K-THEORY FOR CROSSED PRODUCTS AND ROTATION ALGEBRAS

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Abstract. In this paper, we define and discuss basic notions related to discrete crossed products $C^*$-algebras. Then we give a proof of the Pimsner-Voiculescu six-term exact sequence. This is the main tool used for computation of the K-theory groups of a crossed product $C^*$-algebra by the group $\mathbb{Z}$. Next, the rotation algebras $A_\theta$ are introduced and their basic properties are discussed. It is shown that $K_i(A_\theta) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $i = 0, 1$, and we conclude with an explicit calculation of the generators of these groups.

1. Discrete Crossed Products

Suppose $A$ is a $C^*$-algebra and $G$ is a discrete group acting on $A$ by automorphisms. That is, we have a group homomorphism $\alpha : G \to \text{Aut}(A)$. Let $\alpha_g$ denote the automorphism $\alpha(g)$ of $A$. We shall construct a new $C^*$ algebra built out of $A$ and the action of $G$.

To do this, let $AG$ be the vector space of finite formal sums of the form $\sum_{g \in G} a_g g$ where $a_g \in A$. Addition and scalar multiplication are defined by

$$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g$$

and

$$c(\sum_{g \in G} a_g g) = \sum_{g \in G} ca_g g, c \in \mathbb{C}.$$ 

Multiplication is determined by the requirement that $gag^{-1} = \alpha_g(a)$. So for finite sums, we have

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) = \sum_{g \in G} \sum_{h \in G} a_g gb_h h$$

$$= \sum_{g \in G} \sum_{h \in G} a_g gb_h g^{-1} gh$$

$$= \sum_{g \in G} \sum_{h \in G} a_g \alpha_g(b_h) gh$$

$$= \sum_{h \in G} \left(\sum_{g \in G} a_g \alpha_g(b_{g^{-1}h})\right) h$$

We give $AG$ the structure of a $*$-algebra with the adjoint defined by

$$(ag)^* = \alpha_{g^{-1}}(a^*) g^{-1}$$
and extended by linearity. Then \( \ast \) is a conjugate-linear involution and moreover,
\[
((ag)(bh))^* = (\alpha g(b)(gh))^* = \alpha_{(gh)^{-1}}(\alpha g(b))^* (gh)^{-1} = \alpha_{h^{-1}g^{-1}}(\alpha g(b^*a^*)^{-1} \cdot g^{-1}) = \alpha_{h^{-1}}(b^*)\alpha_{h^{-1}g^{-1}}(a^*)^{-1} g^{-1} = \alpha_{h^{-1}}(b^*)^{-1}a_{h^{-1}}(a^*)^{-1} g^{-1} = (bh)^*(ag)^*
\]
for all \( a, b \in A \) and \( g, h \in G \).

We define a C*-norm on \( AG \) by setting
\[
\|f\| = \sup_{\pi} \|\pi(f)\|
\]
where the supremum ranges over all *-representations of \( AG \). To show the supremum is bounded, first note that a *-representation \( \pi \) of \( AG \) gives rise to a representation \( \hat{\pi} \) of \( A \) defined by \( \hat{\pi}(a) = \pi(ea) \) where \( e \in G \) is the identity. We have that
\[
\|\pi(ae)\| = \|\hat{\pi}(a)\| \leq \|a\|.
\]
Moreover, by the C*-identity,
\[
\|\pi(ae)\| = \|\pi(a)\|^{1/2} \leq \|a\|^{1/2}
\]
Thus, we see that for any representation \( \pi \) of \( AG \),
\[
\left\| \sum_{g \in G} a_g g \right\| \leq \sum_{g \in G} \|\pi(a_g g)\| \leq \sum_{g \in G} \|a_g\|
\]
and consequently,
\[
\left\| \sum_{g \in G} a_g g \right\| \leq \sum_{g \in G} \|a_g\| < \infty.
\]

To show that the supremum in the norm is well defined, we will show that there actually are *-representations of \( AG \). Let \( \hat{\pi} \) be a representation of \( A \) on a Hilbert space \( \mathcal{H} \), which exists by the GNS construction. We will construct a representation of \( AG \) on the Hilbert space \( \mathcal{K} = \ell^2(G, \mathcal{H}) \) of square summable functions from \( G \) to \( \mathcal{H} \). We define a representation \( \pi : AG \rightarrow B(\mathcal{K}) \) by
\[
(\pi(ag)f)(h) = \hat{\pi}(\alpha_{h^{-1}}(a))(f(g^{-1}h))
\]
where \( f \in K \), and extend by linearity. We verify that \( \pi \) is \(*\)-preserving:

\[
\langle \pi((ag)^*)f, f' \rangle_K = \langle \pi(a_g^{-1}(a^*)g^{-1})f, f' \rangle_K = \sum_{h \in G} \langle (\pi(a_g^{-1}(a^*)g^{-1})f)(h), f'(h) \rangle_H
\]

\[
= \sum_{h \in G} \langle \hat{\pi}(\alpha_{h^{-1}g^{-1}}(a^*))(f(gh)), f'(h) \rangle_H
\]

\[
= \sum_{h \in G} \langle f(gh), \hat{\pi}(\alpha_{gh^{-1}}(a))(f'(h)) \rangle_H
\]

\[
= \sum_{h \in G} \langle f(h), \hat{\pi}(\alpha_{h^{-1}}(a))(g^{-1}h))(f') \rangle_H
\]

\[
= \sum_{h \in G} \langle f(h), (\pi(ag)f')(h) \rangle_H
\]

\[
= \langle f, \pi(ag)f' \rangle_K
\]

and so \( \pi((ag)^*) = \pi(ag)^* \).

Moreover, \( \pi \) preserves multiplication, as

\[
(\pi(ag)\pi(bh)f)(k) = \hat{\pi}(\alpha_{k^{-1}}(a))((\pi(bh)f)(g^{-1}k))
\]

\[
= \hat{\pi}(\alpha_{k^{-1}}(a))\hat{\pi}(\alpha_{k^{-1}g}(b))(f(h^{-1}g^{-1}k))
\]

\[
= \hat{\pi}(\alpha_{k^{-1}}(aa_g(b)))(f((gh)^{-1}k))
\]

\[
= (\pi(aa_g(b)(gh))f)(k)
\]

\[
= (\pi(abgh)f)(k).
\]

The \( C^* \)-algebra completion of \( AG \) in this norm is the crossed product \( A \rtimes \alpha G \). As a consequence of the definition of the norm, every \(*\)-representation \( \pi \) of \( AG \) extends uniquely by continuity to a representation \( \hat{\pi} \) of \( A \rtimes \alpha G \).

Remark. The crossed product \( A \rtimes \alpha G \) that we constructed is called the full crossed product. When defining the norm in \( AG \), if one only uses representations of the form \((1), \) then upon completing to a \( C^* \)-algebra, one obtains what is called the reduced crossed product. In general, the full and reduced crossed products may differ, but it is a theorem that if \( G \) is an amenable group, then they are equal.

1.1. Crossed Products by \( Z \). Consider the case where \( A \) is unital with unit 1 and \( G = Z \). Let \( e \in Z \) denote the identity and let \( g \) denote the generator. The action of \( Z \) on \( A \) is determined by the automorphism \( \alpha(g) \), which we denote by \( \alpha \). There is an isometric embedding \( i : A \to A \rtimes \alpha Z \) given by \( i(a) = ae \). Let \( u \in A \rtimes \alpha Z \) be the unitary defined by \( u = 1g \). Then since every element of \( AZ \) is a finite sum of the form \( \sum_n a_n g^n \), it is easily seen that the image \( i(A) \) and the unitary \( u \) generate \( AZ \). So as a \( C^* \)-algebra, \( A \rtimes \alpha Z \) is generated by a copy of \( A \) and a unitary \( u \) such that \( uau^* = \alpha(a) \) for all \( a \in A \). In fact, more than that is true:

**Proposition 1.1.** \( A \rtimes \alpha Z \) is the universal \( C^* \)-algebra generated by \( A \) and a unitary \( u \) subject to the relation \( uau^* = \alpha(a) \) for all \( a \in A \). In other words, given a \( C^* \)-algebra \( B \), a unitary \( v \in B \) and a \(*\)-homomorphism \( \phi : A \to B \) such that
vφ(a)v* = φ(α(a)) for all a ∈ A, there is a unique surjective *-homomorphism ψ : A ×₀ Z → C*(φ(A), v) such that φ = ψ ∘ i and ψ(u) = v.

Proof. Let π : B → B(H) be a faithful representation, which exists by the Gelfand-Naimark Theorem. We define a representation σ : A ⊗ B → B(H) by σ(a ⊗ b)(x) = a ⊗ π(b)x ⊗ 1 for all a ∈ A, b ∈ B, x ∈ B(H). We verify that σ is multiplicative:

\[ σ(a ⊗ b)(x ⊗ y) = σ(a ⊗ b)(x ⊗ y) = σ(a ⊗ b)v ⊗ 1 \]

A representation σ of AG extends uniquely by continuity to a representation ˜σ of A ×₀ Z. We define ψ : A ×₀ Z → B by ψ = π⁻¹ ∘ ˜σ. Then

\[ (ψ ∘ i)(a) = ψ(ae) = π⁻¹(˜σ(1g)) = π⁻¹(π(φ(a))) = φ(a) \]

and also

\[ ψ(u) = π⁻¹(˜σ(1g)) = π⁻¹(π(v)) = v. \]

The uniqueness of ψ follows from the requirements that ψ(1) = φ(1) and ψ(u) = v combined with the fact that Ae and u generate AZ, which is dense in A ×₀ Z.

