggest what to think of, what to visualize, but they are not usable definitions since they involve other concepts that are no less mysterious than the objects they purport to explain. In fact, there is no explicit argument anywhere in the Elements that uses the definitions of point and line. In diagrams, of course, the points and lines are drawn as suggested by the definitions. If a definition is not used, then it must be superfluous and a theory can be developed without it.

A modern axiomatic theory begins with certain ***undefined terms*** and a number of ***axioms*** which are statements involving these undefined terms and possibly terms defined by means of the initial undefined terms. The undefined terms are not known, all that is assumed are the axioms. The axioms are not considered objective truths, they may or may not be true when the undefined concepts are ***interpreted*** in some way, but *if* they are true in some interpretation or ***model***, then all their logical consequences will also be true. In a modern axiom system for geometry an interpretation of point could be the visual picture which Euclid tries to describe in his definition, but it could also be a pair (x, y) of real numbers. What matters is whether the axioms become true or not for these interpretations. The difficulty of the axiomatic method is its ***abstractness***, the fact that we are not talking about concrete objects which we can visualize and become familiar with. The advantage of the method rests in the fact that we obtain a body of truths for any interpretation of the undefined terms for which the axioms become true.

3. Axioms for Geometry

A completely satisfactory system of axioms for Euclidean geometry was established by David Hilbert (1862–1943) in his booklet "Foundations of Geometry" which appeared in 1898. The following system is a variant of Hilbert's axioms.

Definition 3.1. *Let $\mathbb{E}$ be a set whose elements are called* ***points*** *and denoted by $A, B, C, \dots$ Three points A, B, C may or may not satisfy a relation ABC, read as B* ***is between*** *A* ***and*** *C. This relationship among points is called* ***order relation****.*

For two distinct points A, B the set

$$\overleftrightarrow{AB} =$$

$$\{X : X = A \text{ or } X = B \text{ or } XAB \text{ or } AXB \text{ or } ABX\}$$

is called the ***straight line through*** *A and B. The set*

$$AB = \{X : AXB\}$$

is called the ***line segment*** *with* ***endpoints*** *A, B. Any point X with AXB is also called an* ***interior point*** *of AB, the points of $\overleftrightarrow{AB}$ not belonging to $AB \cup \{A, B\}$ are* ***exterior*** *to AB. Straight lines are denoted by $a, b, c, \dots$*

Given the points A, B, C, there are six possible relations which may be true or not: ABC, ACB, BAC, BCA, CAB, CBA. We postulate the following.

3.2. *Axioms of Order*

I1. $\mathbb{E}$ *contains three points* A_0, B_0, C_0 *which are not related by order.*

I2. *For any two (distinct) points* $A, B \in \mathbb{E}$*, there is* $C \in \mathbb{E}$ *such that* ABC*.*

I3. *If* ABC *then* CBA*.*

I4. *For three points* A, B, C*, at most one of the relations* BAC*,* CBA*, and* ACB *is true.*

I5. *If the points* A, B, C *are related by "between" and the points* A, B, D *are related by "between", then the points* B, C, D *are related by "between".*

I6. (Pasch) *Let* $A, B, C \in \mathbb{E}$ *be different points not related by "between". If* ADB *and* E *is a point of* $\overleftrightarrow{AC}$ *which lies outside* AC*, then* $\overleftrightarrow{DE} \cap BC \neq \emptyset$*.*

Definition 3.3. *Let O, A, B be three distinct points of a line a. We say that A, B are **on different sides of** O if AOB, otherwise we say that A, B are **on the same side of** O. The **half–ray** with **initial point** O **through** A is the set*

$$\overrightarrow{OA} = \{A\} \cup \{X : OXA\} \cup \{X : OAX\}.$$

Definition 3.4. *The **triangle** $\triangle ABC$ is the set of the three points A, B, C provided these are not related by "between". The points A, B, C are the **vertices** of the triangle, the line segments AB, BC, CA are the **sides** of the triangle. The side AB is the side **opposite** C, etc.*

Definition 3.5. *Let A, B, C form a triangle. The **plane** $\overleftrightarrow{ABC}$ determined by A, B, C is the union of all lines $\overleftrightarrow{AD}$ with $D \in BC \cup \{B, C\}$, $\overleftrightarrow{BD}$ with $D \in CA \cup \{C, A\}$, and $\overleftrightarrow{CD}$ with $D \in AB \cup \{A, B\}$.*

Proposition 3.6.

