11th International Conference on

Computability, Complexity, and Randomness

University of Hawaii, Honolulu, January 4–8, 2016

Degrees of the Isomorphism Types of Structures

Valentina Harizanov

Department of Mathematics

George Washington University

harizanv@gwu.edu
http://home.gwu.edu/~harizanv/

- Assume *countable* structures *A* for *computable* (usually finite) languages.

  Turing degree of *A* is the Turing degree of the *atomic diagram of A*, $D(A)$.
  A is *computable* (*recursive*) if its Turing degree is $\mathbf{0}$.
  $D(A)$ may be of much lower Turing degree than $Th(A)$.

- (Tennenbaum) If *A* is a nonstandard model of PA, then *A* is not computable.

- (Harrington, Knight) There is a nonstandard model *A* of PA such that *A* is *low* and $Th(A) \equiv_T \emptyset^{(\omega)}$.

- (Downey and Jockusch) Every Boolean algebra of *low* Turing degree has a computable copy.

- The *Turing degree spectrum* of A is

$$DgSp(A) = \{\mathbf{deg}(B) : B \cong A\}.$$

- (Knight)
  A structure A is *automorphically trivial* if there is a sequence $\overrightarrow{c} \in A^{<\omega}$ such that every permutation of A that fixes $\overrightarrow{c}$ pointwise is an automorphism of A.
  (i) If A is automorphically trivial, then

$$|DgSp(A)| = 1.$$

  (ii) If A is automorphically nontrivial, then
  $DgSp(A)$ is closed upwards.

- (Harizanov, Knight and Morozov)
  (i) If A is automorphically trivial, then

$$(\forall B \simeq A)[D^e(B) \equiv_T D(B)].$$

  (ii) If A is automorphically nontrivial, and $X \geq_T D^e(A)$, there exists $B \cong A$ such that

$$D^e(B) \equiv_T D(B) \equiv_T X$$

- (Harizanov and R. Miller) If the language of A is finite, then A is automorphically trivial iff $DgSp(A) = \{\mathbf{0}\}$.

- (Hirschfeldt, Khoussainov, Shore and Slinko) For every automorphically nontrivial structure A, there is a structure B, which can be:
  a symmetric irreflexive graph,
  a partial order,
  a lattice,
  a ring,
  an integral domain of arbitrary characteristic,
  a commutative semigroup,
  a 2-step nilpotent group, such that

$$DgSp(A) = DgSp(B)$$

- (R. Miller, Poonen, Schoutens and Shlapentokh)
  a field

- $\mathcal{D}$=the set of all Turing degrees

- (Wehner; Slaman)
  There is a structure A such that

$$DgSp(A) = \mathcal{D} - \{\mathbf{0}\}$$

- (Hirschfeldt)
  There is a complete decidable theory with all types computable, prime model of which has no computable copy, but has an X-decidable copy for every $X >_T \emptyset$.

- (R. Miller)
  There is a linear order A such that

$$DgSp(A) \cap \mathbf{\Delta}^0_2 = \mathbf{\Delta}^0_2 - \{\mathbf{0}\}$$

- (Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon)
  For every $n \in \omega$, there is a structure A such that

$$DgSp(A) = \{\mathbf{c} \in \mathcal{D} \colon \mathbf{c}^{(n)} > \mathbf{0}^{(n)}\}.$$

  A degree $\mathbf{c}$ is *nonlow*$_n$ if $\mathbf{c}^{(n)} > \mathbf{0}^{(n)}$.

- (Frolov, Harizanov, Kalimullin, Kudinov and R. Miller)
  Let $n \geq 2$. For every Turing degree $\mathbf{d}$, there is a linear order with spectrum $\{\mathbf{c} \in \mathcal{D} : \mathbf{c}^{(n)} > \mathbf{d}\}$.

  In particular, there is a linear order the spectrum of which contains exactly the *nonlow*$_n$ degrees.

- (Harizanov and R. Miller)
  There exists a structure A such that its spectrum consists of the degrees that are high-or-above:

$$DgSp(A) = \{\mathbf{c} \in \mathcal{D}\colon\ \mathbf{c}' \geq \mathbf{0}''\}.$$

  A degree $\mathbf{c}$ is *high* if $\mathbf{c}' = \mathbf{0}''$.

