REPRESENTATIONS OF POSETS AND DECOMPOSITIONS OF TORSION-FREE ABELIAN GROUPS

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1. Background

Motivation.

Kaplansky.

- ***torsion–free abelian group*** G = additive subgroup of a $\mathbb{Q}$–vector space V, $A \leq V$.
- ***divisible hull*** $\mathbb{Q}A$ = subspace of V spanned by A.

  rank $\operatorname{rk} A := \dim \mathbb{Q}A$.
- ***rational group*** = additive subgroup of $\mathbb{Q}$.
- **Examples.**
  - $\mathbb{Z}$, $\mathbb{Q}$.
  - For P a set of prime numbers,

$$\mathbb{Z}[P^{-1}] := \langle 1/p^n \mid p \in P, n \in \mathbb{N} \rangle.$$

- $\mathbb{Z}[P_1^{-1}] \cong \mathbb{Z}[P_2^{-1}]$ if and only if $P_1 = P_2$. There are $2^{\aleph_0}$ non–isomorphic rational groups of this "idempotent type", but there are many more rational groups.
- A ***rank–one group*** is a group isomorphic with a rational group.

Bad news #1

- A ***type*** is an isomorphism class of rank–one groups.
- $[A]$ is the type of the rank–one group A. The set $\mathbb{T}$ of all types is a partially ordered set (poset) via $[A] \leq [B]$ if and only if $\mathrm{Hom}(A, B) \neq 0$.

- ***completely decomposable group*** = direct sum of rank–one groups. Completely decomposable group of ***finite rank***:

$$A = A_1 \oplus \cdots \oplus A_n, \quad \operatorname{rk} A_i = 1.$$

- ***homogenous decomposition***

$$A = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(A)} A_\rho, \quad A_\rho = A_{\rho 1} \oplus \cdots \oplus A_{\rho n_\rho} \neq 0.$$

- (Reinhold Baer 1940)

2. Almost Completely Decomposable Groups

- (Lee Lady 1974) ***Almost completely decomposable group*** = torsion–free group X containing a completely decomposable group A of **finite rank** that has **finite index** in X.

Example 2.1.

$$\begin{aligned} A &:= \mathbb{Z}[5^{-1}]v_1 \oplus \mathbb{Z}[5^{-1}]v_2 \oplus \mathbb{Z}[7^{-1}]v_3 \oplus \mathbb{Z}[7^{-1}]v_4; \\ X_1 &:= (\mathbb{Z}[5^{-1}]v_1 \oplus \mathbb{Z}[7^{-1}]v_3) + \mathbb{Z}\frac{1}{2}(v_1 + v_3); \\ X_2 &:= (\mathbb{Z}[5^{-1}]v_2 \oplus \mathbb{Z}[7^{-1}]v_4) + \mathbb{Z}\frac{1}{3}(v_2 + v_4). \end{aligned}$$

Basis change in the homogeneous components of A:

$$\mathbb{Z}[5^{-1}]v_1 \oplus \mathbb{Z}[5^{-1}]v_2 = \mathbb{Z}[5^{-1}](3v_1 + 2v_2) \oplus \mathbb{Z}[5^{-1}](v_1 + v_2);$$

$$\mathbb{Z}[7^{-1}]v_3 \oplus \mathbb{Z}[7^{-1}]v_4 = \mathbb{Z}[7^{-1}](3v_3 + 2v_4) \oplus \mathbb{Z}[5^{-1}](v_3 + v_4);$$

$$Y := (\mathbb{Z}[5^{-1}](3v_1+2v_2) \oplus \mathbb{Z}[7^{-1}](3v_3+2v_4)) + \mathbb{Z}\frac{1}{6}((3v_1+2v_2)+(3v_3+2v_4));$$

X_1, X_2, Y *are indecomposable and*

$$X := X_1 \oplus X_2 = Y \oplus \mathbb{Z}[5^{-1}](v_1 + v_2) \oplus \mathbb{Z}[7^{-1}](v_3 + v_4).$$

Bad news #2: "pathological decompositions".