1. *Every line segment contains infinitely many points.*
2. *There are infinitely many lines passing through every point.*
3. *On every line there are infinitely many points on both sides of any point of the line.*

3.7. *Axiom of Dimension*

II. $\mathbb{E} = \overleftrightarrow{A_0B_0}C_0$.

Definition 3.8. *Let a be a straight line and A, B two points not on a. We say that A, B* ***lie on the same side of*** *a if $AB \cap a = \emptyset$. We say that A, B* ***lie on different sides of*** *a if $AB \cap a \neq \emptyset$.*

Theorem 3.9. *Let a be a straight line. Then there are unique subsets α_1, α_2 of $\mathbb{E}$, called* ***the half–planes of*** *a, such that*

(1) *All points of α_1 are on the same side of a and all points of α_2 are on the same side of a. If $A \in \alpha_1$ and $B \in \alpha_2$, then A and B are on different sides of a.*
(2) $\alpha_1 \cap \alpha_2 = \emptyset$.
(3) $\mathbb{E} = \alpha_1 \cup a \cup \alpha_2$.

With these basic results we can define the "interior of a triangle".

Definition 3.10. *Let $\Delta\, ABC$ be given. The* ***interior*** *of $\Delta\, ABC$ is the intersection of the half–plane of $\overleftrightarrow{AB}$ which contains C and the half–plane of $\overleftrightarrow{BC}$ which contains A and the half–plane of $\overleftrightarrow{CA}$ that contains B.*

We now introduce angles.

Definition 3.11. *Let O be a point and h and k two half–rays emanating from O. The set $\angle(h,k) = \{h,k\} = \angle(k,h)$ is the* ***angle with legs*** *h, k and* ***vertex*** *O. If $A \in h$ and $B \in k$, then we set $\angle AOB = \angle BOA = \angle(h,k)$. Let α_k be the half–plane of $\overleftrightarrow{k}$ containing h and α_h the half–plane of $\overleftrightarrow{h}$ containing k. Then $\alpha_k \cap \alpha_h$ is the* ***interior*** *of $\angle(h,k)$.*

Theorem 3.12. (Crossbar Theorem) *Let $\angle AOB$ be given and let C be a point in the interior of the angle. Then $\overleftrightarrow{OC}$ intersects AB.*

In addition to the undefined relation "between" for triples of points we have an undefined relation called "congruence" between two line segments, and an undefined relation, also called "congruence" between two angles.

Definition 3.13. *Two line segments AB and CD may or may not be* ***congruent****. If they are, we write $AB \equiv CD$. Two angles $\angle(h,k)$ and $\angle(h',k')$ may or may not be* ***congruent****. If they are, we write $\angle(h,k) \equiv \angle(h',k')$.*

3.14. *Axioms of Congruence (1)*

III1. *Let AB be any line segment, a any straight line and A' be any point on a. Then there exist points B' and B'' on a such that $B'A'B''$ and $AB \equiv B'A'$ and $AB \equiv A'B''$.*

III2. *If $A'B' \equiv AB$ and $A''B'' \equiv AB$, then $A'B' \equiv A''B''$.*

III3. *If $B \in AC$, $B' \in A'C'$, $AB \equiv A'B'$, and $BC \equiv B'C'$, then $AC \equiv A'C'$.*

3.15. *Axioms of Congruence (2)*

III4. *Let $\angle(h,k)$ be an angle, a a line, α one of the half–planes of a, and h' a half–ray of a. Then there exists precisely one half–ray k' such that $\angle(h,k) \equiv \angle(h',k')$ and k' belongs to α.*

III5. $\angle(h,k) \equiv \angle(h,k)$.

III6. *Given $\Delta\, ABC$ and $\Delta\, A'B'C'$, if $AB \equiv A'B'$, $AC \equiv A'C'$, and $\angle BAC \equiv \angle B'A'C'$, then $\angle ABC \equiv \angle A'B'C'$.*

The following proposition says that congruence of line segments is an equivalence relation.

Proposition 3.16. *For any line segments AB, $A'B'$, $A''B''$ the following hold.*

(1) $AB \equiv AB$.

(2) *If* $AB \equiv A'B'$, *then* $A'B' \equiv AB$.

(3) *If* $AB \equiv A'B'$ *and* $A'B' \equiv A''B''$, *then* $AB \equiv A''B''$.