- (Csima and Kalimullin)
  The set of *hyperimmune* degrees is the degree spectrum of a structure.

- (Jockusch and Richter)
  The (*Turing*) *degree* of the *isomorphism type* of *A*, if it exists, is the *least* Turing degree in $DgSp$(*A*).

- *Effective Extendability Condition* for *A*

  For every finite structure *C* isomorphic to a substructure of *A*, and every embedding f of *C* into *A*, there is an algorithm that determines whether a given finite structure *D* extending *C* can be embedded into *A* by an embedding extending f.

- (Richter)
  Assume that a structure *A* satisfies the effective extendability condition. If the degree of the isomorphism type of *A* exists, then it must be $\mathbf{0}$. ($DgSp$(*A*) will contain a minimal pair of degrees.)

- (Richter)
  (i) A *linear order* without a computable copy does not have the isomorphism type degree.

  (ii) A *tree* without a computable copy does not have the isomorphism type degree.

- Abelian p-group G

  $x \in (G - \{0\}) \Rightarrow (\exists n)[order(x) = p^n]$

- (A. Khisamiev)
  An *abelian p-group* without a computable copy does not have the isomorphism type degree.

- *Richter's Combination Method*

  Let T be a theory in a finite language L such that there is a computable sequence $A_0, A_1, A_2, \ldots$ of *finite* structures for L, which are *pairwise nonembeddable*. Assume that for every $X \subseteq \omega$, there is a model A_X of T such that

  $$A_X \leq_T X,$$

  and for every $i \in \omega$:

  $$A_i \text{ is embeddable in } A_X \Leftrightarrow i \in X.$$

  Then for every Turing degree $\mathbf{d}$, there is a model of T the isomorphism type of which has degree $\mathbf{d}$.

- (Richter)
  For every Turing degree $\mathbf{d}$, there is a *torsion abelian group* the isomorphism type of which has the degree $\mathbf{d}$. There is such a group the isomorphism type of which does not have a degree.

- (Dabkowska, Dabkowski, Sikora and Harizanov)
  There are *centerless groups* that have arbitrary Turing degrees for their isomorphism types, as well as no degrees.

- (Calvert, Harizanov and Shlapentokh)
  Let $\mathcal{C}$ be a class of countable structures in a finite language L, closed under isomorphism. Assume that there is a computable sequence $(A_i)_{i\in\omega}$ of computable structures in $\mathcal{C}$ satisfying the following conditions. There exists a finitely generated structure $A \in \mathcal{C}$ such that for all $i \in \omega$, we have that $A \subseteq A_i$. For any $X \subseteq \omega$, there is a structure A_X in $\mathcal{C}$ such that $A \subseteq A_X$ and $A_X \leq_T X$, and for every $i \in \omega$, there exists an embedding σ such that

  $$\sigma : A_i \hookrightarrow A_X \ \wedge \ \sigma \upharpoonright A = id$$

  iff $i \in X$. Suppose that any A_X is isomorphic to some structure B under isomorphism $\tau : A_X \simeq B$. Consider a pair of structures A_i, A_j such that exactly one of them embeds in B via σ with $(\tau^{-1} \circ \sigma) \upharpoonright A = id$. Then there is a uniform procedure with oracle B to decide which of the two structures embeds in B.
  Then for every Turing degree $\mathbf{d}$, there is a structure in $\mathcal{C}$ the isomorphism type of which has degree $\mathbf{d}$.

(Calvert, Harizanov and Shlapentokh)

- There are various algebraic fields and torsion-free abelian groups of any finite rank greater than 1, the isomorphism types of which have arbitrary Turing degrees.
  There are structures in each of these classes the isomorphism types of which do not have Turing degrees.

- Ringed spaces corresponding to unions of varieties, ringed spaces corresponding to unions of subvarieties of certain fixed varieties, and schemes over a fixed field can have arbitrary Turing degrees for their isomorphism types, as well as no degrees.

