MATH 207 ♢ History of Mathematics ♢ Spring 2009

Symbolic (formal, mathematical) Logic

- Logic is common sense.

Example.

(1) **Whenever** it is sunny, George is on the beach.
If it is sunny, **then** George is on the beach.
(2) It is sunny.
(3) Conclusion: George is on the beach.
(4) **Since** it is sunny, George is on the beach.
Because it is sunny, George is on the beach.
It is sunny. **Therefore** George is on the beach.
(5) It being sunny is **sufficient** for George being on the beach.
George being on the beach is **necessary** for it being sunny.
It being sunny **implies** that George is on the beach.

Item (1) and (5) logically mean exactly the same thing in spite of the different wording. Item (4) says more then the others: It says in addition that it is in fact sunny.

A mathematical example.

(1) **Whenever** a triangle has two congruent sides, the angles opposite the congruent sides are also congruent.
If a triangle has two congruent sides, **then** the angles opposite the congruent sides are also congruent.
(2) The triangle ΔABC has two congruent sides.
(3) Conclusion: the angles opposite the congruent sides are also congruent.
(4) **Since** The triangle ΔABC has two congruent sides, it also has two congruent angles.
Because the triangle ΔABC has two congruent sides, it also has two congruent angles.
The triangle ΔABC has two congruent sides. **Therefore** it also has two congruent angles.
(5) Having two congruent angles is **sufficient** for having two congruent angles.
Having two congruent angles is **necessary** for having two congruent sides.
Having two congruent sides **implies** having two congruent angles.

Aristotle (384-322 BC) formulates logical principles.

- Aristotle was not primarily a mathematician but made important contributions by systematising deductive logic.
- Aristotle determined the orientation and the content of Western intellectual history. He was the author of a philosophical and scientific system that

through the centuries became the support and vehicle for both medieval Christian and Islamic scholastic thought until the end of the 17th century.

- In 367 BC Aristotle, at the age of seventeen, became a student at Plato's Academy in Athens. The Academy was highly involved in the politics of the time, and the politics of the Academy and of the whole region would play a major role in influencing the course of Aristotle's life.

  After being a student, Aristotle soon became a teacher at the Academy and he was to remain there for twenty years.
- In 343 BC Aristotle reached the Court of Macedonia and he was to remain there for seven years. The often quoted story that he became tutor to the young Alexander the Great, the son of Philip, is almost certainly a later invention.
- In 335 BC Aristotle founded his own school the Lyceum in Athens. Aristotle pursued a broad range of subjects. Prominence was given by Aristotle to the detailed study of nature. He carried out his researches in company, and he thought that a man could not claim to know a subject unless he was capable of transmitting his knowledge to others, and he regarded teaching as the proper manifestation of knowledge. According to a tradition Aristotle lectured on logic, physics, astronomy, meteorology, zoology, metaphysics, theology, psychology, politics, economics, ethics, rhetoric, poetics; and wrote down these lectures, expanding them and amending them several times, until they reached the stage in which we read them. However, still more astounding is the fact that the majority of these subjects did not exist as such before him, so that he would have been the first to conceive of and establish them, as systematic disciplines.
- After the death of Alexander the Great in 323 BC, Aristotle retired to Chalcis where he died the following year from a stomach complaint at the age of 62.
- Aristotle's works were first published in about 60 BC by Andronicus of Rhodes, the last head of the Lyceum. There are important works on logic. Aristotle believed that logic was not a science but rather had to be treated before the study of every branch of knowledge.

  The sciences - at any rate the theoretical sciences - are to be axiomatised. What, then, are their axioms to be? What conditions must a proposition satisfy to count as an axiom? By what rules will theorems be deduced from axioms? Those are among the questions which Aristotle poses in his logical writings.

  Aristotle proposed the now famous Aristotelian syllogistic, a form of argument consisting of two premises and a conclusion. His example is:-

  (i) Every Greek is a person. (ii) Every person is mortal. (iii) Every Greek is mortal.