Now congruent triangles can be defined and the congruence theorems can be proved pretty much the way we have done it following Euclid. However, we are now in a position to argue the relations which we had to read off pictures previously.

We depart from Euclid in introducing measurement before we talk about area. This requires the following axioms.

3.17. *Axioms of Continuity*

IV1. (Axiom of Archimedes) *For any line a, any line segment AB on a, and any point A_1 between A and B, there exist $n-1$ points $A_2, A_3, \ldots, A_n \in a$ such that*

$$AA_1 \equiv A_1A_2 \equiv A_2A_3 \equiv A_3A_4 \equiv \cdots \equiv A_{n-1}A_n,$$

AA_1A_2, $A_iA_{i+1}A_{i+2}$ and $A_{n-1}BA_n$ or $B = A_n$.

IV2. (Cantor's Axiom) *For any line segments A_iB_i, $i = 1, 2, \ldots$, such that both A_{i+1} and B_{i+1} are between A_i and B_i for all i, there is a point contained in all line segments A_iB_i.*

Definition 3.18. *Two non–intersecting lines are called* ***parallel****. Every line is also considered parallel to itself.*

3.19. *Axiom on Parallels*
V. (Euclid) *For any line a and any point $A \notin a$ there is at most one parallel to a through A.*

While Euclid proved the similarity theorems and the Pythagorean Theorem using the area concept, we make repeated use of the congruence and the axioms of Archimedes and Cantor in order to prove the theorems on an angle cut by parallel lines. These theorems are equivalent to the similarity theorems and the Pythagorean Theorem follows from these.

The area concept is introduced axiomatically and the usual area formulas are derived easily.

4. Consistency, Completeness, Independence of Axioms

An axiom system for plane Euclidean geometry intends to capture Euclidean geometry exactly. Every theorem of Euclidean geometry ideally is derivable from the axiom system.

Saying this presupposes that we know what Euclidean geometry really is. This is, of course, a problem. However, we do live in a world and we are in some way or other involved with geometry, we have experiences of space, and science and mathematics have successfully used the geometry which Euclid suggests and describes. These geometric ideas are so overpowering that the eminent philosopher Immanuel Kant (1724–1804) taught that they are built into our minds and that the world is Euclidean since this is the only way we can perceive it.

In setting up an axiom system we are guided by our preconceived ideas and we certainly want that the axioms are "obvious truths". However, having achieved this, the axiom system and its consequences assume a life of their own. From a logical point of view there is one main problem: The axioms must be ***consistent***, i.e., there may not be any hidden contradiction that could show up somewhere along the line when some statement is derived and also its negation.

How can one be sure that an axiom system is consistent? One way of doing it, is to come up with a "real life example" for which the axioms are all true. In the case of Euclidean geometry, the real life example can be ***analytic geometry*** where a point is interpreted as a pair (x, y) of real numbers, and all other concepts are interpreted as is usually done. This ***model*** of Euclidean geometry satisfies all axioms. We can draw the following conclusion.

Theorem 4.1. *If the real number system is consistent, then so is Euclidean geometry.*

Is the real number system consistent? All experience for thousands of years points to an affirmative answer, but it was shown by Kurt Gödel in 1931 that it is not possible to prove the consistency of something like the real number system from itself. This means that we have to live with a degree of uncertainty even in the certain world of mathematics.

It should be noted that there is a considerable amount of choice in axiomatizing a theory. If we have two axiom systems $\mathcal{S}_1$ and $\mathcal{S}_2$ which deal with the same undefined and defined terms, then the systems are logically equivalent if every axiom of $\mathcal{S}_1$ can be proved on the basis of the axioms of $\mathcal{S}_2$ and conversely. Given such freedom, what considerations guide the choice of axioms?

To begin with unnecessary axioms should be avoided, i.e., no axiom should be provable from the other axioms. An axiom system which satisfies this condition is called ***independent***. With this terminology we can say that the century long search for a proof of the Parallel Axiom was to show its dependence. At the end it turned out to be independent of the other axioms.

Another consideration is to make the axioms as simple and basic as possible. From a puristic standpoint, the axioms are supposed to be as weak as possible, but this is a minor consideration when one is interested mainly in efficiently developing a rigorous theory. In the above axiom system, it would have been enough to postulate IV2. and V. for a single particular line rather than for all lines. The advantage of weak axioms is that it is easier to establish a model for the theory, the disadvantage is that it is harder to develop the theory.