Theorem 2.2. (A.L.S. Corner 1961) *Given integers* $n \geq k \geq 1$, *there exists a (an almost completely decomposable) group* X *of rank* n *such that for any partition* $n = r_1 + \cdots + r_k$, *there is a decomposition of* X *into a direct sum of* k *indecomposable subgroups of ranks* $r_1, \ldots, r_k$ *respectively.*

3. Regulating Subgroups and the Regulator

Definition 3.1. (Lee Lady 1974) X *almost completely decomposable.*

- A = ***regulating subgroup*** *of* X*, if* A *is a completely decomposable subgroup of* X *and the index* $[X : A]$ *is minimal.*
- *The* ***regulator*** $\mathrm{R}(X)$ *is the intersection of all regulating subgroups*

Theorem 3.2. (Rolf Burkhardt 1984) *Let* X *be an almost completely decomposable group. Then* $\mathrm{R}(X)$ *is a fully invariant, completely decomposable subgroup of* X *that has finite index in* X*.*

Definition 3.3. (Lee Lady 1975) *G and H, torsion-free of finite rank, are **isomorphic at** p if there is an integer n prime to p and homomorphisms $f : G \to H$ and $g : H \to G$ with $fg = n$ and $gf = n$. The groups G and H are **nearly isomorphic**, $G \cong_{\mathrm{nr}} H$ if they are isomorphic at p for every prime p.*

Theorem 3.4. (David Arnold 1982) *$X \cong_{\mathrm{nr}} Y$ and $X = X_1 \oplus X_2$, then $Y = Y_1 \oplus Y_2$ for some subgroups $Y_1 \cong_{\mathrm{nr}} X_1$ and $Y_2 \cong_{\mathrm{nr}} X_2$.*

Theorem 3.5. (Ted Faticoni and Phill Schultz 1995) *The "indecomposable" decompositions of an almost completely decomposable group X with $X/\,\mathrm{R}(X)$ a primary group are unique up to near–isomorphism.*

Example 3.6.

$$
\begin{aligned}
A &:= \mathbb{Z}[5^{-1}]v_1 \oplus \mathbb{Z}[5^{-1}]v_2 \oplus \mathbb{Z}[7^{-1}]v_3 \oplus \mathbb{Z}[7^{-1}]v_4;\\
X_1 &:= (\mathbb{Z}[5^{-1}]v_1 \oplus \mathbb{Z}[7^{-1}]v_3) + \mathbb{Z}\frac{1}{2}(v_1 + v_3);\\
X_2 &:= (\mathbb{Z}[5^{-1}]v_2 \oplus \mathbb{Z}[7^{-1}]v_4) + \mathbb{Z}\frac{1}{3}(v_2 + v_4).
\end{aligned}
$$

$$Y := (\mathbb{Z}[5^{-1}]w_1 \oplus \mathbb{Z}[7^{-1}]w_2) + \mathbb{Z}\frac{1}{6}(w_1 + w_2);$$

$$X := X_1 \oplus X_2 = Y \oplus \mathbb{Z}[5^{-1}](v_1 + v_2) \oplus \mathbb{Z}[7^{-1}](v_3 + v_4).$$

Program.

Lemma 3.7. *X, Y almost completely decomposable groups. If $X \cong_{\mathrm{nr}} Y$, then $\mathrm{R}(X) \cong \mathrm{R}(Y)$ and $X/\mathrm{R}(X) \cong Y/\mathrm{R}(Y)$.*

Definition 3.8. *A completely decomposable, e positive integer.*

$$\mathrm{RFEE}(A, e) := \{X \leq \mathbb{Q}A : A = \mathrm{R}(X), eX \subseteq A\}.$$

(RFEE = "regulated finite essential extension")

4. Representations

Rigid homocyclic case (*):

- $A = A_{\tau_1} \oplus \cdots \oplus A_{\tau_n}$, anti-chain $\{\tau_i\}$, where $A_{\tau_i} \neq pA_{\tau_i}$,
- $X \in \mathrm{RFEE}(A, p^m)$, $X/A \cong \langle g_1 \rangle \oplus \cdots \oplus \langle g_r \rangle$
  where $\langle g_i \rangle \cong \mathbb{Z}/p^m\mathbb{Z}$, ***homocyclic regulator quotient***.

Definition 4.1. $\mathrm{RH}(A, p^m) := \{X \in \mathrm{RFEE}(A, p^m) \mid (*)\}$.

Definition 4.2. *A completely decomposable*

$\overline{} : A \to A/p^m A : \overline{a} = a + p^m A$, $\overline{A} = A/p^m A$,

$\overline{} : \operatorname{End} A \to \operatorname{End} \overline{A}$ *induced map.*

Definition 4.3. $X \in \mathrm{RH}(A, p^m)$, $A = A_{\tau_1} \oplus \cdots \oplus A_{\tau_n}$.

