Being low for K along sequences and elsewhere

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Lowness for K

The term word refers to finite binary sequences.

The term sequence refers to infinite binary sequences, which, as usual, can be viewed as a set of words or as a real.

Let K denote prefix-free Kolmogorov complexity and recall the following standard lowness notions with respect to K.

Definition Lowness and weak lowness for K

A sequence X is LOW FOR $\mathbf{K}$ in case access to X as an oracle does not improve the prefix-free complexity of any word by more than an additive constant c, i.e., we have for all words w

$$\mathrm{K}(w) - \mathrm{K}^{\mathrm{X}}(w) \leq c. \quad (*)$$

A sequence X is WEAKLY LOW FOR $\mathbf{K}$ in case $(*)$ holds for some constant c and an infinite set of words w

In the sequel, we consider similar notions where condition $(*)$ is required only for all words w in a certain set, e.g., the set of all initial segments of some fixed sequence.

Lowness for K on a set

Definition Lowness for K on a set and along a sequence

A sequence X is LOW FOR $\mathbf{K}$ ON A SET D (OF WORDS) in case access to X as an oracle does not improve the prefix-free Kolmogorov complexity of any word in D by more than an additive constant c, i.e.,

$$\text{for all words } w \text{ in } D, \text{ we have } \quad \mathrm{K}(w) - \mathrm{K}^{\mathrm{X}}(w) \leq c.$$

A sequence X is LOW FOR $\mathbf{K}$ ALONG A SEQUENCE A in case X is low for K on the set of initial segments of A.

A sequence X is LOW FOR $\mathbf{K}$ ALONG A SEQUENCE A ON A SET I (OF NATURAL NUMBERS) in case X is low for K on the set of initial segments of A with length in I.

Weak lowness for K on a set

Definition Weak lowness for K on a set and along a sequence

A sequence X is WEAKLY LOW FOR **K** ON A SET D in case X is low for K on some infinite subset of D.

A sequence X is WEAKLY LOW FOR **K** ALONG A SEQUENCE A in case X is weakly low for K on the set of initial segments of A. (i.e., in case X is low for K on some infinite set of initial segments of A).

A sequence X is WEAKLY LOW FOR **K** ALONG A SEQUENCE A ON A SET I in case X is weakly low for K on the set of initial segments of A with length in I.

Lowness for Ω and for ML-randomness

In case X is low for K along A, this can be described as follows:

> *access to oracle X does not allow to compress the initial segments of A significantly better than without X.* $(**)$

Assertion $(**)$ also yields descriptions of the situations where

(1) oracle X is LOW FOR $\mathbf{\Omega}$
(i.e., Ω remains Martin-Löf random with respect to X),

(2) oracle X is LOW FOR MARTIN-LÖF RANDOMNESS.

In both cases, apply the Levin-Schnorr characterization of Martin-Löf randomness in terms of incompressibility and adjust accordingly what it means to compress significantly better.

In the first case it then suffices to let A be equal to Ω.

In the second case, we have to consider a whole set of sequences A, i.e., all Martin-Löf random ones in place of a single one.

Lowness for Ω and for ML-randomness

By the discussion above, the notions of being low for Ω and being low for Martin-Löf randomness both resemble being low for K along a sequence.

Now the two former notions yield well-known characterizations of lowness and weak lowness for K.

Theorem (Joe Miller)

A sequence X is weakly low for K if and only if X is low for Ω.

Theorem (André Nies)

A sequence X is low for K if and only if X is low for Martin-Löf randomness.

It is suggesting to ask whether results of this type can be obtained for the notion of being low for K along a sequence.

Lowness and lowness along sequences

Theorem

For any sequence X, the following assertions are equivalent.

(a) X is low for K.

(b) X is low for K along all sequences on all infinite computable sets.

(c) X is low for K along some sequence on some infinite computable set.

Sketch of proof. The implications (a)$\rightarrow$(b) and (b)$\rightarrow$(c) are immediate, so it remains to show the implication (c)$\rightarrow$(a).

We give the proof for the case $I = \mathbb{N}$. Since I is computable, this does not change much.

Proof of the lemma

Sketch of proof (cont.). By contraposition, it suffices to prove

If X is not low for K, then for all A, X is not low for K along A.

