**Problem 8.** Prove that a set, \( X \), is finite iff there is a relation such that it and its inverse order both well-order \( X \).

**Definition 2.** A set of strings over alphabet \( A \) is a subset of \( \bigcup_{n \in \omega} A^n \). In other words, it is a set of functions whose domains are natural numbers and whose ranges are contained in \( A \). We denote by \( \lambda \) the unique function with domain 0. The alphabet will not be mentioned if it is clear from context or does not need to be specified. If \( S \) is a set of strings and \( x,y \in S \), we write \( x \preceq y \) if \( y \in \text{dom}(x) = x \). If \( \neg(x \preceq y \lor y \preceq x) \), then we write \( x \perp y \) and say that \( x \) and \( y \) are incomparable; otherwise, we write \( x \parallel y \) and say that \( x \) and \( y \) are comparable. For a set of strings, \( S, T[S] = \{ x : (\exists y \in S)(x \preceq y) \} \) is the prefix closure of \( S \).

**Definition 3.** Let \( S \) and \( P \) be two sets of strings.
- \( P \ast S = \{ xy : x \in P \land y \in S \} \).
- \( P^{-1}S = \{ y : (\exists x \in P)(xy \in S) \} \).

For notational simplicity, we define \( x^{-1}S = \{ x \}^{-1}S \), \( P^{-1}x = P^{-1}\{x\} \), \( x \ast S = \{ x \} \ast S \) and \( P \ast x = P \ast \{x\} \) for a string \( x \).

**Definition 4.** Given a set of strings, \( S \), we call \( P \subseteq T[S] \) a maximal antichain of \( S \) if \( (\forall x,y \in P)(x \perp y \lor x = y) \) and \( (\forall x \in S)(\exists y \in P)(y \parallel x) \). \( P \) is a valid antichain of \( S \) if \( P \) is a maximal antichain of \( S \) and \( (\forall x,y \in P)(x^{-1}T[S] = y^{-1}T[S]) \). We define, \( \text{Vac}(S) = \{ P : P \text{ is a valid antichain of } S \} \).

**Example 1.** Consider the following set of strings over the alphabet \( \{a,b\} \):
\[
S = \{ a^5, a^4b, a^2ba, a^2b^2, ba^4, ba^3b, baba, bab^2, b^2a^3, b^2a^2b, b^3a, b^4 \}.
\]

Graphically, we can represent \( S \) as a tree where branching left indicates an \( a \) and branching right indicates a \( b \). In the picture below to the right, we highlight the four valid antichains of \( S \): \( P_0 = \{ \lambda \}, \quad P_1 = \{ a^2, ba, b^2 \}, \quad P_2 = \{ a^4, a^2b, ba^3, bab, b^2a^2, b^3 \} \) and \( P_3 = S \). Note that \( S \) is only a valid antichain of itself because it contains no comparable strings. The members of the four valid antichains are connected via dotted lines in the right picture (\( P_0 \) has only one member and therefore includes no dotted lines). For reference a maximal antichain that is not valid is included in the picture on the left and its members are joined with a dotted line.

![Figure 1](image_url)
In the next figure, we focus on the valid antichain $P_1$.

Observe that the portions of the tree below each of $a^2$, $ba$ and $b^2$ are identical; the terminal nodes of all three sub-trees are $\{a^3, a^2b, ab, b^2\}$. It is this equivalence of suffixes that makes $P_1$ a valid antichain.

**Problem 9.** Suppose that $P$ is a valid antichain of a set of strings $S$ and $Q$ is a valid antichain of $P$. Prove that $Q$ is a valid antichain of $S$. 

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**Figure 2.** The identical subtrees below the elements of the valid antichain $P_1$. 

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