2. The Pimsner-Voiculescu Six-Term Exact Sequence

In this section, we give a proof of the Pimsner-Voiculescu six-term exact sequence. It is a particular exact sequence in K-theory that allows computation of the K-theory groups of crossed products by Z.

2.1. The Extension. Suppose A is a unital C*-algebra and let α be an automorphism of A. By Proposition 1.1, A ×₀ Z is the universal C*-algebra generated by A and a unitary u such that uαu* = α(u) for all u ∈ A. Let T denote the Toeplitz algebra, which is the universal C*-algebra generated by an isometry v. Let K denote the C*-algebra of compact operators on an infinite dimensional separable Hilbert space. The first step in proving the Pimsner-Voiculescu Theorem is to construct a short exact sequence of C*-algebras:

\[ 0 \rightarrow K ⊗ A \rightarrow T_α \rightarrow A ×₀ Z \rightarrow 0 \]

where T_α is the C*-subalgebra of (A ×₀ Z) ⊗ T generated by A ⊗ 1 and u ⊗ v.

2.1.1. The *-homomorphisms. In T_α, we will write a to denote the element a ⊗ 1 for a ∈ A. Let V = u ⊗ v and P = 1 − VV* = 1 ⊗ (1 − vv*). We have the following basic algebraic results in T_α.

Lemma 2.1. The following hold in T_α:

1. V is an isometry.
2. P is a projection.
3. Va = α(a)V for any a ∈ A.
4. V*a = α⁻¹(a)V* for any a ∈ A.
5. aP = Pa for any a ∈ A.
Proof. For (1), \( V^*V = (u^* \otimes v^*)(u \otimes v) = u^*u \otimes v^*v = 1 \otimes 1 \). Since \( V \) is an isometry, \( VV^* \) is a projection and hence \( P = 1 - VV^* \) is a projection. For (3), we have
\[
V a = (u \otimes v)(a \otimes 1) = ua \otimes v = uaa^*u \otimes v = a(a)u \otimes v = (a(a) \otimes 1)(u \otimes v) = a(a)V.
\]
For (4), let \( b = a^{-1}(a^*) \). Then by taking the adjoint equation of (3), we have \( V^*a = V^*a(b^*) = b^*V^* = a^{-1}(a)V^* \). Finally, for (5), we have \( aP = (a \otimes 1)(1 \otimes 1 - vv^*) = a \otimes (1 - vv^*) = (a \otimes 1 - vv^*)(a \otimes 1) = Pa. \)

In \( \mathcal{K} \), let \( e_{i,j} \) denote the usual matrix units for \( i, j \geq 0 \), that is, \( e_{i,j} = \delta_{i,j}e_{i,i} \) and \( e_{i,j}^* = e_{j,j} \). In \( \mathcal{T}_\alpha \), we define “matrix units” by \( e_{i,j}(a) = \alpha^i(a)V^iPV^j \) for \( i, j \geq 0 \).

Lemma 2.2. There is an injective \( * \)-homomorphism \( \varphi_n : M_n(\mathbb{C}) \otimes A \to \mathcal{T}_\alpha \) such that \( \varphi(e_{i,j} \otimes a) = e_{i,j}(a) \).

Proof. Define \( \varphi_n : M_n(\mathbb{C}) \to \mathcal{T}_\alpha \) by \( \varphi(e_{i,j} \otimes a) = e_{i,j}(a) \) and extend by linearity. Then \( \varphi \) is multiplicative because
\[
e_{i,j}(a)e_{j,k}(b) = \alpha^i(a)V^iPV^j\alpha^j(b)V^jPV^k = \alpha^i(a)V^iPbV^jV^jPV^k = \alpha^i(a)V^iPbPV^k = \alpha^i(a)\alpha^j(b)V^iPV^k = \alpha^i(ab)V^iPV^k = e_{i,k}(ab).
\]

When \( j \neq k \),
\[
e_{i,j}(a)e_{k,l}(b) = \alpha^i(a)V^iPV^j\alpha^k(b)V^kPV^l = \alpha^i(a)V^iPbV^jV^kPV^l = \alpha^i(a)V^iPbPV^jV^kPV^l = 0
\]
because \( V^*P = PV = 0 \), and one of those two appears in that expression when \( j \neq k \). We also have
\[
e_{i,j}(a)^* = (\alpha^i(a)V^iPV^j)^* = V^jPV^j\alpha^j(a^*) = V^jP\alpha^jV^j = V^j\alpha^jPV^j = \alpha^j(a^*)V^jPV^j = e_{j,i}(a^*)
\]
Moreover, \( \varphi_n \) is injective because if \( \varphi_n(x) = 0 \), where \( x = \sum_{i,j} c_{i,j} \otimes a_{i,j} \), then

\[
0 = \varphi_n(x) = \varphi_n\left( \sum_{i,j} c_{i,j} \otimes a_{i,j} \right) = \sum_{i,j} \alpha^i(a_{i,j})V^iPV_j^* \]

Since the elements \( v^i(1 - vv^*)v^j \) are linearly independent in \( \mathcal{T} \), we conclude that \( \alpha^i(a_{i,j})u^{i-j} = 0 \) for each \( i, j \). Since \( \alpha \) is an automorphism and \( u \) is a unitary, then \( a_{i,j} = 0 \) for all \( i, j \). Therefore, \( x = 0 \) and \( \varphi_n \) is injective as claimed.

**Lemma 2.3.** There is a unique injective \( * \)-homomorphism \( \varphi : \mathcal{K} \otimes A \rightarrow \mathcal{T}_d \) such that \( \varphi(e_{i,j} \otimes a) = e_{i,j}(a) \).

**Proof.** The \( C^* \)-algebra \( \mathcal{K} \otimes A \) is the direct limit of the sequence \( M_n(\mathbb{C}) \otimes A \) under the inclusion maps

\[
M_1(\mathbb{C}) \otimes A \hookrightarrow M_2(\mathbb{C}) \otimes A \hookrightarrow M_3(\mathbb{C}) \otimes A \hookrightarrow \ldots \rightarrow \mathcal{K} \otimes A.
\]

Let \( j_n : M_n(\mathbb{C}) \otimes A \rightarrow M_{n+1}(\mathbb{C}) \otimes A \) and \( k_n : M_n(\mathbb{C}) \otimes A \rightarrow \mathcal{K} \otimes A \) denote the inclusion maps. Notice that \( \varphi_n = \varphi_{n+1} \circ j_n \). By the universal property of the direct limit, there is a unique \( * \)-homomorphism \( \varphi : \mathcal{K} \otimes A \rightarrow \mathcal{T}_d \) such that \( \varphi_n = \varphi \circ k_n \).

We will show that \( \varphi \) is isometric. Notice that since \( \varphi_n \) is an injective \( * \)-homomorphism, it is isometric. Let \( x \in \mathcal{K} \otimes A \) and let \( x = \lim x_k \) where \( x_k \in M_n(\mathbb{C}) \otimes A \). Then

\[
\|\varphi(x)\| = \lim \|\varphi(x_k)\| = \lim \|\varphi(x_k)\| = \lim \|\varphi_n(x_k)\| = \|x_k\| = \|x\|.
\]

Since \( \varphi \) is isometric, it is injective.

**Lemma 2.4.** There is a unique \( * \)-homomorphism \( \psi : \mathcal{T}_d \rightarrow A \times_\alpha \mathbb{Z} \) such that \( \psi(a) = a \) for all \( a \in A \) and \( \psi(V) = u \).

**Proof.** Let \( f : \mathcal{T} \rightarrow A \times_\alpha \mathbb{Z} \) be the unique \( * \)-homomorphism given by \( f(v) = 1 \). By the universal property of the tensor product, there is a unique \( * \)-homomorphism \( \text{id} \otimes f : (A \times_\alpha \mathbb{Z}) \otimes \mathcal{T} \rightarrow A \times_\alpha \mathbb{Z} \) such that \( \text{id} \otimes f : x \otimes y \rightarrow x f(y) \) for \( x \in A \times_\alpha \mathbb{Z} \), \( y \in \mathcal{T} \).

Let \( \psi \) be the restriction of \( \text{id} \otimes f \) to \( \mathcal{T}_d \). Then \( \psi(a) = \psi(a \otimes 1) = a f(1) = a \) and \( \psi(V) = \psi(u \otimes v) = u f(v) = u \). Since \( A \otimes 1 \) and \( u \otimes v \) generate \( \mathcal{T}_d \), any other \( * \)-homomorphism that agrees with \( \psi \) on this generating set must agree with \( \psi \) everywhere. Thus, \( \psi \) is unique.