  Aristotle suggested an axiom system for each science. Notice that Euclid and his axiom system for geometry came after Aristotle.

- The importance of a proper understanding of the mathematics in Aristotle lies principally in the fact that most of his illustrations of scientific method are taken from mathematics.

  Aristotle had a thorough grasp of elementary mathematics and believed mathematics to have great importance as one of three theoretical sciences. However, he did not agree with Plato, who elevated mathematics to such a prominent place of study that there was little room for the range of sciences studied by Aristotle. The other two theoretical sciences, Aristotle claimed, were (using modern terminology) philosophy and theoretical physics.
- Aristotle's idea that 'continuous' could not be made up of indivisible parts; the continuous is that in which the boundary or limit between two consecutive parts, where they touch, is one and the same...

  As to the infinite Aristotle believed that it did not actually exist but only potentially exists. Aristotle writes: But my argument does not anyhow rob mathematicians of their study, although it denies the existence of the infinite in the sense of actual existence as something increased to such an extent that it cannot be gone through; for, as it is, they do not need the infinite or use it, but only require that the finite straight line shall be as long as they please. ... Hence it will make no difference to them for the purpose of proofs.

Gottfried Wilhelm Leibniz (1646-1716) suggests a calculus of logic.

- Parents: Friedrich Leibniz, professor of moral philosophy at Leipzig. Friedrich Leibniz died when Leibniz was only six years old and he was brought up by his mother. Certainly Leibniz learnt his moral and religious values from her which would play an important role in his life and philosophy.
- At the age of seven, Leibniz entered the Nicolai School in Leipzig. Although he was taught Latin at school, Leibniz had taught himself far more advanced Latin and some Greek by the age of 12. He was taught Aristotle's logic and theory of categorising knowledge. Leibniz was clearly not satisfied with Aristotle's system and began to develop his own ideas on how to improve on it. In later life Leibniz recalled that he was trying to find orderings on logical truths which, although he did not know it at the time, were the ideas behind rigorous mathematical proofs. Leibniz also studied his father's books. In particular he read metaphysics books and theology books from both Catholic and Protestant writers.
- In 1661, at the age of fourteen, Leibniz entered the University of Leipzig. It may sound today as if this were a truly exceptionally early age for anyone to enter university, but it is fair to say that by the standards of the time he was quite young but there would be others of a similar age. He studied philosophy (well taught), mathematics (poorly taught), rhetoric, Latin, Greek and Hebrew. He graduated with a bachelors degree in 1663 with a thesis De Principio Individui (On the Principle of the Individual).

- After being awarded a bachelor's degree in law, Leibniz worked on his habilitation in philosophy. His work was to be published in 1666 as Dissertatio de arte combinatoria (Dissertation on the combinatorial art). In this work Leibniz aimed to reduce all reasoning and discovery to a combination of basic elements such as numbers, letters, sounds and colours.
- Leibniz was refused the doctorate in law at Leipzig. Leibniz went immediately to the University of Altdorf where he received a doctorate in law in February 1667 for his dissertation De Casibus Perplexis (On Perplexing Cases).
- Leibniz served as secretary to the Nuremberg alchemical society for a while then he met Baron Johann Christian von Boineburg. By November 1667 Leibniz was living in Frankfurt, employed by Boineburg. During the next few years Leibniz undertook a variety of different projects, scientific, literary and political. He also continued his law career taking up residence at the courts of Mainz before 1670. One of his tasks there, undertaken for the Elector of Mainz, was to improve the Roman civil law code for Mainz.
- Leibniz began to study motion, and began with abstract ideas of motion. In 1671 he published Hypothesis Physica Nova (New Physical Hypothesis). In this work he claimed, as had Kepler, that movement depends on the action of a spirit. He communicated with Oldenburg, the secretary of the Royal Society of London, and dedicated some of his scientific works to the Royal Society and the Paris Academy.
- In Paris Leibniz studied mathematics and physics under Christiaan Huygens beginning in the autumn of 1672. Diplomacy.
- The Royal Society of London elected Leibniz a fellow on 19 April 1673.
- In Paris Leibniz developed the basic features of his version of the calculus. In 1673 he was still struggling to develop a good notation for his calculus and his first calculations were clumsy. On 21 November 1675 he wrote a manuscript using the $\int f(x)dx$ notation for the first time and gives the product rule for differentiation. By autumn 1676 Leibniz discovered the familiar $d(x^n) = nx^{n-1}dx$ for both integral and fractional n.
- Newton wrote a letter to Leibniz, which took some time to reach him. The letter listed many of Newton's results but it did not describe his methods. Leibniz replied immediately but Newton, not realising that his letter had taken a long time to reach Leibniz, thought he had had six weeks to work on his reply. Certainly one of the consequences of Newton's letter was that Leibniz realised he must quickly publish a fuller account of his own methods.

