Georg Cantor and Set Theory

1. Life

- Father, Georg Waldemar Cantor, born in Denmark, successful merchant, and stock broker in St Petersburg. Mother, Maria Anna Böhm, was Russian.
- In 1856, because of father's poor health, family moved to Germany.
- Georg graduated from high school in 1860 with an outstanding report, which mentioned in particular his exceptional skills in mathematics, in particular trigonometry.
- "Höhere Gewerbeschule" in Darmstadt from 1860, Polytechnic of Zürich in 1862. Cantor's father wanted Cantor to become:-

  ... a shining star in the engineering firmament.
- 1862: Cantor got his father's permission to study mathematics.
- Father died. 1863 Cantor moved to the University of Berlin where he attended lectures by Weierstrass, Kummer and Kronecker.
- Dissertation on number theory in 1867.
- Teacher in a girls' school.
- Professor at Halle in 1872.
- Friendship with Richard Dedekind.

- 1874: marriage with Vally Guttmann, a friend of his sister. Honeymoon in Interlaken in Switzerland where Cantor spent much time in mathematical discussions with Dedekind.
- Starting in 1877 papers in set theory. "Grundlagen einer allgemeinen Mannigfaltigkeitslehre". Theory of sets not finding the acceptance hoped for.
- May 1884 Cantor had the first recorded attack of depression. He recovered after a few weeks but now seemed less confident.
- Turned toward philosophy and tried to show that Francis Bacon wrote the Shakespeare plays.
- International Congress of Mathematicians 1897. Hurwitz openly expressed his great admiration of Cantor and proclaimed him as one by whom the theory of functions has been enriched. Jacques Hadamard expressed his opinion that the notions of the theory of sets were known and indispensable instruments.
- Paradoxes of set theory appear.
- Retirement 1913, frequently ill, died of a heart attack.
- Hilbert: ...the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity.

2. Set Theory

What do we count?

the birds of a flock
the fish in a school
the students of the student body
the students of the class of 2007
the bisons of a herd
the chairs in Bil 152
the students in Math 100
the members of a tribe
the members of a congregation
the residents of Hawaii
the soldiers in a regiment
a band of Indians
the members of the Mafia
the geese in a gaggle
the members of the middle class

What are sets?

$$\left.\begin{array}{l}\text{flock of birds}\\ \text{school of fish}\\ \text{student body}\\ \text{class of 2007}\\ \text{herd of bison}\\ \text{herd of sheep}\\ \text{tribe}\\ \text{congregation}\\ \text{people}\\ \text{regiment of soldiers}\\ \text{band of Indians}\\ \text{Mafia}\\ \text{gaggle of geese}\\ \text{the middle class}\end{array}\right\} \overset{\text{abstraction}}{\Rightarrow} \boldsymbol{SET}.$$

Definition 2.1. *(Cantor) By a **set** we are to understand any collection into a whole M of definite and separate objects m of our intuition or our thought. Notation: $M = \{m\}$.*

Examples.

(1) $\{0, 1, 2, 3\}$.
(2) $\{\}$, ***the empty set.***
(3) $\{(x, y) \mid 3x - 5y + 3 = 0\}$
(4) $\{f \mid f \text{ is a factor of } 240\}$

Definition 2.2. *Two sets are equal if and only if they contain exactly the same elements.*

Example. $\{2,4\} = \{2, 4/2, 2 \cdot 2\} = \{4,2\} = \{6/3, 2^2\}$

Note. $a = b$ means that a and b are names of the same object.

Names contain information.

Definition 2.3. *A **pair** (a,b) of objects a,b is defined to be $(a,b) = \{\{a\},\{a,b\}\}$.*

Lemma 2.4. $(a,b) = (a',b')$ *if and only if* $a = a'$ *and* $b = b'$.

Set builder scheme.

$$S = \{x \in U \mid P(x)\}$$

"S is the set consisting of all elements x in the universe U such that the condition $P(x)$ is satisfied."