2.1.2. The Short Exact Sequence. We will show that

\[
0 \rightarrow \mathcal{K} \otimes A \xrightarrow{\varphi} \mathcal{T}_d \xrightarrow{\psi} A \times_\alpha \mathbb{Z} \rightarrow 0
\]

is a short exact sequence. By Lemmas 2.3 and 2.4, the sequence is exact at \( \mathcal{K} \otimes A \) and \( A \times_\alpha \mathbb{Z} \). It remains to be shown that the sequence is exact at \( \mathcal{T}_d \).

Let \( S_d \) denote that dense \( * \)-subalgebra of \( \mathcal{T}_d \) generated by all elements of \( A \) and the element \( V \).

**Lemma 2.5.** Every element of \( S_d \) can be written uniquely as a finite sum \( \sum_{i,j} a_{i,j} V^i V_j^* \).
Proof. Every element of $S_\alpha$ is a finite sum of words constructed from $V_i V^*$, and elements of $A$, so it suffices to prove the result for each of these words. If $x$ is such a word, then by Lemma 2.1 parts (3) and (4), $x = ay$, where $a \in A$ and $y$ is a word in $V$ and $V^*$. Since $V^* V = 1$, any such $y$ can be written as $V^i V^* j$ for some $i$ and $j$.

To show uniqueness, suppose $x \in S_\alpha$, and $x = \sum_{i,j} a_{i,j} V^i V^* j = \sum_{i,j} b_{i,j} V^i V^* j$. Then

\[
0 = \sum_{i,j} a_{i,j} V^i V^* j - \sum_{i,j} b_{i,j} V^i V^* j = \sum_{i,j} (a_{i,j} - b_{i,j}) V^i V^* j = \sum_{i,j} (a_{i,j} - b_{i,j}) u^i j \otimes v^i j
\]

Since the elements $u^i j \otimes v^i j$ are linearly independent in $T_\alpha (a_{i,j} - b_{i,j}) u^i j = 0$ for all $i,j$. Because $u$ is a unitary, we conclude that $a_{i,j} = b_{i,j}$ for all $i,j$. \hfill $\square$

Let $(P)_{alg}$ denote the two-sided ideal generated by $P$ in $S_\alpha$, and let $(P)$ be the closed two-sided ideal generated by $P$ in $T_\alpha$. Then the closure of $(P)_{alg}$ in $T_\alpha$ is $(P)$. This is because any finite sum of the form $\sum_{i,j} x_i P y_j$, where $x_i, y_j \in T_\alpha$ can be approximated by an element in $(P)_{alg}$ by choosing suitable approximations in $S_\alpha$ for each $x_i$ and $y_j$. Hence the ideal generated by $P$ in $T_\alpha$ contains the closure of $(P)_{alg}$. Therefore the closure of $(P)_{alg}$ is $(P)$.

Lemma 2.6. In $T_\alpha$, the ideal $(P)$ is the image $\varphi(K \otimes A)$.

Proof. For any elementary tensor of the form $x = e_{i,j} \otimes a \in K \otimes A$, $\varphi(x) = \alpha^i(a)V^i V^* j \in (P)$. Since $K \otimes A$ is the closure of the span of elements of this form, $\varphi(K \otimes A) \subseteq (P)$, as $(P)$ is a closed ideal.

Conversely, suppose $z \in (P)_{alg}$. Then $z$ is a finite sum of the form $z = \sum_n x_n P y_n$. By Lemma 2.5, for each $n$ we can write $x_n = \sum_{i,j} a_{i,j,n} V^i V^* j$ and $y_n = \sum_{k,l} b_{k,l,n} V^k V^* l$. So

\[
z = \sum_n (\sum_{i,j} a_{i,j,n} V^i V^* j) P (\sum_{k,l} b_{k,l,n} V^k V^* l) = \sum_{n,i,j,k,l} a_{i,j,n} V^i V^* j P b_{k,l,n} V^k V^* l.
\]

So to show $z \in \varphi(K \otimes A)$, it suffices to show that $z_0 = aV^i V^* j P b V^k V^* l \in \varphi(K \otimes A)$ for $a, b \in A$. Using Lemma 2.1, $z = aV^i V^* j P b V^k V^* l = a\alpha^i j (b) V^i V^* j P V^k V^* l$. Unless $j = k = 0$, then $V^j P V^k = 0$ and $z_0 = 0$. So we may assume $j = k = 0$, and $z_0 = a\alpha^{i-j} (b) V^i V^* l = \varphi(e_{1,1} \otimes \alpha^{-1}(a) \alpha^{-j} (b))$. This shows $(P)_{alg} \subseteq \varphi(K \otimes A)$. Since the closure of $(P)_{alg}$ is $(P)$, and the image of a $*$-homomorphism is closed, $(P) \subseteq \varphi(K \otimes A)$. \hfill $\square$

Proposition 2.7. The sequence

\[
0 \longrightarrow K \otimes A \overset{\varphi}{\longrightarrow} T_\alpha \overset{\psi}{\longrightarrow} A \times_\alpha Z \longrightarrow 0
\]
is exact.

Proof. It is clear that $\text{im } \varphi \subseteq \text{ker } \psi$, because $\varphi(K \otimes A) = (P)$ and $\psi(P) = \psi(1 - VV^*) = 1 - uu^* = 0$.

To show the converse, we will first construct a *-homomorphism $\rho : A \times_\alpha \mathbb{Z} \to \mathcal{T}_\alpha/(P)$. Note that in $\mathcal{T}_\alpha$,

$$V^*V - VV^* = 1 - VV^* = P,$$

and

$$\alpha(a) - VaV^* = \alpha(a) - \alpha(a)VV^* = \alpha(a)(1 - VV^*) = \alpha(a)P.$$ 

So $V$ is unitary mod $(P)$, and $VaV^* = \alpha(a)$ mod $(P)$. Thus, there is a unique *-homomorphism $\rho : A \times_\alpha \mathbb{Z} \to \mathcal{T}_\alpha/(P)$ such that $\rho(a) = a + (P)$ and $\rho(u) = V + (P)$.

Consider the following commutative diagram:

$$\begin{array}{cccc}
0 & \longrightarrow & K \otimes A & \xrightarrow{\varphi} & \mathcal{T}_\alpha & \xrightarrow{\psi} & A \times_\alpha \mathbb{Z} & \longrightarrow & 0 \\
\varphi \downarrow \cong & & & & \rho \downarrow & & & & \\
0 & \longrightarrow & (P) & \xrightarrow{i} & \mathcal{T}_\alpha & \xrightarrow{p} & \mathcal{T}_\alpha/(P) & \longrightarrow & 0
\end{array}$$

If $x \in \text{ker } \psi$, then $p(x) = \rho(\psi(x)) = 0$, and so $x \in (P) = \text{im } \varphi$. We conclude that $\ker \psi = \text{im } \varphi$.

\[\square\]

2.2. The Isomorphism.

2.2.1. Quasihomomorphisms. Let $A$ and $B$ be two C*-algebras. For our purposes, a quasihomomorphism from $A$ to $B$ is a pair of *-homomorphisms $(f_0, f_1)$ from $A$ to $E$, where $E$ is a C*-algebra containing $K \otimes B$ as an ideal, such that $f_0(a) - f_1(a) \in K \otimes B$ for all $a \in A$. We will write $(f_0, f_1) : A \to E \triangleright K \otimes B$ for such a quasihomomorphism.

Given a quasihomomorphism $(f_0, f_1) : A \to E \triangleright K \otimes B$, one can construct a K-theory map $(f_0, f_1)_* : K_i(A) \to K_i(B)$ as follows. Let $\overline{E}$ be the C*-subalgebra of $A \oplus E$ given by $\overline{E} = \{(a, e) | f_0(a) = e \text{ mod } K \otimes B\}$. Define the maps $\iota : K \otimes B \to \overline{E}$ and $\pi : \overline{E} \to A$ by $\iota(x) = (0, x)$ and $\pi(a, e) = a$. Then we have the following split exact sequence:

$$0 \longrightarrow K \otimes B \xrightarrow{\iota} \overline{E} \xrightarrow{\pi} A \longrightarrow 0$$

with two splittings $f_0, f_1 : A \to \overline{E}$ given by $f_0(a) = (a, f_0(a))$ and $f_1(a) = (a, f_1(a))$.

So there is a split exact sequence

$$0 \longrightarrow K_i(K \otimes B) \xrightarrow{\iota_*} K_i(\overline{E}) \xrightarrow{\pi_*} K_i(A) \longrightarrow 0.$$

Since $\pi \circ f_0 = \text{id} = \pi \circ f_1$, we have that $\pi_* f_0_* = \pi_* f_1_*$ and so $\pi_*(f_0_* - f_1_*) = 0$. Thus for any element $x \in K_i(A)$, $f_0_*(x) - f_1_*(x) \in \text{ker } \pi_*$. So $f_0_*(x) - f_1_*(x) = \iota_*(y)$ for some unique $y \in K_i(K \otimes B)$. We define $(f_0, f_1)_*(x) = \kappa_B^{-1}(y)$ where $\kappa_B : B \to K \otimes B$ is the canonical inclusion.
2.2.2. The inclusion. Our aim is to show that the inclusion \( j : A \to \mathcal{T}_\alpha \) induces an isomorphism \( j_* : K_i(A) \to K_i(\mathcal{T}_\alpha) \).