- A magma $(M, *)$ is a *quandle* if:

  1. $(\forall x)[x * x = x]$ (idempotence);
  2. $(\forall x, y)(\exists! z)[z * x = y]$
  i.e., for every x, the mapping $f_x : M \longrightarrow M$ defined by $f_x(z) = z * x$ is bijective
  3. $(\forall x, y, z)[(x * y) * z = (x * z) * (y * z)]$ (right self-distributivity).

- A magma $(M, *)$ is a *rack* if only 2 and 3 are satisfied.

- A quandle $(M, *)$ is called *trivial* if the operation $*$ is defined by $(\forall x, y)[x * y = x]$.

- A trivial quandle is automorphically trivial.

- Let $(Z_n, +)$ be a cyclic group of order n.

  Let $S_n = (Z_n, *)$, where

$$x * y = x + 1$$

- S_n is a rack, but not a quandle if $n > 1$.
  $(y - 1) * x = y$
  $(x * y) * z = (x + 1) * z = x + 2$
  $(x * z) * (y * z) = (x + 1) * (y * z) = x + 2$

  $x * x = x + 1$

- S_n is a simple rack; that is, has no proper subracks.

  Hence $(S_n)_{n \geq 1}$ forms an antichain of racks with respect to embeddability.

- Let $(A_i, *_i)$, $i \in I$, be racks with pairwise disjoint domains. Their disjoint union, $\bigsqcup\limits_{i \in I} A_i$, is a rack with domain $\bigcup\limits_{i \in I} A_i$ and with the rack operation defined as follows:

$$x * y = \begin{cases} x *_i y & \text{if } x, y \in A_i, \\ x & \text{if } x \in A_i,\ y \in A_j, \text{ and } i \neq j. \end{cases}$$

- (Harizanov and Maeda)

  For every Turing degree $\mathbf{d}$, there is *rack* the isomorphism type of which has degree $\mathbf{d}$.

  There is a *rack* the isomorphism type of which has no degree.

- $Q_{p,2} = (Z_p, *)$, where $y * x = 2x - y \pmod{p}$ for any positive p is a quandle (Takasaki kei).

- $Q_{p,n} = (Z_p, *_n)$, where $y * x = nx - (n-1)y \pmod{p}$ where p is prime and $p \nmid n - 1$ is a quandle (Alexander quandle)

  $x * x = nx - (n-1)x = x$

  $y_1 * x = y_2 * x \Rightarrow nx - (n-1)y_1 = nx - (n-1)y_2 \pmod{p}$
  $\Rightarrow (n-1)(y_1 - y_2) = 0 \pmod{p} \Rightarrow y_1 = y_2 \pmod{p}$

$$(x * y) * z = (ny - (n-1)x) * z = nz - n(n-1)y + (n-1)^2 x$$

$$\begin{aligned}
&(x * z) * (y * z) = n(y * z) - (n-1)(x * z) \\
&= n(nz - (n-1)y) - (n-1)(nz - (n-1)x) \\
&= nz - n(n-1)y + (n-1)^2 x
\end{aligned}$$

- Let $Q_{p,q} = (Z_p, *_q)$, where p, q are primes such that $p \nmid q - 1$ and $ord_p(q) = p - 1$ (that is, q is a generator of the multiplicative group of integers modulo p).

- Consider $(Q_{p_i,q})_{i\in\omega}$, where q is fixed and p_i's are primes as above.

  $(Q_{p_i,q})_{i\in\omega}$ forms an antichain with respect to embeddability.

- Let $(A_i, *_i)$ be quandles with fixed $b_i \in A_i$, $i \in I$. The direct sum, $\bigoplus_{i \in I} A_i$, is a quandle the elements of which are sequences $(a_i)_{i \in I}$ with $a_i \in A_i$ and for all but finitely many i, $a_i = b_i$. Its operation is defined pointwise.

- (Harizanov and Maeda)
  For every Turing degree $\mathbf{d}$, there is a *quandle* the isomorphism type of which has degree $\mathbf{d}$.
  There is a *quandle* the isomorphism type of which has no degree.

THANK YOU!