Noncommutative Tori and the Gauss-Manin Connection

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Given \( \theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z} \), the \textit{rotation algebra} \( A_\theta \) is the universal \( C^* \)-algebra generated by two unitaries \( u, v \) such that

\[ vu = e^{2\pi i \theta} uv. \]

When \( \theta = 0 \), the \( C^* \)-algebra \( A_0 \) is commutative, and

\[ A_0 \cong C(\mathbb{T}^2). \]

Thus, \( A_\theta \) are also called \textit{noncommutative tori}. 
The isomorphism $A_0 \cong C(\mathbb{T}^2)$ is given by

$$u \mapsto u(s, t) = e^{2\pi i s}, \quad v \mapsto v(s, t) = e^{2\pi i t}.$$ 

By Fourier analysis, $\sum_{m,n\in\mathbb{Z}} c_{mn}u^m v^n$ represents a smooth function on $\mathbb{T}^2$ if and only if the coefficients $(c_{mn})$ are of *rapid decay*. That is, $(c_{mn})$ is in the Schwartz space $\mathcal{S}(\mathbb{Z}^2)$.

The **smooth noncommutative 2-torus** $A_\theta$ is the subalgebra

$$A_\theta = \left\{ \sum_{m,n\in\mathbb{Z}} c_{mn}u^m v^n \in A_\theta \mid (c_{mn}) \in \mathcal{S}(\mathbb{Z}^2) \right\}.$$
Calculus on noncommutative tori

As vector spaces, $\mathcal{A}_\theta \cong \mathcal{S}(\mathbb{Z}^2)$. Thus $\mathcal{A}_\theta$ has a natural Fréchet topology and $\mathcal{A}_\theta$ is a Fréchet algebra.

(Partial differential operators) There are continuous derivations $\delta_1, \delta_2 : \mathcal{A}_\theta \rightarrow \mathcal{A}_\theta$ given by

$$
\delta_1(u^m v^n) = 2\pi im \cdot u^m v^n, \quad \delta_2(u^m v^n) = 2\pi in \cdot u^m v^n.
$$

(Integrating over the noncommutative torus) There is a canonical trace $\tau_0 : \mathcal{A}_\theta \rightarrow \mathbb{C}$ given by

$$
\tau_0 \left( \sum_{m,n \in \mathbb{Z}} c_{mn} u^m v^n \right) = c_{00}.
$$
Connes first computed that $H^i(A_\theta) \cong \mathbb{C} \oplus \mathbb{C}$ for $i = 0, 1$. In particular, $H^0(A_\theta)$ is spanned by $[\tau_0]$ and $[\tau_2]$, where $\tau_2$ is the cyclic 2-cocycle

$$\tau_2(a_0, a_1, a_2) = \frac{1}{2\pi i} \tau_0(a_0 \delta_1(a_1) \delta_2(a_2) - a_0 \delta_2(a_1) \delta_1(a_2)).$$

Let $p \in A_\theta$ denoted the Powers-Rieffel idempotent.

<table>
<thead>
<tr>
<th>Pairing $H^0(A_\theta)$ and $K_0(A_\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0(1) = 1$</td>
</tr>
<tr>
<td>$\tau_2(1, 1, 1) = 0$</td>
</tr>
<tr>
<td>$\tau_0(p) = \theta$</td>
</tr>
<tr>
<td>$\tau_2(p, p, p) = 1$</td>
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</tbody>
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The values in the bottom row are the derivatives with respect to $\theta$ of the values in the top row.
$A_\theta$ as a deformation

Let $J \subset \mathbb{T}$ be an open interval. The underlying vector space of each $A_\theta$ is $\mathcal{S}(\mathbb{Z}^2)$. Thus, the family of algebras $\{A_\theta\}_{\theta \in J}$ can be thought of as a family of associative multiplications $\{m_\theta\}_{\theta \in J}$ on the space $\mathcal{S}(\mathbb{Z}^2)$.