**Lemma 2.8.** The pair of \( * \)-homomorphisms \((\text{id}, \text{Ad}(1 \otimes v)) : \mathcal{T}_\alpha \to \mathcal{T}_\alpha \rtimes \mathcal{K} \otimes A\) is a quasihomomorphism from \( \mathcal{T}_\alpha \) to \( A \).

**Proof.** For the element \( x = aV^iV^*j = au^{i-j} \otimes v^j v^*j \in \mathcal{T}_\alpha \), we have
\[
\begin{align*}
\text{id}(x) - \text{Ad}(1 \otimes v)(x) &= \text{id}(au^{i-j} \otimes v^j v^*j - au^{i-j} \otimes v^{i+1} v^*j+1) \\
&= a(Y^{i-j} \otimes (v^j v^*j) - v^{i+1} v^*j+1) \\
&= a(Y^{i-j} \otimes v^j(1 - vv^*) v^*j) \\
&= aV^iPV^*j \in K \otimes A
\end{align*}
\]

Since \( \text{id} - \text{Ad}(1 \otimes v) \) is linear and continuous, then for any \( x \in \mathcal{T}_\alpha \), \( \text{id}(x) - \text{Ad}(1 \otimes v)(x) \in \mathcal{K} \otimes A \) as \( \mathcal{T}_\alpha \) is the closure of the span of elements of the form \( aV^iV^*j \). \( \square \)

Let \( \hat{T} \) denote the C*-subalgebra of \( \mathcal{T} \otimes \mathcal{T} \) generated by \( \mathcal{K} \otimes \mathcal{T} \) and \( \mathcal{T} \otimes 1 \). In \( \hat{T} \), let \( \hat{v} = v \otimes 1, \hat{p} = 1 - \hat{v}\hat{v}^* \), and \( \hat{v}' = (1 - vv^*) \otimes v \). The following lemma is due to Cuntz.

**Lemma 2.9.** (Cuntz) There is a continuous path \( \phi_t, t \in [-1,1] \) of \( * \)-homomorphisms from \( \mathcal{T} \) to \( \hat{T} \) such that
- \( \phi_{-1}(v) = \hat{v}(1 - \hat{p}) + \hat{v}' \)
- \( \phi_0(v) = \hat{v} \)
- \( \phi_1(v) = \hat{v}(1 - \hat{p}) + \hat{p} \)

Let \( \hat{\mathcal{T}}_\alpha \) be the C*-subalgebra of \( A \times_\alpha \mathbb{Z} \otimes \hat{T} \) generated by \( A \otimes 1 \) and the elements \( \hat{V} := u \otimes \hat{v} \) and \( V' := u \otimes v' \). Let \( \hat{\mathcal{P}} = 1 - \hat{V}\hat{V}^* \). We will construct an embedding of \( \mathcal{K} \otimes \mathcal{T}_\alpha \) into \( \hat{\mathcal{T}}_\alpha \).

**Lemma 2.10.** There is an injective \( * \)-homomorphism \( \hat{\phi} : \mathcal{K} \otimes \mathcal{T}_\alpha \to \hat{\mathcal{T}}_\alpha \).

**Proof.** We first consider the \( * \)-homomorphism \( k' : \mathcal{T}_\alpha \to \hat{\mathcal{T}}_\alpha \) given by \( k'(a) = \hat{a} \mathcal{P} \) for \( a \in A \) and \( k'(V) = V' \). It is evident that this is a \( * \)-homomorphism when the elements of \( \mathcal{T}_\alpha \) and \( \hat{\mathcal{T}}_\alpha \) are viewed as tensors. Then for the generators of \( \mathcal{T}_\alpha \), we have
\[
k' : a \otimes 1 \mapsto a \otimes p \otimes 1,
\]
where \( p = 1 - vv^* \) is the rank one projection in \( \mathcal{T} \).

Now define the “matrix units” \( e_{i,j}(x) = \hat{V}^*k'(x)\hat{V}^*j \) for an element \( x \in \mathcal{T}_\alpha \). Since \( k'(x)\hat{V} = \hat{V}^*k'(x) = 0 \), we have that \( e_{i,j}(x)e_{k,l}(y) = \delta_{j,k}e_{i,l}(xy) \) and \( e_{i,j}(x)^* = e_{j,i}(x^*) \). Then \( \hat{\phi} : \mathcal{K} \otimes \mathcal{T}_\alpha \to \hat{\mathcal{T}}_\alpha \) is given by \( \phi(c_{e_{i,j}} \otimes x) = e_{j,i}(x) \) and extended by linearity and continuity. The proof that it is an injective \( * \)-homomorphism is similar to the proof of Lemma 2.3. \( \square \)

Let \( k : \mathcal{T}_\alpha \to \hat{\mathcal{T}}_\alpha \) be given by \( k = \text{id} \otimes \phi_0 \) where \( \phi_0 \) is as in Proposition 2.9.

**Lemma 2.11.** The pair of \( * \)-homomorphisms \((k, k \circ \text{Ad}(1 \otimes v)) : \mathcal{T}_\alpha \to \hat{\mathcal{T}}_\alpha \rtimes \mathcal{K} \otimes \mathcal{T}_\alpha \) is a quasihomomorphism from \( \mathcal{T}_\alpha \) to \( \mathcal{T}_\alpha \).
Proof. This is due to Lemma 2.8 and the fact that the following diagram commutes:

\[
\begin{array}{c}
\mathcal{K} \otimes A \xrightarrow{\varphi} \mathcal{T}_\alpha \\
\id \otimes f \downarrow \quad \downarrow k \quad \downarrow \varphi \\
\mathcal{K} \otimes \mathcal{T}_\alpha \xrightarrow{\hat{\varphi}} \hat{\mathcal{T}}_\alpha
\end{array}
\]

\[\square\]

**Theorem 2.12.** The inclusion \( j : A \to \mathcal{T}_\alpha \) induces an isomorphism \( j_* : K_1(A) \to K_1(\mathcal{T}_\alpha) \).

**Proof.** We will show that the inverse of \( j_* \) is \( q := (\id, \Ad(1 \otimes v))_* \). To show that \( q \circ j_* = \id_{K_1(A)} \), we first claim that \( j_* - (\Ad(1 \otimes v) \circ j)_* = \varphi_* \circ \kappa_* \) where \( \kappa_* : A \to \mathcal{K} \otimes A \) is the canonical embedding. For \( a \in A \), \( \Ad(1 \otimes v)(j(a)) + \varphi(\kappa_* (a)) = a \otimes vv^* + a \otimes 1 - vv^* = a \otimes 1 = j(a) \), so \( j_* - (\Ad(1 \otimes v) \circ j)_* \) is well-defined, for if \( x \in K_1(A) \), then \( \varphi_* \circ \kappa_* (x) = \varphi_* (\kappa_* (x)) = 0 \). Thus, \( j_* = (\Ad(1 \otimes v) \circ j + \varphi \circ \kappa)_* = (\Ad(1 \otimes v) \circ j)_* + (\varphi \circ \kappa) \). So for a class \( x \in K_1(A) \), since \( (\id - \Ad(1 \otimes v))_* (j_* (x)) = j_*(x) - \Ad(1 \otimes v)_*(j_*(x)) = \varphi_*(\kappa_*(x)) \), we have \( q_*(j_*(x)) = \kappa_*^{-1}(\kappa_*(x)) = x \). Thus \( q \circ j_* = \id_{K_1(A)} \).

To show that \( j_* \circ q = \id_{K_1(\mathcal{T}_\alpha)} \), we will first show that \( j_* \circ q = (k, k \circ \Ad(1 \otimes v))_* \). Let \( \bar{q} = (k, k \circ \Ad(1 \otimes v))_* \). Let \( E = \{(x, y) \in \mathcal{T}_\alpha \otimes \mathcal{T}_\alpha | x = y \mod \mathcal{K} \otimes A \} \) and \( F = \{(x, y) \in \mathcal{T}_\alpha \otimes \mathcal{T}_\alpha | k(x) = y \mod \mathcal{K} \otimes \mathcal{T}_\alpha \} \). Define \( f_0, f_1 : \mathcal{T}_\alpha \to E \) by \( f_0(x) = (x, x) \), \( f_1(x) = (x, \Ad(1 \otimes v)(x)) \) and define \( g_0, g_1 : \mathcal{T}_\alpha \to F \) by \( g_0(x) = (x, k(x)) \), \( g_1(x) = (x, (k \circ \Ad(1 \otimes v))(x)) \). Let \( \bar{k} : E \to F \) be given by \( k(x, y) = (x, k(y)) \). This is well-defined, for if \( x = y \mod \mathcal{K} \otimes A \), then \( k(x, y) = (x, k(y)) \) mod \( \mathcal{K} \otimes \mathcal{T}_\alpha \) because the following diagram commutes:

\[
\begin{array}{c}
\mathcal{K} \otimes A \xrightarrow{\varphi} \mathcal{T}_\alpha \\
\id \otimes f \downarrow \quad \downarrow k \quad \downarrow \varphi \\
\mathcal{K} \otimes \mathcal{T}_\alpha \xrightarrow{\hat{\varphi}} \hat{\mathcal{T}}_\alpha
\end{array}
\]