  Newton wrote a second letter to Leibniz on 24 October 1676 which did not reach Leibniz until June 1677 by which time Leibniz was in Hanover. This second letter, although polite in tone, was clearly written by Newton believing that Leibniz had stolen his methods. In his reply Leibniz gave some details of the principles of his differential calculus including the rule for differentiating a function of a function.

- Leibniz would have liked to have remained in Paris in the Academy of Sciences, but it was considered that there were already enough foreigners there and so no invitation came. Reluctantly Leibniz accepted a position from the Duke of Hanover, Johann Friedrich, of librarian and of Court Councillor at Hanover. The rest of Leibniz's life, from December 1676 until his death, was spent at Hanover except for the many travels that he made.
- Another of Leibniz's great achievements in mathematics was his development of the binary system of arithmetic. He perfected his system by 1679 but he did not publish anything until 1701 when he sent the paper Essay d'une nouvelle science des nombres to the Paris Academy to mark his election to the Academy. Another major mathematical work by Leibniz was his work on determinants which arose from his developing methods to solve systems of linear equations.
- Leibniz continued to perfect his metaphysical system in the 1680s attempting to reduce reasoning to an algebra of thought. Leibniz published Meditationes de Cognitione, Veritate et Ideis (Reflections on Knowledge, Truth, and Ideas) which clarified his theory of knowledge. In February 1686, Leibniz wrote his Discours de mtaphysique (Discourse on Metaphysics).

  Another major project which Leibniz undertook, this time for Duke Ernst August, was writing the history of the Guelf family, of which the House of Brunswick was a part.

  In 1684 Leibniz published details of his differential calculus in Nova Methodus pro Maximis et Minimis, itemque Tangentibus... in Acta Eruditorum, a journal established in Leipzig two years earlier. The paper contained the familiar d notation, the rules for computing the derivatives of powers, products and quotients. However it contained no proofs and Jacob Bernoulli called it an enigma rather than an explanation.

  In 1686 Leibniz published, in Acta Eruditorum, a paper dealing with the integral calculus with the first appearance in print of the notation.
- Newton's Principia appeared the following year. Newton's 'method of fluxions' was written in 1671 but Newton failed to get it published and it did not appear in print until John Colson produced an English translation in 1736. This time delay in the publication of Newton's work resulted in a dispute with Leibniz.
- It is no exaggeration to say that Leibniz corresponded with most of the scholars in Europe. He had over 600 correspondents.
- In 1710 Leibniz published Thodice a philosophical work intended to tackle the problem of evil in a world created by a good God. Leibniz claims that the universe had to be imperfect, otherwise it would not be distinct from God. He then claims that the universe is the best possible without being perfect. Leibniz is aware that this argument looks unlikely - surely a universe in which nobody is killed by floods is better than the present one, but still not perfect. His argument here is that the elimination of natural disasters, for example,

would involve such changes to the laws of science that the world would be worse. In 1714 Leibniz wrote Monadologia which synthesised the philosophy of his earlier work, the Thodice.