$\overline{A}$ *is a free* $\mathbb{Z}/p^m\mathbb{Z}$ *module,* $\overline{A} = \overline{A_{\tau_1}} \oplus \cdots \oplus \overline{A_{\tau_n}}$.

Representation of X*:* $U_X := (\overline{A}, \overline{A_{\tau_i}}, \overline{p^m X})$

$\alpha \in \operatorname{End} U_X \Leftrightarrow \alpha \in \operatorname{End} \overline{A}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ *with* $\alpha_i \in \operatorname{End} \overline{A_{\tau_i}}$ *and* $\alpha(\overline{p^m X}) \subseteq \overline{p^m X}$.

Theorem 4.4. *There is a bijective correspondence from* $\mathrm{RFEE}(A, e)$ *to (certain) representations such that*

- $X \cong_{\mathrm{nr}} Y$ *if and only if* $U_X \cong U_Y$*,*
- X *is indecomposable if and only if* U_X *is indecomposable,*
- U_X *is indecomposable if and only if the only idempotents in* $\operatorname{End} U_X$ *are* 0 *and* 1*.*

Remark 4.5. Used that $A = \mathrm{R}(X)$ is fully invariant: $\alpha : X \to X$ restricts to $\alpha : A \to A$, induces $\overline{\alpha} : \overline{A} \to \overline{A}$.

5. REPRESENTING MATRICES

Definition 5.1.

proper *basis* $\mathcal{B}$ *of* $\overline{A}$ = *union of bases* $\mathcal{B}_i$ *of the* $\overline{A_{\tau_i}}$.

$M_X = [m_{ij}]$ = ***representing matrix*** *if*

$$p^m g_i = \sum_i \sum \{m_{ib} b : b \in \mathcal{B}_i\}.$$

Remark 5.2. $\overline{p^m X} = \vec{\mathbb{Z}} M_X =$ *row space of* M_X.

Regulator Criterion. $\begin{bmatrix} 1 & 0 & 0 & \| & 0 & 0 & 0 & \| & 1 & 0 & 0 \\ 0 & 1 & 0 & \| & 0 & 0 & p & \| & 0 & 1 & 0 \\ 0 & 0 & p & \| & 0 & 1 & 0 & \| & 1 & 0 & p \\ 0 & 0 & 0 & \| & 1 & 0 & 0 & \| & 0 & 1 & p \end{bmatrix}$

6. Unbounded representation type

Recall: $\alpha \in \operatorname{End} U_X \Leftrightarrow \alpha \in \operatorname{End} \overline{A}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \operatorname{End} \overline{A_{\tau_i}}$ and $\alpha(\vec{\mathbb{Z}} M_X) \subseteq \vec{\mathbb{Z}} M_X$.

Theorem 6.1. Indecomposability Criterion.

- $X \in \mathrm{RH}(A, p^m)$,
- U_X, *the representation of* X,
- $M := M_X$ *a representing matrix of* X.

Assume that M^* *is a right inverse of* M. *Let* $f^2 = f \in \operatorname{End} U_X$. *Then* X *is indecomposable if and only if*

$$Mf = MfM^*M$$

implies that $f = 0$ *and* $f = 1$.

Theorem 6.2. *The category* $\mathrm{RH}(4, p)$ *has unbounded representation type.*

Proof. • $A = n \times n$ matrix with coefficients in $\mathbb{Z}_p$ such that $Ax = xA$ implies that $x \in \{0, 1\}$,

• $X \in \mathrm{RH}(4, p)$ with representing matrix

$$M = \begin{bmatrix} I_n & \| & 0 & \| & I_n & \| & I_n \\ 0 & \| & I_n & \| & I_n & \| & A \end{bmatrix}$$

$$M^* = \begin{bmatrix} I_n & 0 \\ 0 & I_n \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M^*M = \begin{bmatrix} I_n & 0 & I_n & I_n \\ 0 & I_n & I_n & A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $f^2 = f$ be a representation idempotent in U_X. Then, for $n \times n$ idempotent matrices a, b, c, d,

$$f = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, \quad Mf = \begin{bmatrix} a & 0 & c & d \\ 0 & b & c & Ad \end{bmatrix}, \quad MfM^*M = \begin{bmatrix} a & 0 & a & a \\ 0 & b & b & bA \end{bmatrix}$$

and $Mf = MfM^*M$. Then $c = a = d = b$ and $Ad = bA$, hence $Ab = bA$ and it follows that $b \in \{0, 1\}$ and that $f \in \{0, 1\}$. $\square$