- The algebras $\{A_\theta\}_{\theta \in J}$ form a smooth field of Fréchet algebras in the sense that for any $x, y \in \mathcal{S}(\mathbb{Z}^2)$, the map
  $$\theta \mapsto m_\theta(x, y)$$
  is smooth.
- The space $\mathcal{A} = C^\infty(J, \mathcal{S}(\mathbb{Z}^2))$ is a Fréchet algebra under the product
  $$(ab)(\theta) = m_\theta(a(\theta), b(\theta)), \quad a, b \in \mathcal{A}.$$  
  We call $\mathcal{A}$ the section algebra of the smooth field $\{A_\theta\}_{t \in J}$.
- The section algebra $\mathcal{A}$ is an algebra over $C^\infty(J)$, by pointwise scalar multiplication.
The section algebra $\mathcal{A}$

- Elements of $\mathcal{A}$ are of the form
  $$a = \sum_{m,n \in \mathbb{Z}} f_{mn} u^m v^n, \quad f_{mn} \in C^\infty(J).$$

- The derivations $\delta_1, \delta_2$ extend to $C^\infty(J)$-linear derivations on $\mathcal{A}$.

- The trace $\tau_0$ extends to a trace $\tau_0 : \mathcal{A} \to C^\infty(J)$.

- There is a connection on $\mathcal{A}$, whose covariant derivative $\nabla : \mathcal{A} \to \mathcal{A}$ is given by
  $$\nabla \left( \sum_{m,n \in \mathbb{Z}} f_{mn} u^m v^n \right) = \sum_{m,n \in \mathbb{Z}} \frac{df_{mn}}{d\theta} u^m v^n.$$
The connection $\nabla$

$$\nabla \left( \sum_{m,n \in \mathbb{Z}} f_{mn} u^m v^n \right) = \sum_{m,n \in \mathbb{Z}} \frac{df_{mn}}{d\theta} u^m v^n$$

- If $\nabla$ were a derivation, then we could identify $\mathcal{A}_{\theta_0} \cong \mathcal{A}_{\theta_1}$ as algebras, provided we can solve the ODEs required for parallel translation. But it is well-known that $\mathcal{A}_{\theta_0}$ and $\mathcal{A}_{\theta_1}$ are non-isomorphic in general.

- "$\nabla$ is a derivation up to derivations." That is,

$$\nabla(ab) = \nabla(a)b + a \nabla(b) + \frac{1}{2\pi i} \delta_2(a) \delta_1(b), \quad \forall a, b \in \mathcal{A}.$$  

- Idea: Construct parallel translation isomorphisms at the level of cyclic theory. This requires extending $\nabla$ to a connection on the periodic cyclic chain complex of $\mathcal{A}$ that is compatible with the boundary map.
In general, we define a *smooth one-parameter deformation of Fréchet algebras* to be a family \( \{m_t\}_{t \in J} \) of continuous associative multiplications on a Fréchet space \( X \) such that the map

\[
t \mapsto m_t(x, y)
\]

is smooth for all \( x, y \in X \). Let \( B_t \) be the algebra \( (X, m_t) \).

**Theorem (Getzler)**

*If \( B \) is the section algebra of a smooth one-parameter deformation with parameter space \( J \subset \mathbb{R} \), then there exists a \( \mathbb{C} \)-linear map \( \nabla_{GM} \) on the \( C^\infty(J) \)-module \( HP_{\bullet}(B) \), such that*

\[
\nabla_{GM}[f \cdot \omega] = f' \cdot [\omega] + f \cdot \nabla_{GM}[\omega], \quad \forall f \in C^\infty(J), [\omega] \in HP_{\bullet}(B).
\]
What is $\nabla_{GM}$?

As a vector space, the section algebra $\mathcal{B} \cong C^\infty(J, X)$. Let $\nabla$ be the operator $\frac{d}{dt}$ on $\mathcal{B}$. Consider the bilinear map $E : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ determined by

$$\nabla(ab) = \nabla(a)b + a\nabla(b) + E(a, b).$$

Given any multilinear map $D : \mathcal{B}^\times n \to \mathcal{B}$, Getzler and others constructed a noncommutative calculus of Lie derivative and contraction operators $L_D, I_D$ on the periodic cyclic complex of $\mathcal{B}$. In terms of these operators,

$$\nabla_{GM} = L_\nabla - I_E.$$

Using Getzler’s Cartan homotopy formula, it follows that

$$[b + B, \nabla_{GM}] = 0.$$
Integrating $\nabla_{GM}$

**Question**

*Can we integrate the connection $\nabla_{GM}$? That is, given $t_0, t_1 \in J$, can we use $\nabla_{GM}$ to parallel transport a homology class $[\omega] \in HP_\bullet(\mathcal{B}_{t_0})$ to a class $[P(\omega)] \in HP_\bullet(\mathcal{B}_{t_1})$?*

If the answer to this question is yes, then there exist parallel transport isomorphisms

$$P : HP_\bullet(\mathcal{B}_{t_0}) \cong HP_\bullet(\mathcal{B}_{t_1})$$

for all $t_0, t_1 \in J$.

