A Minimal Degree Computable from a Weakly 2-Generic

Rod Downey
Victoria University
Wellington
New Zealand

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Basic computability

- Not everyone here is a computability theorist, or even a recursion theorist.
- The arithmetic hierarchy is a collection of sets (languages) defined by number quantifiers.
- Let $R$ be a computable (binary) relation.
- $X$ is $\Sigma_1^0$ if $x \in X$ iff $\exists s R(x, s)$.
- Think halting problem.
- $Y$ is $\Pi_1^1$ if $Y$ is $\Sigma_1^0$.
- $X$ is $\Sigma_{n+1}^0$ if $X$ is "$\Sigma_1^0"$ but we replace $R$ by a $\Pi_n^0$ relation. Equivalently we have $n + 1$ alternating quantifiers beginning with a $\exists$.
- For example $\{ x \mid \varphi_x \text{ has finite domain } \}$ is $\Sigma_2^0$.
- $X$ is $\Delta_n^0$ iff $X \in \Sigma_n^0 \cap \Pi_n^0$.
- $(\text{Post, 1944})$ $X \leq_T \emptyset^{(n)}$ iff $X \in \Delta_{n+1}^0$.
- Theme Definability and computation correlate.
The Basic Concepts

- \( m \neq 0 \) is called minimal if \( b \prec m \) implies \( b = 0 \).
- Minimal degrees were first constructed by Spector, and this construction was clarified by Sacks and Shoenfield as forcing with perfect trees.
- All constructions of minimal degrees use some kind of variation on the theme.
- (Actually a theorem of Jockusch and Posner) A set \( G \) is called \( n \)-generic if \( G \) meets or avoids all \( \Sigma^0_n \) sets of strings \( S \). That is, either there is a \( \sigma \in S \) with \( \sigma \prec G \), or there is a \( \rho \prec G \) such that for all \( \tau \in S \), \( \rho \not\leq \tau \).
- This is the same as Cohen genericity for \( n \) quantifier arithmetic. Forcing where the conditions are finite strings.
- \( G \) is weakly \( n \)-generic if \( G \) meets all \( \Sigma^0_n \) dense sets of strings.
- Weakly \( n + 1 \)-generic implies \( n \)-generic properly.
- Fundamental concepts in computability and logic.
How do these two fundamental concepts relate?

It is not difficult to show that as sets \( n \)-genericity and minimality are incompatible. What about degrees?

Early result of Jockusch (1980): If \( g \) is 2-generic and \( 0 < b \leq g \) then \( b \) computes a 2-generic.

In particular, no 2-generic degree can bound a minimal degree.

All hyperimmune degrees are weakly 1-generic. Thus a minimal degree can be weakly 1-generic.

Question[1980] Can a minimal degree be computable from a 1-generic?
Towards a negative solution

- There was a lot of evidence towards a negative solution:

**Theorem (Chong and Jockusch-1983)**

If $0 < b < g < 0'$ and $g$ is 1-generic, then $b$ bounds a 1-generic degree.

**Theorem (Haught-1986+Thesis)**

If $0 < b < g < 0'$ and $g$ is 1-generic, then $b$ is 1-generic.
A positive solution

Theorem (Kumabe-1990, Chong and Downey-1990)

There exists a minimal degree $m < 0'$ and a 1-generic degree $m < g < 0''$. 
A Characterization

- The Chong-Downey paper together with Chong-Downey 1989, provides a characterization of when a set is computable from a 1-generic degree.

**Definition**

- We say that a set of strings $S$ is a proper cover of a set $X$ iff for all $\sigma \prec X$, there exists $\tau \in S$, such that $\sigma \preceq \tau$, and no $\sigma \in S$ is an initial segment of $X$.
- We say that $X$ has a tight cover $S$ if $S$ is a proper cover and for all proper covers $\hat{S}$, $\exists \sigma \in S \exists \tau \in \hat{S} (\sigma \preceq \tau)$.

**Theorem (Chong and Downey-1990,1989)**

- A set $X$ is computable from a 1-generic set iff $X$ has no tight cover.
- Moreover, there exists a procedure $\Phi$ such that for all sets $X$, if $X$ has no tight cover, then there is a 1-generic $G \leq_T X''$ such that $\Phi^G = X$. 
Some consequences

Theorem (Downey and Yu-2009)
There is a hyperimmune-free minimal degree computable from a 1-generic degree.

Theorem (Chong and Downey, 1989)
There is a minimal degree below $0'$ not computable from a 1-generic degree.

Theorem (Kurtz-thesis)
Almost no degree is computable from a 1-generic.

Clearly this was first obtained by direct methods.
The following result was an implicit question, and first formally articulated by Barmpalias, Day and Lewis-Pye.

**Theorem**

There is a minimal degree computable from a weakly 2-generic degree.

This result is tight because of Jockusch’s result on 2-generics. The following theorem is probably evident to anyone who ever thought about it, but points at difficulties in proving the theorem.

**Theorem**

Suppose that $X \leq_T G, \emptyset'$ and $G$ is weakly 2-generic. Then $X$ is computable.
Proof of the minimal pair theorem

- Suppose that \( \Phi^G = X \) with \( X \leq_T \emptyset' \), \( X = \lim_s X_s \), and \( G \) weakly 2-generic.

- Let
  \[ S = \{ \sigma \mid [\exists s_0 \forall s > s_0 (\Phi^\sigma \downarrow [s] \not\preceq X_s)] \lor (\forall \tau \forall s)(\sigma \preceq \tau \rightarrow \Phi^\tau \uparrow [s])\} \].

- If \( S \) is dense then \( G \) meets \( S \) which is a contradiction. Thus \( S \) is not dense.

- Therefore there is some \( \sigma_0 \) such that for all \( \sigma \in S \), \( \sigma_0 \not\preceq \sigma \).

- Then for all \( \sigma \) extending \( \sigma_0 \) there is some \( \tau \), \( \sigma \preceq \tau \) and \( \Phi^\tau \downarrow \). But also for such a \( \tau \), \( \Phi^\tau \not\prec X \), so that \( X \) is computable.
Notice that the above says that both the weakly 2-generic $G$ and minimal $M$ with $\Gamma^G = M$ must be not $\Delta^0_2$.

The construction is a full approximation of these in $\Delta^0_3 - \Delta^0_2$.

$S_i$ denotes the $i$-th $\Sigma^0_2$ set of strings.

$$R_e : \text{Either } S_e \text{ not dense, or } G \text{ meets } S_e.$$ 

$$\mathcal{N}_e : \Phi^M_e \text{ total } \Rightarrow (\Phi^M_e \equiv_T \emptyset) \lor (M \leq_T \Phi^M_e)$$

$\Gamma$ is thought of as a partial computable map from strings to strings, inducing the funtional $\Gamma^G = M$. 

Look for $\tau \in \mathbb{R}$

$\sigma^{\circ}$ for "Si not dense above $\sigma^*$"
Michael McInerney has worked on “multiply generic” $G$. Here think of $S_i$ as the range of a partial computable function. Replace this with e.g. $\omega$-c.a.

Characterizes when $X$ is computable from a multiply generic. Extends Haught’s Theorem. Relationships with integer valued randomness.

**Question** Is there a characterization of when $X$ is computable from a weakly 2-generic?

**(Guess)** $X$ has no $\Sigma^0_2$ “tight proper dense cover”.

The problem is that the CD proof (even when corrected in McInerney’s Thesis) is already a full approximation $0''$ argument so you would guess that any extension would add another quantifier, as the things that need approximating are very complex. But maybe not...
Thank You