For \( n = 0, 1 \) we have the commutative diagrams

\[
\begin{array}{c}
0 \longrightarrow \mathcal{K} \otimes A \xrightarrow{0 \otimes \varphi} E \xrightarrow{\pi} \mathcal{T}_\alpha \longrightarrow 0 \\
\id \otimes f \downarrow \quad \downarrow k \quad \downarrow \pi \\
0 \longrightarrow \mathcal{K} \otimes \mathcal{T}_\alpha \xrightarrow{0 \otimes \varphi} F \xrightarrow{\hat{\pi}} \mathcal{T}_\alpha \longrightarrow 0 \\
\end{array}
\]

and at the level of K-theory,
For $x \in K_i(T_\alpha)$, there is a unique $y \in K_i(K \otimes A)$ such that

$$\tilde{f}_{0*}(x) - \tilde{f}_{1*}(x) = (0 \otimes \varphi)_*(y)$$

and so $q(x) = \kappa_{A_*}^{-1}(y)$. Applying $k_*$, we get

$$g_{0*}(x) - g_{1*}(x) = k_*((0 \otimes \varphi)_*(y)) = (0 \otimes \varphi)_*((id \otimes j)_*(y))$$

and so $\tilde{q}(x) = \kappa_{\tau_\alpha_*}^{-1}((id \otimes j)_*(y))$. Since the diagram

$$\begin{array}{ccc}
K_i(A) & \xrightarrow{j_*} & K_i(T_\alpha) \\
\kappa_{A_*} & \cong & \kappa_{\tau_\alpha_*} \\
K_i(K \otimes A) & \xrightarrow{(id \otimes j)_*} & K_i(K \otimes T_\alpha)
\end{array}$$

we have that

$$\tilde{q}(x) = \kappa_{\tau_\alpha_*}^{-1}((id \otimes j)_*(y)) = j_*(\kappa_{A_*}^{-1}(y)) = j_*(q(x)).$$

So $\tilde{q} = j_* \circ q$ as claimed.

To finish the proof, we will show that $\tilde{q} = id_{K_i(T_\alpha)}$ is an isomorphism. Let $k'' : T_\alpha \to \tilde{T}_\alpha$ be the restriction of $id \otimes \phi_-$ where $\phi_-$ is as in Lemma 2.9. Recalling the definition of $k$, we see that $k$ is homotopic to $k''$ by Lemma 2.9. We claim that $k \circ \text{Ad}(1 \otimes v)$ and $k'$ are orthogonal $*$-homomorphisms. This is a consequence of the fact that

$$\begin{align*}
(k \circ \text{Ad}(1 \otimes v))(V^i a_1 V^{*j}) &\left( k'(V^k a_2 V^{*l}) \right) \\
&= k(u^i a_1 u^{*j} \otimes v^{i+1} v^{*j+1}) V^{*k} a_2 \hat{P} V^{*l} \\
&= k((u^i \otimes v^j)(a_1 \otimes v^*) (u^{*j} \otimes v^*)) V^{*k} a_2 \hat{P} V^{*l} \\
&= k(V^i a_1 (1 - \hat{P}) V^{*j}) V^{*k} a_2 \hat{P} V^{*l} \\
&= V^i a_1 (1 - \hat{P}) V^{*j} V^{*k} a_2 \hat{P} V^{*l} \\
&= 0
\end{align*}$$
because \( \hat{V}^*V' = 0, \hat{V}^*P = 0, (1 - \hat{P})V' = 0, \) and \( a_2\hat{P} = \hat{P}u_2. \) Moreover, we check that for \( a \in A, \)

\[
k''(a) = a
= a(1 - \hat{P}) + a\hat{P}
= (k \circ \text{Ad}(1 \otimes v))(a) + k'(a)
\]

and

\[
k''(V) = k''(u \otimes v)
= u \otimes (\hat{v}(1 - \hat{p}) + v')
= u \otimes \hat{v}(\hat{v}^*) + u \otimes v'
= (1 \otimes \hat{v})(u \otimes \hat{v}^*)(1 \otimes \hat{v}^*) + u \otimes v'
= (k \circ \text{Ad}(1 \otimes v))(V) + k'(V)
\]

So \( k \circ \text{Ad}(1 \otimes v) + k' = k'' \) and \( k_* = k''_* = (k \circ \text{Ad}(1 \otimes v))_* + k'_*. \) Note that \( k' = \hat{\varphi} \circ \kappa_{\tau_a} \) and so we have \( k_* - (k \circ \text{Ad}(1 \otimes v))_* = k'_* = \hat{\varphi}_* \circ \kappa_{\tau_a} \). For any \( x \in K_i(\tau_a), k_*(x) - (k \circ \text{Ad}(1 \otimes v))(x) = \hat{\varphi}_*(\kappa_{\tau_a}(x)) \).

Thus we have shown that \( j_* \circ q = \hat{q} = \text{id}. \)

2.3. The Pimsner-Voiculescu Six-Term Exact Sequence. We begin this section with a simple lemma.

**Lemma 2.13.** Consider the following commutative diagram of abelian groups,

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi'} & B \\
\downarrow f & & \downarrow g \\
D & \xrightarrow{\varphi} & E \\
\downarrow h & & \downarrow \psi \\
 & C & F
\end{array}
\]

where \( f, g, \) and \( h \) are isomorphisms. If the bottom row is exact at \( E, \) then the top row is exact at \( B. \)

**Proof.** We have \( \psi' \circ \varphi' = h^{-1} \circ \psi \circ g \circ g^{-1} \circ \varphi \circ f = h^{-1} \circ \psi \circ \varphi \circ f = 0 \) because \( \psi \circ \varphi = 0. \) Thus the im \( \varphi' \subseteq \ker \psi'. \)

Suppose that \( x \in \ker \psi'. \) Then

\[
0 = h(0) = h(\psi'(x)) = \psi(g(x))
\]

and so \( g(x) \in \ker \psi. \) Therefore, \( g(x) = \varphi(y) \) for some \( y \in D. \) Then \( f^{-1}(y) \in A, \) and

\[
\varphi'(f^{-1}(y)) = (g^{-1} \circ \varphi \circ f)(f^{-1}(y)) = (g^{-1} \circ \varphi)(y) = g^{-1}(g(x)) = x.
\]

Therefore, \( \ker \psi' \subseteq \text{im} \varphi'. \) \( \Box \)

From the short exact sequence

\[
0 \longrightarrow K \otimes A \xrightarrow{\varphi} T_{\alpha} \xrightarrow{\psi} A \times_{\alpha} Z \longrightarrow 0
\]
we obtain the six term exact sequence of K-theory
\[ K_0(K \otimes A) \xrightarrow{\varphi^*} K_0(T_\alpha) \xrightarrow{\psi^*} K_0(A \times_\alpha \mathbb{Z}) \]
\[ \delta_1 \uparrow \quad \delta_0 \downarrow \]
\[ K_1(A \times_\alpha \mathbb{Z}) \xleftarrow{\psi^*} K_1(T_\alpha) \xleftarrow{\varphi^*} K_1(K \otimes A) \]

Let \( \kappa : A \to K \otimes A \) be given by \( \kappa(a) = e_{0,0} \otimes a \). By stability of K-theory, \( \kappa_* : K_i(A) \to K_i(K \otimes A) \) is an isomorphism. We also have the isomorphism \( j_* : K_i(A) \to K_i(T_\alpha) \) induced by inclusion.

**Proposition 2.14.** There is a unique group homomorphism \( f : K_i(A) \to K_i(A) \) such that the following diagram commutes.

\[
\begin{array}{ccc}
K_i(A) & \xrightarrow{f} & K_i(A) \\
\kappa_* \downarrow & & \downarrow j_* \\
K_i(K \otimes A) & \xrightarrow{\varphi^*} & K_i(T_\alpha)
\end{array}
\]

Moreover, \( f = \text{id} - \alpha_*^{-1} \).

**Proof.** The uniqueness follows from the fact that if \( j_* \circ f = \varphi_* \circ \kappa_* \), then \( f = j_*^{-1} \circ \varphi_* \circ \kappa_*^{-1} \).

Let \( \gamma : A \to T_\alpha \) be the \(*\)-homomorphism given by \( \gamma = \text{Ad} V \circ j \circ \alpha^{-1} \). Then \( \gamma(a) = \text{Ad} V(j(\alpha^{-1}(a))) = u\alpha^{-1}(a)u^* \otimes vv^* = a \otimes vv^* = a(1 - P) \). Note that \( \gamma \circ \kappa_*^{-1}(a) = \varphi(e_{0,0} \otimes a) = aP \). So \( \gamma \) and \( \varphi \circ \kappa_*^{-1} \) are orthogonal and note that \( \gamma + \varphi \circ \kappa_*^{-1} = j_* \). So \( j_* = \gamma + \varphi \circ \kappa_*^{-1} \). Taking \( j_*^{-1} \) of both sides, we obtain \( \text{id}_{K_i(A)} \circ j_*^{-1} = \alpha_*^{-1} + \varphi_* \circ \kappa_*^{-1} = \alpha_*^{-1} + f \). So \( f = \text{id} - \alpha_*^{-1} \). \( \square \)

We are now ready to prove the main theorem.