By the coding theorem, up to some constant factor c we have

$$\sum_{|w|=n} 2^{-\mathrm{K}(w)} = 2^{-\mathrm{K}(n)}.$$

Letting $d_n = \mathrm{K}(n) - \mathrm{K}^X(n) - c$, the d_n are unbounded. We have

$$\sum_{|w|=n} 2^{-(\mathrm{K}(w)-d_n)} \leq 2^{d_n} c 2^{-\mathrm{K}(n)} = c 2^{-c} 2^{-\mathrm{K}^X(n)} \leq 2^{-\mathrm{K}^X(n)}.$$

Up to effectivity issues, it were thus possible to obtain for all n and all words w of length n prefix-free codes of length $\mathrm{K}(w) - d_n$.

The values of the d_n can only be effectively approximated in the limit, hence via an X-computable request set we only achieve code lengths $\mathrm{K}(w) - d_n/k$ for some constant k, which suffices. □

Weak lowness and weak lowness along sequences

Theorem

For any sequence X, the following assertions are equivalent.

(a) X is weakly low for K.

(b) X is weakly low for K along almost some sequence.

(c) X is weakly low for K along almost all sequences.

Sketch of proof: The implications (c)$\to$(b) and (b)$\to$(a) are immediate, so it remains to show (a)$\to$(c).

For all n and sequences Y let

$$\Omega_n^Y = \sum_{|w|=n} 2^{-\mathrm{K}^Y(w)} \qquad \text{and} \qquad \Omega_n = \Omega_n^{\emptyset}.$$

Assuming that X is weakly low for K, there are $n_1 < n_2 < \ldots$ such that up to an additive constant we have $\mathrm{K}(n_i) = \mathrm{K}^{\mathrm{X}}(n_i)$.

By the coding theorem, up to constant factors it then holds that

$$\Omega_{n_i} = 2^{\mathrm{K}(n_i)} = 2^{\mathrm{K}^X(n_i)} = \Omega_{n_i}^X.$$

Weak lowness and weak lowness along sequences

Sketch of proof (cont.): So for some constant c_0 and all i we have

$$\Omega^X_{n_i} \le c_0 \Omega_{n_i} \qquad \text{(where } \Omega^X_{n_i} = \textstyle\sum_{|w|=n_i} 2^{-\mathrm{K}^X(w)}\text{)}.$$

There is a constant c_1 such that for all n and any set D that contains at least half of the strings of length n, we have

$$\Omega_n \le c_1 \textstyle\sum_{w \in D} 2^{-\mathrm{K}(w)}.$$

Let $d = c_0 c_2$. Then for all i, less than half of all words of length n_i are contained in the set

$$D_i = \{w : |w| = n_i \text{ and } \mathrm{K}^X(w) \le \mathrm{K}(w) - d\}.$$

Otherwise, we obtain the contradiction

$$\Omega^X_{n_i} \le c_0 c_1 \sum_{w \in D_i} 2^{-\mathrm{K}(w)} = c_0 c_1 2^{-d} \sum_{w \in D_i} 2^{-(\mathrm{K}(w)-d)} < \Omega^X_{n_i}.$$

Weak lowness and weak lowness along sequences

Sketch of proof (cont.): We have seen that there is a constant d and $n_1 < n_2 < \ldots$ such that for all i we have

$\mathrm{K}^X(w) \leq \mathrm{K}(w) - d$ for less than half of all words w of length $n_i (*)$

We want to show that X is weakly low along almost all sequences.

Assume otherwise. Then by the Kolmogorov 0-1-law the class

$$\{A \colon X \text{ is not weakly low along } A\} = \bigcap_t \bigcup_j \underbrace{\{A \colon \mathrm{K}^X(A \upharpoonright n) \leq \mathrm{K}(A \upharpoonright n) - t \text{ for all } n \geq j\}}_{:= V(t,j)},$$

has measure 1, hence the union of all $V(d, j)$ has measure 1.

So there is some j_0 such that the union of $V(d, 0), \ldots, V(d, j_0)$ has measure at least $2/3$, which contradicts $(*)$ for $n_i > j_0$. □

Lowness and weak lowness on a fixed set

Theorem

Let D be an infinite set of strings.
Almost all sequences X are weakly low for K on D but are not low for K on D.