- Leibniz was one of the last great polymaths - in the sense of one who is a citizen of the whole world of intellectual inquiry. He deliberately ignored boundaries between disciplines, and lack of qualifications never deterred him from contributing fresh insights to established specialisms. Indeed, one of the reasons why he was so hostile to universities as institutions was because their faculty structure prevented the cross-fertilisation of ideas which he saw as essential to the advance of knowledge and of wisdom. The irony is that he was himself instrumental in bringing about an era of far greater intellectual and scientific specialism, as technical advances pushed more and more disciplines out of the reach of the intelligent layman and amateur.

George Boole (1815-1869) realizes Leibniz's dream in "An investigation into the Laws of Thought".

- George Boole's father John made shoes but he was interested in science and in particular the application of mathematics to scientific instruments. The family was not well off, partly because John's love of science and mathematics meant that he did not devote the energy to developing his business in the way he might have done.
- If George was a weak child after his birth, he certainly soon became strong and healthy. George first attended a school when he was less than two years old. After a year he went to a commercial school where he remained until he was seven years old. His early instruction in mathematics was from his father who also gave George a liking for constructing optical instruments. When he was seven George attended a primary school and his interests turned to languages.
- Having learnt Latin from a tutor, George went on to teach himself Greek. By the age of 14 he had become so skilled in Greek that it provoked an argument. He translated a poem by the Greek poet Meleager which his father was so proud of that he had it published. However the talent was such that a local schoolmaster disputed that any 14 year old could have written with such depth. By this time George was attending Bainbridge's Commercial Academy. This school did not provide the type of education he would have wished but it was all his parents could afford. However he was able to teach himself French and German studying for himself academic subjects that a commercial school did not cover.
- Boole did not study for an academic degree, but from the age of 16 he was an assistant school teacher at Heigham's School in Doncaster. This was rather forced on him since his father's business collapsed and he found himself having

to support financially his parents, brothers and sister. He maintained his interest in languages, began to study mathematics seriously.

- In 1834 he opened his own school in Lincoln although he was only 19 years old.
- In 1838 Boole was invited to take over Hall's Academy aand he ran the school together with his parents, brothers and sister. At this time Boole was studying the works of Laplace and Lagrange, making notes which would later be the basis for his first mathematics paper. In the summer of 1840 he began publishing regularly in the Cambridge Mathematical Journal and he began to study algebra.
- Boole had begun to correspond with De Morgan in 1842 and when in the following year he wrote a paper On a general method of analysis applying algebraic methods to the solution of differential equations he sent it to De Morgan for comments. It was published and for this work he received the Society's Royal Medal in November 1844. His mathematical work was beginning to bring him fame.
- Boole's father died in December 1848 before Boole became the first Professor of Mathematics at Queen's College, Cork. He taught there for the rest of his life, gaining a reputation as an outstanding and dedicated teacher. However the position was not without difficulty as the College became embroiled in religious disputes.
- Boole met Mary Everest (a niece of Sir George Everest, after whom the mountain is named) whose uncle was the professor of Greek at Cork and a friend of Boole. Boole began to give Mary informal mathematics lessons on the differential calculus. At this time he was 37 years old while Mary was only 20. In 1855 Mary's father died leaving her without means of support and Boole proposed marriage. They married on 11 September 1855. It proved a very happy marriage with five daughters.
- Boole's most important work was he published in 1854 "An investigation into the Laws of Thought, on Which are founded the Mathematical Theories of Logic and Probabilities". Boole approached logic in a new way reducing it to a simple algebra, incorporating logic into mathematics. He pointed out the analogy between algebraic symbols and those that represent logical statements. It began the algebra of logic called Boolean algebra which now finds application in computer construction, switching circuits etc. Boole himself understood the importance of the work. He wrote:

  I am now about to set seriously to work upon preparing for the press an account of my theory of Logic and Probabilities which in its present state I look upon as the most valuable if not the only valuable contribution that I have made or am likely to make to Science and the thing by which I would desire if at all to be remembered hereafter ...