However, periodic cyclic homology groups are not preserved for all such smooth deformations. Thus the answer, in general, is no. Our goal is to identify conditions under which the answer to this question is yes.
The main result

Let $\mathcal{A}$ denote the section algebra of the deformation $\{\mathcal{A}_\theta\}_{\theta \in J}$ of noncommutative tori. Let $J \subset \mathbb{T}$ be a subinterval such that $J \neq \mathbb{T}$.

**Theorem**

For any $\theta_0 \in J$, and any $[\omega_0] \in HP_\bullet(\mathcal{A}_{\theta_0})$, there exists a unique $[\omega] \in HP_\bullet(\mathcal{A})$ such that

$$\nabla_{GM}[\omega] = 0, \quad [\omega(\theta_0)] = [\omega_0].$$

Thus, we obtain isomorphisms $HP_\bullet(\mathcal{A}_{\theta_0}) \cong HP_\bullet(\mathcal{A}_{\theta_1})$ for any $\theta_0, \theta_1 \in J$. In particular, since $\mathcal{A}_0 \cong C^\infty(\mathbb{T}^2)$,

$$HP_0(\mathcal{A}_\theta) \cong HP_0(C^\infty(\mathbb{T}^2)) \cong H^0_{dR}(\mathbb{T}^2, \mathbb{C}) \oplus H^2_{dR}(\mathbb{T}^2, \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C},$$

$$HP_1(\mathcal{A}_\theta) \cong HP_1(C^\infty(\mathbb{T}^2)) \cong H^1_{dR}(\mathbb{T}^2, \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C}. $$
A few words about the proof

- $\nabla_{GM}$ is difficult to work with, so it is hard to prove this using $\nabla_{GM}$ directly.

- There is an auxiliary connection $\nabla$ on the periodic cyclic chain complex. After passing to a suitable chain equivalent subcomplex, $\nabla$ commutes with the boundary map $b + B$, and thus gives another connection on $HP_{\bullet}(\mathcal{A})$, which is easily seen to be integrable.

- The difference $\nabla_{GM} - \nabla$ is a nilpotent operator on $HP_{\bullet}(\mathcal{A})$. Thus the differential equations for $\nabla_{GM}$ can be solved in terms of the solutions to the differential equations for $\nabla$. 
Dual Gauss-Manin connection

Let $\mathcal{B}$ denote the section algebra of a smooth deformation $\{\mathcal{B}_t\}_{t \in J}$.

- By duality, there is a Gauss-Manin differentiation operator $\nabla^{GM}$ on the periodic cyclic cohomology $HP^\bullet(\mathcal{B})$. In terms of the canonical pairing

$$\langle \cdot, \cdot \rangle : HP^\bullet(\mathcal{B}) \times HP_\bullet(\mathcal{B}) \to C^\infty(J),$$

these operators are compatible in the sense that

$$\frac{d}{dt} \langle [\varphi], [\omega] \rangle = \langle \nabla^{GM} [\varphi], [\omega] \rangle + \langle [\varphi], \nabla^{GM} [\omega] \rangle.$$

- The group $HP^0(\mathcal{B})$ pairs with the K-theory group $K_0(\mathcal{B})$ via

$$\langle [\varphi], [e] \rangle := \langle [\varphi], [\text{ch } e] \rangle,$$

for an idempotent $e \in M_k(\mathcal{B})$. Such an idempotent $e$ is given by a smooth family of idempotents $\{e_t \in M_k(\mathcal{B}_t)\}_{t \in J}$. 
**Proposition**

For any idempotent $e \in M_k(\mathcal{B})$, the Chern character $[\text{ch } e] \in HP_0(\mathcal{B})$ satisfies

$$\nabla_{GM}[\text{ch } e] = 0$$

in $HP_0(\mathcal{B})$.