**Theorem 2.15.** (Pimsner-Voiculescu Sequence) For any C*-algebra \( A \), there is a six-term exact sequence

\[
\begin{array}{ccc}
K_0(A) & \xrightarrow{\text{id} - \alpha_*^{-1}} & K_0(A) \\
\downarrow & & \downarrow \\
K_1(A \times_\alpha \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) \xleftarrow{\text{id} - \alpha_*^{-1}} K_1(A)
\end{array}
\]

where \( \iota : A \to A \times_\alpha \mathbb{Z} \) is the inclusion.
Proof. We first prove it for the case where $A$ is unital. Note that $\psi \circ j = \iota$, and consider the commutative diagram

\[
\begin{array}{ccccccccc}
K_0(A) & \xrightarrow{id-\alpha^{-1}} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \times_\alpha \mathbb{Z}) \\
\kappa_A \downarrow & & \downarrow j_* & & & & \\
K_0(K \otimes A) & \xrightarrow{\phi_*} & K_0(T_\alpha) & \xrightarrow{\psi_*} & K_0(A \times_\alpha \mathbb{Z}) \\
\delta_1 & \uparrow & & & & & & \delta_0 & \\
K_1(A \times_\alpha \mathbb{Z}) & \xleftarrow{\psi_*} & K_1(T_\alpha) & \xleftarrow{\iota_*} & K_1(K \otimes A) \\
\kappa_A^{-1} & \uparrow & & \uparrow j_* & & \kappa_A & \\
K_1(A \times_\alpha \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{id-\alpha^{-1}} & K_1(A) \\
\end{array}
\]

Applying Lemma 2.13 repeatedly, we obtain the following exact sequence from the top and bottom rows of the previous diagram

\[
\begin{array}{ccccccccc}
K_0(A) & \xrightarrow{id-\alpha^{-1}} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \times_\alpha \mathbb{Z}) \\
\kappa_A^{-1} \circ \delta_1 & \downarrow & & & & & & \kappa_A^{-1} \circ \delta_0 & \\
K_1(A \times_\alpha \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{id-\alpha^{-1}} & K_1(A) \\
\end{array}
\]

When $A$ is nonunital, we consider the C*-algebra $\tilde{A} = \{a + \lambda 1 | a \in A, \lambda \in \mathbb{C}\}$ obtained by adjoining a unit. Then there is a split exact sequence

\[
0 \longrightarrow A \xrightarrow{f} \tilde{A} \xrightarrow{g} \mathbb{C} \longrightarrow 0
\]

where $f(a) = a, g(a + \lambda 1) = \lambda$, and the splitting $h : \mathbb{C} \rightarrow \tilde{A}$ is defined by $h(\lambda) = \lambda 1$. This split exact sequence induces a split exact sequence

\[
0 \longrightarrow K_i(A) \xrightarrow{f_*} K_i(\tilde{A}) \xrightarrow{g_*} K_i(\mathbb{C}) \longrightarrow 0
\]

of K-theory groups.

The automorphism $\alpha$ of $A$ extends to an automorphism $\tilde{\alpha}$ given by $\tilde{\alpha}(a + \lambda 1) = \alpha(a) + \lambda 1$. Let $\beta : \mathbb{C} \rightarrow \mathbb{C}$ denote the trivial automorphism. Then, using the universal property of crossed products along with the maps $f, g,$ and $h$, we obtain a split exact sequence

\[
0 \longrightarrow A \times_\alpha \mathbb{Z} \longrightarrow \tilde{A} \times_{\tilde{\alpha}} \mathbb{Z} \longrightarrow \mathbb{C} \times_\beta \mathbb{Z} \longrightarrow 0
\]

which induces the split exact sequence

\[
0 \longrightarrow K_i(A \times_\alpha \mathbb{Z}) \longrightarrow K_i(\tilde{A} \times_{\tilde{\alpha}} \mathbb{Z}) \longrightarrow K_i(\mathbb{C} \times_\beta \mathbb{Z}) \longrightarrow 0.
\]
Next, we form the large commutative diagram whose horizontal rows consist of the Pimsner-Voiculescu sequence for $A$, $\tilde{A}$, and $C$ respectively.

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
\cdots & K_1(A) & K_1(A \times \alpha \mathbb{Z}) & K_0(A) & \cdots \\
\downarrow & \downarrow & \downarrow & \\
\cdots & K_1(\tilde{A}) & K_1(\tilde{A} \times \tilde{\alpha} \mathbb{Z}) & K_0(\tilde{A}) & \cdots \\
\downarrow & \downarrow & \downarrow & \\
\cdots & K_1(C) & K_1(C \times \beta \mathbb{Z}) & K_0(C) & \cdots \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 \\
\end{array}
\]

In this diagram, the columns are exact and the rows corresponding to $\tilde{A}$ and $C$ are exact because we proved the exactness of the Pimsner-Voiculescu sequence for unital $C^*$-algebras. To show the exactness of the Pimsner-Voiculescu sequence for $A$, it suffices to complete the following diagram chase:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
\cdots & A & B & C & D \\
\downarrow & n & p & q & \\
\cdots & E & F & G & H \\
\downarrow & s & t & u & \\
\cdots & I & J & K & L \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Suppose that in this commutative diagram, the columns and the second and third rows are exact. We will show that the first row is exact at $C$. We have that $q \circ c \circ b = g \circ f \circ n = 0$. But since $q$ is injective, $c \circ b = 0$. This proves that $\text{im } b \subseteq \text{ker } c$. Conversely, suppose $x \in \text{ker } c$. Then $(g \circ p)(x) = (q \circ c)(x) = 0$, and so there exists $y \in F$ such that $f(y) = p(x)$. So $(j \circ s)(y) = (t \circ f)(y) = (t \circ p)(x) = 0$. Since $s(y) \in \text{ker } j$, there exists $z \in I$ such that $i(z) = s(y)$. Since $r$ is surjective, there is $w \in E$ such that $r(w) = z$. We have $(s \circ c)(w) = (i \circ r)(w) = i(z) = s(y)$. So $y - e(w) \in \text{ker } s$ and hence there is $v \in B$ such that $n(v) = y - e(w)$. Thus $(p \circ b)(v) = (f \circ n)(v) = f(y - e(w)) = f(y) = p(x)$. Since $p$ is injective, $b(v) = x$. So $\text{ker } c \subseteq \text{im } b$. \qed
3. Rotation Algebras

For a real number \( \theta \), the rotation algebra \( \mathcal{A}_\theta \) is defined to be the universal 
\( C^* \)-algebra generated by unitaries \( u \) and \( v \) such that
\[
uv = e^{2\pi i \theta} vu.
\]
The rotation algebra \( \mathcal{A}_\theta \) has a well-known interpretation as a crossed product. 
Consider the homeomorphism of the circle \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \) given by \( t \mapsto t - \theta \). This rotation induces an automorphism \( R_\theta \) of \( C(\mathbb{T}) \) given by \( (R_\theta f)(t) = f(t - \theta) \).

**Proposition 3.1.** The rotation algebra \( \mathcal{A}_\theta \) is isomorphic to the crossed product
\( C(\mathbb{T}) \times_{R_\theta} \mathbb{Z} \).

**Proof.** Let \( u, v \in \mathcal{A}_\theta \) denote the generating unitaries, and let \( w \in C(\mathbb{T}) \times_{R_\theta} \mathbb{Z} \) denote the unitary such that \( wfw^* = R_\theta(f) \) for all \( f \in C(\mathbb{T}) \). Let \( z \in C(\mathbb{T}) \) denote the function \( z(t) = e^{2\pi i t} \). In \( C(\mathbb{T}) \times_{R_\theta} \mathbb{Z} \),
\[
z(t)w = wu z(t)w = w(R_\theta^{-1}(z))(t) = w e^{2\pi i (t+\theta)} = e^{2\pi i \theta} wz(t).
\]
By universality of \( \mathcal{A}_\theta \), there is a \( * \)-homomorphism \( \phi : \mathcal{A}_\theta \to C(\mathbb{T}) \times_{R_\theta} \mathbb{Z} \) such that
\( \phi(u) = z \) and \( \phi(v) = w \).

The functional calculus provides a \( * \)-homomorphism \( \rho : C(\mathbb{T}) \to \mathcal{A}_\theta \) such that \( \rho(f) = f(u) \). Since \( v \rho(z)v^* = vuv^* = e^{-2\pi i \theta}u = \rho(R_\theta(z)) \) and \( z \) generates \( C(\mathbb{T}) \),
we have that \( v \rho(f)v^* = \rho(R_\theta(f)) \) for any \( f \in C(\mathbb{T}) \). By Proposition 1.1, there is a
\( * \)-homomorphism \( \psi : C(\mathbb{T}) \times_{R_\theta} \mathbb{Z} \to \mathcal{A}_\theta \) such that \( \psi(z) = u \) and \( \psi(w) = v \). Clearly, \( \phi \) and \( \psi \) are inverses. \( \square \)

3.1. \( K \)-theory of \( \mathcal{A}_\theta \). Using the interpretation of \( \mathcal{A}_\theta \) as a crossed product, we can use the Pimsner-Voiculescu exact sequence to determine the \( K \)-theory groups of \( \mathcal{A}_\theta \).