- Many honours were given to Boole as the genius in his work was recognised. He received honorary degrees from the universities of Dublin and Oxford and was elected a Fellow of the Royal Society (1857). However his career, which was started rather late, came to an unfortunately early end when he died at the age of 49.

  One day in 1864 he walked from his residence to the College, a distance of two miles, in the drenching rain, and lectured in wet clothes. The result was a feverish cold which soon fell upon his lungs and terminated his career ....

- Formal Logic analyses, clarifies, explains, and completes common logical thinking and develops an appropriate language. (Analogy: grammar).
- Logic deals with ***statements*** that are unambiguously either ***true*** (T) or ***false*** (F).
- Example.
  (1) $\Delta \Rightarrow \Box$ where Δ is short for "it is sunny" and $\Box$ is short for "George goes to the beach" and $\Rightarrow$ is short for "implies", another way of saying "if ... then ... ".
  (2) ***Direct Reasoning***: $\Delta \Rightarrow \Box$ is TRUE, Δ is TRUE, conclusion $\Box$ is TRUE.
  In common usage, if a statement is said or written, then it is implied that it is TRUE: $\Delta \Rightarrow \Box$, Δ, conclusion $\Box$.
- Logic deals with statements that are either true or false. Such statements may be combined by ***logical connectives*** to obtain new statements.
- Formal logic found, despite of the many ways to express connections between statements in spoken languages, that there are only four basic connectives, namely ***NOT*** ($\neg$), ***AND*** ($\wedge$), ***OR*** ($\vee$), and ***if ... then ...*** ($\Rightarrow$).
- Formal logic must decide when a compound statement such as

  $$\textbf{NOT } \Delta, \quad \Delta \textbf{ AND } \Box, \quad \Delta \textbf{ OR } \Box, \quad \textbf{if } \Delta \textbf{ then } \Box$$

  is true of false. This is done with ***truth tables***.
- Example.

Δ	$\Box$	$\Delta \Rightarrow \Box$
T	T	T
T	F	F
F	T	T
F	F	T

Recall: **If** it is sunny, **then** George is on the beach.

- Example.

Δ	$\Box$	Δ **OR** $\Box$
T	T	T
T	F	T
F	T	T
F	F	F

Δ	$\Box$	Δ **AND** $\Box$
T	T	T
T	F	F
F	T	F
F	F	F

Example: A student qualifies of a scholarship if the student is over 40 **OR** has a 3.5 grade point average.
- Having agreed on truth tables (correctly) the truth value T or F of a compound statement can be determined mechanically, e.g. by computers.
- Example: $((\Delta_1 \vee \Delta_2) \wedge (\Box \vee \Delta_2)) \Rightarrow ((\Delta_1 \wedge \Delta_2) \vee (\Box \wedge \Delta_2))$ is a "well-formed" term in formal logic that could not or only with great difficulty expressed in a spoken language.
- The two De Morgan Laws can be understood using common sense or by constructing a truth table:

$$\textbf{NOT}\,(\Delta\ \textbf{OR}\ \Box) \Leftrightarrow (\textbf{NOT}\,\Delta)\ \textbf{AND}\ (\textbf{NOT}\,\Box),\ \text{“neither } \Delta \text{ nor } \Box\text{”}$$
$$\textbf{NOT}\,(\Delta\ \textbf{AND}\ \Box) \Leftrightarrow (\textbf{NOT}\,\Delta)\ \textbf{OR}\ (\textbf{NOT}\,\Box).\ \ \text{“not both } \Delta \text{ and } \Box\text{”}$$