As a consequence, we obtain the differentiation formula

$$\frac{d}{dt} \langle [\varphi], [e] \rangle = \langle \nabla_{GM}^e [\varphi], [e] \rangle.$$
Cocycles in the rotation algebras

\[ \frac{d}{d\theta} \langle [\varphi], [e] \rangle = \langle \nabla^{GM} [\varphi], [e] \rangle. \]

Let us return to the example where \( \mathcal{A} \) is the section algebra of the rotation algebra deformation.

**Calculations of \( \nabla^{GM} \)**

Direct computations in \( \text{HP}^0(\mathcal{A}) \) show

- \( \nabla^{GM} [\tau_0] = -\frac{1}{2} [\tau_2] \)
- \( \nabla^{GM} [\tau_2] = 0 \)
Consequently, for any idempotent $e \in M_k(A)$,

\[
\frac{d}{d\theta} \tau_0(e) = -\frac{1}{2} \langle [\tau_2], [\text{ch } e] \rangle = \tau_2(e, e, e),
\]

\[
\frac{d^2}{d\theta^2} \tau_0(e) = 0.
\]

**Corollary**

Assuming $0 \in J$, any idempotent $e \in M_k(A)$ satisfies

\[
\tau_0(e)(\theta) = A + B\theta,
\]

where $A$ and $B$ are the integers

\[
A = \tau_0(e(0)), \quad B = \tau_2(e(0), e(0), e(0)).
\]

(Here, $e(0)$ is an idempotent in $M_k(C^\infty(\mathbb{T}^2))$.)
Consequences

- The conclusion $\tau_0(e)(\theta) = A + B\theta$ is interesting because it suggests the existence of idempotents in $M_k(A\theta)$ of trace $\theta$, e.g., the Powers-Rieffel projection.

- One consequence of this formula is that if the parameter space $J = \mathbb{T}$, then every idempotent in $M_k(A)$ has constant integer-valued trace. Indeed,

  \[ \tau_0(e)(\theta) = A + B\theta \in C^\infty(\mathbb{T}) \]

  implies $B = 0$.

- In contrast, if $J \subset \mathbb{T}$ is an open interval containing 0 and $J \neq \mathbb{T}$, then there exists an idempotent $e \in M_2(A)$ such that

  \[ \tau_0(e) = 1 + \theta. \]
Generalizations

- (Higher noncommutative tori) If $\mathcal{A}_\Theta$ is a noncommutative $n$-torus, then a similar argument shows that

$$HP_\bullet(\mathcal{A}_\Theta) \cong HP_\bullet(\mathcal{C}^\infty(T^n)).$$

- (Entire cyclic cohomology) The Gauss-Manin connection $\nabla^{GM}$ exists on the entire cyclic cohomology $HE^\bullet(\mathcal{A})$. The same argument shows that $HE^\bullet(\mathcal{A}_\Theta)$ is independent of $\Theta$. By a theorem of Puschnigg, the canonical map

$$HP^\bullet(\mathcal{C}^\infty(T^n)) \rightarrow HE^\bullet(\mathcal{C}^\infty(T^n))$$

is an isomorphism. Consequently, the canonical map

$$HP^\bullet(\mathcal{A}_\Theta) \rightarrow HE^\bullet(\mathcal{A}_\Theta)$$

is an isomorphism.
Thank you!
Odd cocycles

Similar calculations can be done with odd cocycles. For $j = 1, 2$, 

$$\varphi^j(a_0, a_1) = \frac{1}{2\pi i} \tau_0(a_0 \delta_j(a_1))$$

is a cyclic 1-cocycle. A computation shows that 

$$\nabla^{GM}[\varphi^j] = 0.$$ 

If $u \in M_k(\mathcal{A})$ is invertible, then 

$$\nabla_{GM}[\text{ch} \, u] = 0.$$ 

Consequently, 

$$\frac{d}{d\theta} \langle [\varphi^j], [\text{ch} \, u] \rangle = 0.$$ 

Thus, the function $\varphi^j(u^{-1}, u)$ is constant and integer-valued.