- $\{n \in \mathbb{Z} \mid 0 \leq n \leq 5\} = \{0, 1, 2, 3, 4, 5\}$.
- $\{f \in \mathbb{N} \mid f$ is a factor of $60 = 2^2 \cdot 3 \cdot 5\} = \{1, 5, 3, 15, 2, 10, 6, 30, 4, 20, 12, 60\}$.
- $\{(x, y) \mid \frac{x^2}{16} + \frac{y^2}{9} = 1\}$ is an ellipse in a Cartesian coordinate system.

What is counting?

- We know certain sets very well, e.g., sets of fingers. We understand “more fingers”, “fewer fingers”.
- We can compare arbitrary sets with sets of fingers by ***matching*** them with sets of fingers.
- We can compare arbitrary sets with other arbitrary sets and arrive at the concepts “same size”, “more elements”, “fewer elements”. This is the “first abstraction” according to Cantor.
- We invent “numbers” to go with our “model sets”, e.g. a hand has ***five*** fingers. This is the “second abstraction” according to Cantor. There is nothing sacred or natural about the names and symbols used for these counting numbers.
- Numerals and numeration schemes were developed to measure the size or count of any (finite) set.

Definition 2.5. *A set A is **finite** if it can be matched with an **initial segment** of $\mathbb{N}$, say $\{1, 2, \ldots, n\}$, and if so, we say that the **count** of A is n. Notation: $|A| = n$.*

The arithmetic of counting numbers

- Why is $3 + 4 = 7$? We take a set of three elements, say $\{a, b, c\}$ and a ***disjoint*** set with four elements, say, $\{d, e, f, g\}$, and combine them into a new set $\{a, b, c, d, e, f, g\}$ and count to get 7. This is why $3 + 4 = 7$.
- Why is $2 \cdot 3 = 6$? We take a set of two elements, say $\{a, b\}$ and a set with three elements, say, $\{a, b, c\}$, and form all pairs

$$\left\{ \begin{array}{ccc} (a, a) & (a, b) & (a, c) \\ (b, a) & (b, b) & (b, c) \end{array} \right\}$$

  and count to get 6. This is why $2 \cdot 3 = 6$.

Definition 2.6. *Let $m, n \in \mathbb{N}$. Choose disjoint sets A and B such that $|A| = m$ and $|B| = n$. Then, by definition, $m+n = |A \cup B|$ and $m \times n = |A \times B|$ where $A \times B = \{(a, b) \mid a \in A, b \in B\}$, the* **Cartesian Product** *of A and B.*

Transfinite cardinals and cardinal arithmetic

Carl Friedrich Gauss in a letter: "I protest against an infinite quantity as an actual entity; this is never allowed in mathematics. The infinite is only a manner of speaking."

Definition 2.7. *Two sets M and N are said to contain the* ***same number of elements*** *if the elements of M and N can be matched one-to-one. The sets are then* ***equinumerous****.*

Example. $\mathbb{N} = \{1, 2, 3, \ldots\}$ and its subset of squares $\{1^2, 2^2, 3^2, \ldots\}$ are equinumerous. The matching is given by $n \longrightarrow n^2$, i.e., $n \in \mathbb{N}$ gets the partner n^2, and every square s gets the partner $\sqrt{s}$. Observed by Galileo Galilei.

Example. Any two closed real line segments $[a, b]$ and $[a', b']$ with $a < b$ and $a' < b'$ are equipotent.

Example. Any line segement $[a, b]$ with $a < b$ is equipotent with a square or a cube.

Definition 2.8. *A set M is* ***countable*** *if there is a one-to-one matching of the elements of M with the natural numbers in $\mathbb{N}$. In this case the count is $\aleph_0$ or $\mathfrak{a}$.*

Definition 2.9. (Cantor) *Every set M has a definite "power", which we also call its "cardinal number". The "cardinal" of M is the general concept which, by means of our active faculty of thought arises from M when we make abstraction from the nature of its various elements and of the order in which they are given. We denote the result of this double act of abstraction by $|M|$.*

- The rational numbers are countable.
- The algebraic numbers, i.e. the numbers which are roots of polynomial equations with integer coefficients, are countable.
- The real numbers $\mathbb{R}$ are not countable. Set $\mathfrak{c} = |\mathbb{R}|$.
- The idea of a one-one matching (correspondence) appears implicitly for the first time.