Here $\Leftrightarrow$ stand for "if and only if" or "is logically equivalent to".
- Related "if ... then ..." statements.
Positive: If it is sunny, then George goes to the beach, or $\Delta \Rightarrow \Box$.
Converse: If George goes to the beach, then it is sunny, or $\Box \Rightarrow \Delta$.
Contrapositive: If George is **NOT** on the beach, then it is **NOT** sunny, or **NOT** $\Box \Rightarrow$ **NOT** Δ.
- **Possibilities:**
Positive T, converse T ("if and only if", "logically equivalent"),
positive T, converse F,
positive F, converse T.
The positive and the converse are NOT logically equivalent.
- The positive and contrapositve are either both true or both false. They are logically equivalent.
- Example.

Δ	$\Box$	$\Delta \Rightarrow \Box$	$\Box \Rightarrow \Delta$	NOT $\Box$	NOT Δ	NOT $\Box \Rightarrow$ NOT Δ
T	T	T	T	F	F	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	T	T

- Example.

Δ	$\square$	$\Delta \Rightarrow \square$	NOT Δ	$\square$ OR NOT Δ
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Frequently statement involve one or more variables representing objects from a **universe of discourse** U. The formula

$$p(x): \qquad 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1},$$

where x is a real number not equal to 1, is an example with $U = \mathbb{R}$ where $\mathbb{R}$ denotes the system of real numbers.

1. Quantification.

(1) $\forall x \in U,\ p(x)$ *("for all* $x \in U$, $p(x)$*") is true if* $p(a)$ *is true for each object* a *from the universe* U *of discourse.*
(2) $\exists x \in U,\ p(x)$ *("there is* $x \in U$ *such that* $p(x)$*") is true if* $p(a)$ *is true for some object* a *from the universe* U *of discourse.*
(3) $\neg(\forall x \in U\ p(x)) \Leftrightarrow (\exists x \in U\ \neg p(x))$.
(4) $\neg(\exists x \in U\ p(x)) \Leftrightarrow (\forall x \in U\ \neg p(x))$.

The English provides many ways for expressing quantified statements. Here are some.

2. Common expressions.

(1) *"No X is a Y" means "For all X, X is not a Y."*
(2) *Example: No man is an island.*
(3) *"Some X are Y" means "There is an X that is Y."*
(4) *Example: Some people are friendly.*

Exercise 3. *Translate the following statements into quantified formulas of symbolic logic. Identify the implied universe of discourse and choose suitable symbols. E.g.,* $p(x)$ *may stand for "*x *can do logic".*

(1) *Everyone can do logic.*
(2) *None of his numbers is rational.*
(3) *All UH students like mathematics.*
(4) *Without exception, rational numbers are real.*
(5) *Someone cheated.*
(6) *To err is human.*
(7) *Koalas are cute.*
(8) *You can't believe everything you hear. [You can't believe it = it is not true.]*

Statements may involve several variables and several quantifiers.

Example 4. *The universe of discourse is the set of real numbers. Some of the following statements are true, some false.*

(1) $\forall x\ \forall y\ x^2 + y^2 \geq 0$.
(2) $\forall x\ \exists y$ *such that* $y^2 = x^2$ *and* $y \geq 0$.
(3) $\exists y$ *such that* $\forall x$ *we have* $y^2 = x^2$ *and* $y \geq 0$.
[**False.** Assume that y is a number with the required properties. Then $y^2 = 0^2$ and $y^2 = 1^2$, so $0 = 1$, which is a contradiction. Note that this was a typical proof by contradiction.]
(4) $\forall x\ \forall y\ \forall z \quad x(y + z) = xy + xz$.
(5) $\forall a, b, c\ \exists x$ *such that* $ax^2 + bx + c = 0$.

Exercise 5. *Negate the quantified statements of Example 4.*

The negation of 4.3 must be true and it is

$$\forall\, y\ \exists\, x \text{ such that } y^2 \neq x^2 \text{ or } y < 0.$$