Theorem 6.3. *The category* $\mathrm{RH}(4,p)$ *has unbounded representation type.*

Theorem 6.4. *The category* $\mathrm{RH}(3,p^3)$ *has unbounded representation type.*

Corollary 6.5. *The category* $\mathrm{RH}(S,p^m)$*-groups with* S *an antichain has unbounded representation type if*

- $|S| \geq 4$, $m \geq 1$,
- $S = (1,1,1)$ *and* $m \geq 3$.

7. Indecomposable $((1,1,1),p^m)$-groups

Theorem 7.1. *For a given anti-chain of types $\{\tau_1, \tau_2, \tau_3\}$, there are, with critical typeset $\{\tau_1, \tau_2, \tau_3\}$, up to near-isomorphism one indecomposable $((1,1,1),p)$-group in $\mathrm{RH}(3,p)$ with representing matrix $\begin{bmatrix}1 \| 1 \| 1\end{bmatrix}$, and one indecomposable group with representing matrix*

$$\begin{bmatrix} 1 & \| & 0 & \| & 1 \\ 0 & \| & 1 & \| & 1 \end{bmatrix}.$$

Theorem 7.2. *A group in* $\mathrm{RH}(3, p^2)$ *is indecomposable if and only if it is nearly isomorphic to a group having one of the following representing matrices* $\left[M_1 \,\|\, M_2 \,\|\, M_3\right]$ *or permutations of these.*

(1) *Groups with cyclic regulator quotient:*

(a) $M_{G_{31}} = \left[1\|1\|1\right]$, $\mathrm{rk}(G_{31}) = 3$,
one near-isomorphism class,

(b) $M_{G_{32}} = \left[1\|1\|p\right]$, $\mathrm{rk}(G_{32}) = 3$,
three near-isomorphism classes,

(2) *Groups with 2-generated regulator quotient:*

(a) $M_{G_{21}} = \begin{bmatrix} 1 & \| & 0 & \| & 1 \\ 0 & \| & 1 & \| & 1 \end{bmatrix}$, $\mathrm{rk}(G_{21}) = 3$,
one near-isomorphism class,

(b) $M_{G_{22}} = \begin{bmatrix} 1 & 0 & \| & 0 & \| & 1 \\ 0 & p & \| & 1 & \| & 1 \end{bmatrix}$, $\mathrm{rk}(G_{22}) = 4$,
three near-isomorphism classes,

(c) $M_{G_{23}} = \begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 \\ 0 & p & \| & 1 & 0 & \| & 1 \end{bmatrix}$, $\mathrm{rk}(G_{23}) = 5$,
three near-isomorphism classes,

(d) $M_{G_{24}} = \begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 & 0 \\ 0 & p & \| & 1 & 0 & \| & 1 & p \end{bmatrix}$, $\mathrm{rk}(G_{24}) = 6$,
one near-isomorphism class,

(3) *Groups with* 3*-generated regulator quotient:*

(a) $$M_{G_{31}} = \begin{bmatrix} 1 & 0 & \| & 0 & 0 & \| & 1 & 0 \\ 0 & 1 & \| & 0 & 1 & \| & 0 & 1 \\ 0 & 0 & \| & 1 & 0 & \| & 1 & p \end{bmatrix}, \ \mathrm{rk}(G_{31}) = 6,$$

one near-isomorphism class,

(4) *Groups with* 4*-generated regulator quotient:*

(a) $$M_{G_{41}} = \begin{bmatrix} 1 & 0 & 0 & \| & 0 & 0 & 0 & \| & 1 & 0 & 0 \\ 0 & 1 & 0 & \| & 0 & 0 & p & \| & 0 & 1 & 0 \\ 0 & 0 & p & \| & 0 & 1 & 0 & \| & 1 & 0 & p \\ 0 & 0 & 0 & \| & 1 & 0 & 0 & \| & 0 & 1 & p \end{bmatrix}, \ \mathrm{rk}(G_{41}) = 9,$$

one near-isomorphism class.