**Proposition 3.2.** For any real number \( \theta \), \( K_0(\mathcal{A}_\theta) \cong \mathbb{Z} \oplus \mathbb{Z} \) and \( K_1(\mathcal{A}_\theta) \cong \mathbb{Z} \oplus \mathbb{Z} \).

**Proof.** The Pimsner-Voiculescu sequence becomes
\[
\begin{array}{cccccc}
K_0(C(\mathbb{T})) & \overset{id - R_{-\theta} \ast}{\longrightarrow} & K_0(C(\mathbb{T})) & \longrightarrow & K_0(C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}) & \\
\uparrow & & & \downarrow & & \\
K_1(C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}) & \leftarrow & K_1(C(\mathbb{T})) & \overset{id - R_{-\theta} \ast}{\longrightarrow} & K_1(C(\mathbb{T})) & \\
& & & & &
\end{array}
\]
Since a rotation of the circle is homotopic to the identity map, it follows that the
map \( R_{-\theta} \), which is induced by a rotation, is homotopic to the identity automorphism. As a consequence, the map \( \text{id} - R_{-\theta} \ast = \text{id} - \text{id} = 0 \). From the six-term exact sequence, we can extract two short exact sequences:
\[
\begin{array}{cccccc}
0 & \longrightarrow & K_0(C(\mathbb{T})) & \longrightarrow & K_0(C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}) & \longrightarrow & K_1(C(\mathbb{T})) & \longrightarrow & 0 \\
0 & \longrightarrow & K_1(C(\mathbb{T})) & \longrightarrow & K_1(C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}) & \longrightarrow & K_0(C(\mathbb{T})) & \longrightarrow & 0
\end{array}
\]
Since \( K_0(C(\mathbb{T})) \cong K_1(C(\mathbb{T})) \cong \mathbb{Z} \), we have
\[
\begin{array}{cccccc}
0 & \longrightarrow & Z & \longrightarrow & K_0(C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}) & \longrightarrow & Z & \longrightarrow & 0 \\
0 & \longrightarrow & Z & \longrightarrow & K_1(C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}) & \longrightarrow & Z & \longrightarrow & 0
\end{array}
\]
from which it follows that \( K_i(C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \) for \( i = 0, 1 \). \( \square \)
Despite the fact that the isomorphism class of the K-theory groups of $A_\theta$ is independent of $\theta$, the structure of $A_\theta$ depends greatly on $\theta$. The most striking of these differences depend on the irrationality of $\theta$.

3.2. The case $\theta = 0$.

**Proposition 3.3.** In the case where $\theta = 0$, we have $A_0 \cong C(\mathbb{T}^2)$.

**Proof.** In this case, $A_0$ is the universal $C^*$-algebra generated by two commuting unitaries. So in particular, $A_0$ is a unital commutative $C^*$-algebra. By the Gelfand-Naimark Theorem, $A_0 \cong C(X)$ for some compact Hausdorff space $X$. Recall that $X$ in the space of multiplicative linear functionals on $A_0$ with the weak-* topology. We just need to show that $X$ is homeomorphic to $\mathbb{T}^2$.

By universality of $A_0$, there is a map $\phi : A_0 \to C(\mathbb{T}^2)$ such that $\phi(u) = z_1$ and $\phi(v) = z_2$, where $z_1$ and $z_2$ are the coordinate functions on $\mathbb{T}^2$. This induces a continuous map of Gelfand duals $\phi^* : \mathbb{T}^2 \to X$. Explicitly, $(\phi^*(s_1, s_2))(u) = s_1$ and $(\phi^*(s_1, s_2))(v) = s_2$. There is also a continuous map $f : X \to \mathbb{T}^2$ given by $f(\rho) = (\rho(u), \rho(v))$ for a multiplicative linear functional $\rho$ on $A_0$. It is clear that $\phi^*$ and $f$ are inverses. □

As a consequence of this proposition, the algebras $A_\theta$, have earned the nickname “noncommutative tori.”

We remark at some of the $C^*$-algebraic properties of $C(\mathbb{T}^2)$. Notice first, that it is far from being simple. Indeed, for any closed subset $Y \subseteq \mathbb{T}^2$, $I_Y = \{f | f(y) = 0, \forall y \in Y\}$ is a closed ideal. Moreover, $C(\mathbb{T}^2)$ has many traces, as integration with respect to any reasonable measure is a trace. However, $C(\mathbb{T}^2)$ is projectionless. If $f \in C(\mathbb{T}^2)$ is a projection, then $f(w)^2 = f(w)$ for all $w \in \mathbb{T}^2$. So the only values $f$ can take are 0 or 1. But since $\mathbb{T}^2$ is connected, then either $f = 0$ or $f = 1$. Thus, $C(\mathbb{T}^2)$ has no nontrivial projections.

3.3. Irrational Rotation Algebras.

3.3.1. Basic Properties. It can be shown that for irrational $\theta$, $A_\theta$ possesses a unique unital faithful trace $\tau : A_\theta \to \mathbb{C}$. This trace has the property that

$$\tau(\sum_{i,j} c_{ij} u^i v^j) = c_{0,0}$$

for finite sums. Viewing $A_\theta$ as $C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}$,

$$\tau(\sum_i f_i(t)v^i) = \int_0^1 f_0(t) dt$$

where again the sum is finite. Here $v$ denotes the unitary in $C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}$ for which $vf(t)v^* = f(t-\theta)$. Using this trace, one can also show that $A_\theta$ contains no nontrivial closed ideals, that is, $A_\theta$ is simple. Already, this is a stark contrast to the structure of $C(\mathbb{T}^2)$.

3.3.2. Projections in $A_\theta$. It has been conjectured in the past that like $C(\mathbb{T}^2)$, $A_\theta$ is projectionless. However, this could not be farther from the truth, as projections in $A_\theta$ exist in abundance.

**Theorem 3.4.** For every $\eta \in (\mathbb{Z} + \mathbb{Z} \theta) \cap [0,1]$, there exists a projection $p \in A_\theta$ such that $\tau(p) = \eta$. 
Proof. We first consider the case \( \eta = \theta \), and we shall construct the so-called Powers-Rieffel projection. Thinking of \( A_\theta \) as \( C(\mathbb{T}) \times \mathbb{R}_+ \mathbb{Z} \), the idea is to look for a projection of the form \( p = v^*g(t) + f(t) + g(t)v \), where \( f(t) \) and \( g(t) \) are real valued functions in \( C(\mathbb{T}) \). Thus \( p = p^* \) is satisfied. We must choose \( f(t) \) and \( g(t) \) so that \( p^2 = p \). Recalling that \( vh(t) = h(t-\theta)v \) and \( v^*h(t) = h(t+\theta)v^* \), we calculate

\[
p^2 = (v^*g(t) + f(t) + g(t)v)^2
\]

\[
= v^*g(t)v^*g(t) + v^*g(t)f(t) + v^*g(t)f(t) + f(t)v^*g(t) + f(t) + f(t)v^*g(t)f(t) + f(t)v^*g(t)v = g(t+\theta)g(t+2\theta)v^2 + (f(t)g(t+\theta) + f(t+\theta)g(t+\theta))v^2 \\
= (f(t)^2 + g(t)^2 + g(t+\theta)^2 + f(t)g(t) + f(t+\theta)g(t))v
\]

Thus, we will have \( p^2 = p \) if \( f(t) \) and \( g(t) \) are chosen to satisfy

\[
g(t)g(t-\theta) = 0 \\
f(t)g(t) + f(t-\theta)g(t) = g(t) \\
f(t)^2 + g(t)^2 + g(t-\theta)^2 = f(t)
\]

The last requirement we make is that \( \tau(p) = \theta \). But \( \tau(p) = \int_0^1 f(t)dt \), so we demand that \( \int_0^1 f(t)dt = \theta \). These requirements can all be satisfied with the following functions:

\[
f(t) = \begin{cases} 
\epsilon t, & \text{if } 0 \leq t \leq \epsilon \\
1, & \text{if } \epsilon \leq t \leq \theta \\
\epsilon^{-1}(\theta + \epsilon - t), & \text{if } \theta \leq t \leq \theta + \epsilon \\
0, & \text{if } \theta + \epsilon \leq t \leq 1
\end{cases}
\]

\[
g(t) = \begin{cases} 
\sqrt{f(t) - f(t)^2}, & \text{if } 0 \leq t \leq \theta + \epsilon \\
0, & \text{if } \theta + \epsilon \leq t \leq 1
\end{cases}
\]

For any other \( \eta = m + k\theta \in [\mathbb{Z} + \mathbb{Z} \theta] \cap [0, 1] \), we note that \( u^kv = e^{2\pi i k \theta}v^k \), and so the \( C^* \)-subalgebra on \( A_\theta \) generated by \( u^k \) and \( v \) is isomorphic to \( A_{k\theta} \). We can construct the Powers-Rieffel projection in \( A_{k\theta} \subseteq A_\theta \), which will have trace equal to the fractional part of \( k\theta \), which is \( \eta \). \( \square \)

3.3.3. Generators of the K-theory of \( A_\theta \). We have already calculated that \( K_i(A_\theta) \cong \mathbb{Z} \oplus \mathbb{Z} \) for \( i = 0, 1 \). In this section we will calculate the classes that can be taken as the generators.