Definition 2.10. *An* ***algebraic number*** *is a number that is the root of a polynomial with integer coefficients. A* ***transcendental number*** *is a number that is not a root of any polynomial equation with integer coefficients.*

Liouville established in 1851 that transcendental numbers exist. In 1874 Charles Hermite proved e to be transcendental, Ferdinand Lindemann proved that π is transcendental in 1882.

Remark. *A set is matched one-to-one with* $\mathbb{N}$ *if and only if the elements can be listed in a sequence.*

(1) Counting $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$. It suffices to list the fractions $\frac{a}{b}$ with $a, b > 0$. First list those fractions $\frac{a}{b}$ with $a + b = 1$, then those with $a + b = 2$, then $a + b = 3$, etc. List the fractions $\frac{a}{b}$ with $a + b = s$ according to the size of the numberator. To wit:

$$\begin{array}{llllll}
\frac{1}{1} \\[1em]
\frac{1}{2} & \frac{2}{1} \\[1em]
\frac{1}{3} & \frac{2}{2} & \frac{3}{1} \\[1em]
\frac{1}{4} & \frac{2}{3} & \frac{3}{2} & \frac{4}{1} \\[1em]
\frac{1}{5} & \frac{2}{4} & \frac{3}{3} & \frac{4}{2} & \frac{5}{1} \\[1em]
\frac{1}{6} & \frac{5}{4} & \frac{3}{4} & \frac{4}{3} & \frac{5}{2} & \frac{6}{1} \\[1em]
\vdots
\end{array}$$

(2) Polynomials $a_n x^n + \cdots + a_1 x + a_0$ linearly ordered using $N = n - 1 + |a_n| + \cdots + |a_1| + |a_0|$.

(3) $\mathbb{R}$ is NOT countable: famous diagonal argument.

(4) Every subset of $\mathbb{N}$ is either finite or countable.

Proof. Start with a listing of $\mathbb{N}$ and omit all elements not belonging to the subset.
$\aleph_0$ is the smallest infinite cardinal. □

(5) Addition and multiplication for arbitrary cardinals is defined as it was for finite cardinals, i.e., the ordinary counting numbers.
(6) Laws of cardinal arithmetic (associative, commutative, distributive, etc., hold but subtraction is tricky.).
(7) $\aleph_0$ is the smallest transfinite cardinal.
(8) $\aleph_0 + 1 = \aleph_0$. (Hilbert's hotel)
(9) $\aleph_0 + \aleph_0 = \aleph_0$.
(10) $\aleph_0 \aleph_0 = \aleph_0$.
(11) A set is finite if it is not equinumerous with any of its (proper) parts.
(12) A set is infinite if it is equinumerous with one of its (proper) parts.
(13)

Theorem 2.11. (Cantor) *The set $\mathcal{P}(X)$ of all subsets of a set X has a larger cardinality (number of elements) than the original set X.*

Proof. Suppose they have the same number of elements. Let $f : X \to \mathcal{P}(X)$ be a bijection between X and $\mathcal{P}(X)$.

(1) Let $D = \{x \in X : x \notin f(x)\}$.
(2) Since D is a subset of X and f is onto, $D = f(d)$ for some $d \in X$.
(3) Thus $d \in f(d)$ iff (by (2)) $d \in D$ iff (by (1)) $d \notin f(d)$.

This is a contradiction. $\square$

A problem

Let U be the set of all sets. Then, for any set X, $|U| \geq |X|$. But $|\mathcal{P}(X)| > |U|$, a contradiction.

Russel's Paradox

$$S := \{X \mid X \text{ is a set and NOT } X \in X\}.$$

The Continuum Hypothesis

- The set of cardinal numbers is ***well-ordered***, i.e. every non-void set of cardinal numbers contains a smallest element.
- Is $\mathfrak{c}$ the smallest cardinal greater than $\mathfrak{a}$?