Problem 17. Show that \(|C(\mathbb{R}, \mathbb{R})| = 2^{\aleph_0}\) and \(|\mathbb{R}^\mathbb{R}| = 2^{2^{\aleph_0}}\). (See exercise I.15.8 and the associated hint.)

Proof. Suppose \(f\) is continuous. Because it is continuous, the values of \(f\) at all points are determined by its values at rational points. Therefore, we consider the sequence \(f(q_0), f(q_1), \ldots\) thwere \(q_0, q_1, \ldots\) is an enumeration of \(\mathbb{Q}\). Again, because \(\mathbb{Q}\) is dense in \(\mathbb{R}\), each of the values \(f(q_i)\) is the unique limit of some sequence of rational numbers, \(p_0, p_1, \ldots\). Thus, \(f\) is uniquely determined by a sequence of sequences of rational numbers, a set with cardinality \(2^{\aleph_0}\). To show that \(|C(\mathbb{R}, \mathbb{R})| \geq 2^{\aleph_0}\), note that there are \(2^{\aleph_0}\) lines \(y = rx\) for \(r \in \mathbb{R}\).

First, note that \(|\mathbb{R}^2| = 2^{\aleph_0}\). Every member of \(\mathbb{R}^\mathbb{R}\) is subset of \(\mathbb{R}^2\), thus, \(|\mathbb{R}^\mathbb{R}| \leq 2^{|\mathbb{R}^2|} = 2^{2^{\aleph_0}}\). On the other hand, the characteristic function of any subset of \(\mathbb{R}\) is an element of \(\mathbb{R}^\mathbb{R}\), so we also have the reverse inequality. \(\square\)

Problem 18. If \(P\) and \(Q\) are maximal antichains of the same finite set of strings, then there is a relation \(R \subseteq P \times Q\) such that

- \(\text{dom}(R) = P\),
- \(\text{ran}(R) = Q\),
- \(xRy \iff x \parallel y\).

Furthermore, if \(|P| = |Q|\) and \(P \parallel_{ac} Q\) (that is \(P <_{ac} Q\) or \(Q <_{ac} P\), then \(R\) is a well-defined and bijective function. (Hint: You may have already proved nearly this for previous antichain exercises!)

Proof. Define \(R = \{(x, y) : x \in P \land y \in Q \land x \parallel y\}\). Since \(P\) and \(Q\) are maximal, \((\forall x \in P)(\exists y \in Q)(x \parallel y)\), thus, \(\text{dom}(R) = P\). Similarly, \((\forall y \in Q)(\exists x \in P)(x \parallel y)\), thus \(\text{ran}(R) = Q\).

Now suppose that \(|P| = |Q|\) and \(P \parallel_{ac} Q\). If \(x \in P\) and \(y, z \in Q\) with \(y \neq z\), \(x \parallel y\) and \(x \parallel z\), then \(x < y\) and \(x < z\). Thus, every \(p \in P\) and \(q \in Q\) with \(p \parallel q\) are such that \(p < q\). Thus, for every \(p \in P\) there is at least one \(q \in Q\) that is comparable with no other element of \(P\). Thus, \(|P| + 1 \leq |Q|\), which is a contradiction. We conclude that there is a one-to-one correspondence between the members of \(P\) and \(Q\), as desired. \(\square\)

Problem 19. If \(S\) is a finite set of strings, then \(\left(\text{Vac}(S); <_{ac}\right)\) is a finite linear order.

Proof. Consider a finite set of strings, \(S\), and let \(T = T[S]\). We begin by fixing \(P, Q \in \text{Vac}(S)\). We may assume that \(|P| = |Q|\); if \(|P| \neq |Q|\), then \(P <_{ac} Q\) or \(Q <_{ac} P\). We pick an element \(x \in P\) and observe that, by the preceding problem, there is a unique \(y \in Q\) such that \(x \parallel y\).

Suppose that \(x = y\) and let \(x'\) be any other member of \(P\). By the preceding problem, there is a unique \(y' \in Q\) such that \(x' \parallel y'\). Since \(P\) and \(Q\) are valid antichains and \(x = y\), \(x'^{-1}T = x^{-1}T = y^{-1}T = y'^{-1}T\). Given that \(x' \parallel y'\), \(T\) is finite and \(x'^{-1}T = y'^{-1}T\) we conclude that \(x' = y'\). Now assume \(x < y\). In the case \(y < x\) simply exchange the roles of \(x\) and \(y\). As above, we pick \(x' \in P\) and its unique comparable element \(y' \in Q\). Clearly \(y'^{-1}T\) is a strict subtree of \(x'^{-1}T\) and hence, \(y'^{-1}T\) is a strict subtree of \(x'^{-1}T\). We conclude that \(x' < y'\).

We have shown that any two members of \(\text{Vac}(S)\) are comparable. The remaining order properties follow immediately from the definitions. \(\square\)