The problems that did not make it onto the midterm.

**Problem 1.** If $\mathcal{F}$ is of finite character, $X \in \mathcal{F}$ and $Y \subseteq X$, show that $Y \in \mathcal{F}$.

*Proof.* Let $F$ be an arbitrary finite subset of $Y$. Since $Y \subseteq X$, $F$ is a finite subset of $X$. Since $\mathcal{F}$ is of finite character, this means that $F$ is an element of $\mathcal{F}$. Thus, every finite subset of $Y$ is an element of $\mathcal{F}$ and, again because $\mathcal{F}$ is of finite character, $Y$ is an element of $\mathcal{F}$. □

**Problem 2.** Suppose every subset of a partial order $(A, \preceq)$ has a finite number of $\preceq$-minimal elements. Use recursion to define a well-order on $A$. (This is a nice problem, but, sadly, it is too hard for a short exam.)

*Proof.* Define $f$ by recursion on the ordinals as follows:

1. $f(0) = \{ x \in A : x \text{ is } \preceq\text{-minimal in } A \}$
2. $f(\alpha) = \{ x \in A : x \text{ is } \preceq\text{-minimal in } A \setminus \bigcup_{\beta < \alpha} f(\beta) \}$ if $A \setminus \bigcup_{\beta < \alpha} f(\beta) \neq \emptyset$ and $\alpha > 0$.
3. $f(\alpha) = \emptyset$ otherwise.

Define $g_\alpha : f(\alpha) \to n_\alpha$ for some $n_\alpha \in \omega$ to be a bijection (this exists because the set of $\preceq$-minimal elements of a nonempty set is finite). Finally, for $x, y \in A$, define $x \preceq' y$ if and only if there are $\alpha$ and $\beta$ such that $x \in f(\alpha)$, $y \in f(\beta)$ and $\alpha < \beta$ or there is an $\alpha$ such that $x, y \in f(\alpha)$ and $g_\alpha(x) < g_\alpha(y)$. That $\preceq'$ is a total order is clear. To see it is a well-order, consider $B \subseteq A$. Let $\delta$ be least in $\{ \alpha : (\exists x \in B)(x \in f(\alpha)) \}$. Only finitely many members of $B$ are in $f(\alpha)$. Among them, let $x$ be such that $g_\alpha(x)$ is least. We see that $x$ is the $\preceq'$-least member of $B$. □

**Problem 3.** If $\lambda$ is a limit ordinal, then $\text{cf}(\aleph_\lambda) \leq \text{cf}(\lambda)$. (Note: In fact, $\text{cf}(\aleph_\lambda) = \text{cf}(\lambda)$, but you only have to prove one inequality.)

*Proof.* Suppose $A \subseteq \lambda$ is such that $\text{type}(A) = \text{cf}(\lambda)$ and $\text{sup}(A) = \lambda$. If $\beta < \lambda$, then there is a $\delta \in A$ with $\delta \geq \beta$ – hence, also $\aleph_\delta \in \aleph_\delta$. Thus, $\aleph_\lambda = \text{sup}\{ \aleph_\delta : \delta \in A \}$ and we have a set with order type $\text{cf}(\lambda)$ whose supremum is $\aleph_\lambda$. □

**Problem 4.** For any $A$ and $B$ with $A, B \neq \emptyset$, prove that there is a $g : B \to A$ which is injective if and only if there is an $f : A \to B$ which is surjective. (Use one of the versions of the Axiom of Choice.)

*Proof.* If there is such a $g$, then fix $b \in B$ and define $f = g^{-1} \cup \{(a, b) : a \not\in \text{ran}(g)\}$. If there is an $f$ as in the statement, then consider $X = \{ f^{-1}(b) : b \in B \}$, where $f^{-1}(b)$ is the preimage of the singleton $\{b\}$ under $f$. By the Axiom of Choice, there is a choice function, $g$ of $X$ because the preimages are disjoint. Since the preimages are disjoint, $g$ is injective as desired. □