**Theorem 3.5.** The generators of \( K_0(A_\theta) \) are \([1]\) and \([p]\).

**Proof.** Recall that we have the short exact sequence

\[
0 \longrightarrow K_0(C(\mathbb{T})) \overset{\iota}{\longrightarrow} K_0(C(\mathbb{T}) \times_{\mathbb{R}_+} \mathbb{Z}) \overset{\kappa \circ \delta_0}{\longrightarrow} K_1(C(\mathbb{T})) \longrightarrow 0
\]

where \( \iota : C(\mathbb{T}) \rightarrow C(\mathbb{T}) \times_{\mathbb{R}_+} \mathbb{Z} \) is the inclusion, \( \kappa : C(\mathbb{T}) \rightarrow \mathcal{K} \otimes C(\mathbb{T}) \) is the canonical embedding, and \( \delta_0 : K_0(C(\mathbb{T}) \times_{\mathbb{R}_+} \mathbb{Z}) \rightarrow K_1(\mathcal{K} \otimes C(\mathbb{T})) \) is the exponential map associated to the short exact sequence

\[
0 \rightarrow \mathcal{K} \otimes C(\mathbb{T}) \overset{\varphi}{\rightarrow} T_{\mathbb{R}_+} \overset{\psi}{\rightarrow} C(\mathbb{T}) \times_{\mathbb{R}_+} \mathbb{Z} \longrightarrow 0
\]
Proposition 3.6. (Rordam et al. Proposition 12.2.2) Let $K$ be a short exact sequence of C$^*$-algebras where $A$ and $B$ are unital. Let $\varphi : I \to A$ be given by $\varphi(x + \alpha 1) = \varphi(x) + \alpha I_A$. Suppose that $g \in K_0(B)$ and $g = [p]$ for some projection in $M_n(B)$, and let $a$ be a self-adjoint element of $M_n(I)$ such that $\psi(a) = p$. Then $\varphi(u) = \exp(2\pi i a)$ for precisely one unitary element $u$ in $M_n(I)$, and $\delta_0(g) = -[u]$.

Let $a = V^*g + f + gV \in \mathcal{T}_{R_0}$, which is a self-adjoint lift of $p$. We shall calculate $\exp(2\pi i a)$. To do this, we first evaluate $a^2$. Recall that in $\mathcal{T}_{R_0}$, $Vh(t) = h(t - \theta)V$ and $V^*h(t) = h(t + \theta)VA$ for any $h \in C(\mathbb{T})$. We have that

$$a^2 = (V^*g(t) + f(t) + g(t)V)^2$$

$$= V^*g(t)V^*g(t) + V^*g(t)f(t) + V^*g(t)^2V + f(t)V^*g(t) + f(t)^2$$

$$+ f(t)g(t)V + g(t)VV^*g(t) + g(t)Vf(t) + g(t)Vg(t)V$$

$$= g(t + \theta)g(t + 2\theta)V^*V + (f(t)g(t + \theta) + f(t + \theta)g(t + \theta))V^*$$

$$(f(t)^2 + g(t - \theta)^2) + (f(t)g(t) + f(t - \theta)g(t))V + g(t)^2VV^*$$

$$+ g(t)g(t - \theta)V^*V$$

$$= V^*g(t) + f(t) - g(t)^2 + g(t)V + g(t)V^*V$$

$$= (V^*g(t) + f(t) + g(t)V) - g(t)^2(1 - VV^*)$$

$$= a - g(t)^2(1 - VV^*)$$

using the relations between $f(t)$ and $g(t)$. If we denote the projection $E = 1 - VV^*$, we have the simple relationship $a^2 = a - g(t)^2E$. We will prove by induction that

$$a^n = a - \left(\sum_{k=0}^{n-2} f(t)^kg(t)^2E\right).$$

As we have already established this for $n = 2$, assume it holds for some $n \geq 2$. Note that

$$g(t)^2Ea = g(t)^2E(V^*g(t) + f(t) + g(t)V)$$

$$= g(t)^2g(t + \theta)EV^* + g(t)^2Ef(t) + g(t)^3EV$$

$$= g(t)^2Ef(t)$$
because $g(t)g(t + \theta) = 0$ and $EV = 0$. So we have that

\[
a^{n+1} = (a - \sum_{k=0}^{n-2} f(t^k)g(t)^2E)a
\]

\[
= a^2 - \sum_{k=0}^{n-2} f(t^k)g(t)^2Ea
\]

\[
= a - g(t)^2E - \sum_{k=0}^{n-2} f(t^k)g(t)^2Ef(t)
\]

\[
= a - \sum_{k=0}^{n-1} f(t^k)g(t)^2E
\]

and the claim is proved.

We calculate

\[
\exp(2\pi ia) = \sum_{n=0}^{\infty} \frac{(2\pi i)^n}{n!}a^n
\]

\[
= 1 + 2\pi ia + \sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!}a^n
\]

\[
= 1 + 2\pi ia + \sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!}(a - \sum_{k=0}^{n-2} f(t^k)g(t)^2E)
\]

\[
= 1 + \left( \sum_{n=1}^{\infty} \frac{(2\pi i)^n}{n!} \right)a - \sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!}\left( \sum_{k=0}^{n-2} f(t^k)g(t)^2E \right)
\]

\[
= 1 - \sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!}\left( \sum_{k=0}^{n-2} f(t^k)g(t)^2E \right)
\]

\[
= 1 - h(t)E
\]
where \( h(t) = \sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!} (\sum_{k=0}^{n-2} f(t)^k g(t))^2 \). If \( t \notin (\theta, \theta + \epsilon) \), then \( g(t) = 0 \) and so \( h(t) = 0 \). If \( t \in (\theta, \theta + \epsilon) \), then by definition, \( g(t)^2 = f(t) - f(t)^2 \). So

\[
\begin{align*}
\hat{h}(t) &= \sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!} (\sum_{k=0}^{n-2} f(t)^k (f(t) - f(t)^2)) \\
&= \sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!} f(t)^n f(t) - 1 \frac{f(t)(1 - f(t))}{f(t)} \\
&= -\sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!} f(t)^n (f(t) - f(t)) \\
&= -\sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!} f(t)^n + \sum_{n=2}^{\infty} \frac{(2\pi i)^n}{n!} f(t) \\
&= -(\exp(2\pi i f(t)) - 2\pi i f(t) - 1) + (\exp(2\pi i) - 2\pi i - 1)f(t) \\
&= 1 - \exp(2\pi i f(t))
\end{align*}
\]

Thus we can write \( \exp(2\pi i a) = \hat{h}(t)E + (1 - E) \) where

\[
\hat{h}(t) = \begin{cases} 
\exp(2\pi i f(t)), & \text{if } t \in [\theta, \theta + \epsilon] \\
0, & \text{if } t \notin [\theta, \theta + \epsilon]
\end{cases}
\]

We see that \( \hat{h}(t) \) has winding number \(-1\), and so \( \exp(2\pi i a) \) corresponds to a generator of \( K_1(K \otimes C(\mathbb{T})) \). Thus \( (\kappa^{-1} \circ \delta_0)[p] \) is a generator of \( K_1(C(\mathbb{T})) \) as desired.

**Theorem 3.7.** The generators of \( K_1(A_0) \) are \([u]\) and \([v]\).

**Proof.** We have the short exact sequence

\[
0 \longrightarrow K_1(C(\mathbb{T})) \xrightarrow{\iota_*} K_1(C(\mathbb{T}) \times_{R_\theta} \mathbb{Z}) \xrightarrow{\kappa^{-1} \circ \delta_1} K_0(C(\mathbb{T})) \longrightarrow 0
\]

The identity function in \( C(\mathbb{T}) \) can be taken as the generator of \( K_1(C(\mathbb{T})) \). It’s image under \( \iota_* \) is just the class \([u]\). So we can take \([u]\) as one of the generators of \( K_1(A_0) \). To finish, we must show that \([v]\) maps to the generator of \( K_0(C(\mathbb{T})) \) under the map \( \kappa^{-1} \circ \delta_1 \). We shall use the following concrete description of the index map:

**Proposition 3.8.** *(Rordam et al. Proposition 9.2.3)* Let

\[
0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} B \longrightarrow 0
\]

be a short exact sequence of \( C^* \)-algebras where \( A \) and \( B \) are unital. Let \( \varphi : I \rightarrow A \) be given by \( \varphi(x + \alpha 1) = \varphi(x) + \alpha 1_A \). Let \( v \) be a unitary in \( B \) and \( V \) be a partial isometry in \( A \) such that \( \psi(V) = v \). Then \( 1 - V^*V = \varphi(1) \) and \( 1 - VV^* = \varphi(1) \) for some projections \( p, q \in I \), and \( \delta_1([v]) = [p] - [q] \).

We have that \( \psi(V) = v \), where \( V \) is the isometry in \( T_{R_\theta} \). Also, \( 1 - V^*V = 0 \) and \( 1 - VV^* = \varphi([e_{0,0} \otimes 1]) \). Thus \( (\kappa^{-1} \circ \delta_1)([v]) = \kappa^{-1}([-e_{0,0} \otimes 1]) = -[1] \). As \([1]\) is the generator of \( K_0(C(\mathbb{T})) \), this completes the proof.

\( \square \)