INDECOMPOSABLE VECTOR-VALUED MODULAR FORMS AND PERIODS OF MODULAR CURVES

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Abstract. We classify the three-dimensional representations of the modular group that are reducible but indecomposable, and their associated spaces of holomorphic vector-valued modular forms. We then demonstrate how such representations may be employed to compute periods of modular curves. This technique obviates the use of Hecke operators, and therefore provides a method for studying noncongruence modular curves as well as congruence.

1. Introduction

For a very long while, modular forms have been an indispensable tool in the theory of numbers. Perhaps in part because Frobenius was so separated in time from Jacobi, Eisenstein, and the other early adopters of modularity, the use of representation theory of the modular group as a means for studying modular forms has been a much more recent development. In a certain sense, this point of view is entirely natural since if G is a normal subgroup of the modular group $\Gamma = PSL_2(\mathbb{Z})$ and k is any even integer, then Γ acts on the space of weight k modular forms for G (either meromorphic or holomorphic) via the usual "slash" action

$$f \mapsto f|_k\gamma(\tau) = f\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-k}$$

in weight k that defines (weak) modularity on G. Selberg [Sel65], for example, made good use of this point of view in improving bounds on the growth of Fourier coefficients of cusp forms for congruence subgroups of Γ, and later on Eichler and Zagier [EZ85] pointed out that one may define Jacobi forms using this point of view together with Jacobi theta functions.

More recently, the importance of this action of the full modular group on spaces of modular forms has been made clear in the growing unification of number theory and high energy physics, where e.g. the work of Zhu [Zhu96] shows that certain vector-valued modular forms have Fourier coefficients that count the dimensions of graded spaces of modules for rational vertex operator algebras (VOAs). Largely motivated by the connection with rational VOAs, Knopp and Mason [KM03] initiated a formal study of vector-valued modular forms, and this has led to a significant amount of ongoing research in the subject.

One of the more novel aspects of this representation theoretic approach to modular forms is that one may study more general functions that attain modular invariance only on an infinite index subgroup of Γ. Since the Riemann surface that arises from the action of such a subgroup on the union $\mathbb{H}^*$ of the complex upper half-plane $\mathbb{H}$ and the cusps $\mathbb{Q} \cup \{i\infty\}$ of Γ is not compact, these functions are not constrained by e.g. Liouville's theorem, and consequently one may observe in this situation modular functions that are holomorphic throughout all of $\mathbb{H}^*$. A classic example of this occurs when one fixes a

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base point $\tau_0 \in \mathbb{H}^*$ and integrates a weight two cusp form f on a finite index subgroup $G \leq \Gamma$, obtaining a holomorphic modular function $u(\tau) = \int_{\tau_0}^{\tau} f(z)\,dz$ on some infinite index subgroup of G. The periods associated with f for the modular curve $G\backslash\mathbb{H}^*$ are then obtained by evaluating u at $\sigma\tau_0$ for appropriate hyperbolic elements $\sigma \in G$. In general it is not so easy to determine explicitly these periods, but as we show in Section 5 below one may obtain them by studying the action of Γ on the full space $S_2(G)$ of weight two cusp forms. In this way, one realizes these periods as columns of the matrices $\rho(\sigma)$, where ρ is a representation of Γ that encodes the action of Γ on the integrals of the weight two cusp forms and σ denotes the above-mentioned hyperbolic elements of G. This provides what appears to be a novel method for obtaining periods of modular curves, one that in particular does not require the use of Hecke operators. This allows one to obtain periods for noncongruence modular curves, i.e. curves $G\backslash\mathbb{H}^*$ with G a finite index, noncongruence subgroup of Γ, even though the action of the Hecke algebra associated to such a subgroup is known to be deficient [Sch97].

The representations of the modular group that occur in this context are indecomposable but reducible, a phenomenon that can occur only in the infinite image setting, and in order to reliably compute periods of modular curves of genus g via this method one requires a classification of representations $\rho : \Gamma \to GL_{g+1}(\mathbb{C})$ of the form

$$0 \to \rho_0 \to \rho \to \rho_1 \to 0, \tag{1}$$

where the g-dimensional subrepresentation ρ_0 gives the action of Γ on $S_2(G)$ (here the genus G subgroup is assumed to be normal in Γ for simplicity) and the quotient $\rho_1 = 1$ is trivial. Thus we are motivated to study the more general problem of classifying representations (1) together with their associated spaces $M(\rho)$ of holomorphic vector-valued modular forms. The *free module theorem* for vector-valued modular forms [MM10] asserts that in this context $M(\rho)$ is free of rank $r = \dim\rho$ over the graded ring $R = M(1)$ of scalar-valued modular forms for Γ. Thus the classification of the $M(\rho)$ comes down to determining the weights of the r free generators for $M(\rho)$, the so-called *generating weights* for ρ. For representations of low dimension, it is indeed possible to list all such representations ρ and their corresponding generating weights. For dimension two, this classification has already been carried out in [MM10, Sec 4], so a primary aim of this article is to accomplish a similar classification for representations (1) of dimension three. In Section 2 below we classify all such extensions, and the results may be summarized as follows.

Theorem 1.1. *Let $\rho : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{GL}_3(\mathbb{C})$ be as in (1) with ρ_0 two-dimensional and suppose $\rho\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is unitarizable. Then up to equivalence there are*

(i) *18 ρ with ρ_0 a direct sum of characters $\Gamma \to \mathbb{C}^\times$, as tabulated in Table 1 below.*
(ii) *30 ρ with ρ_0 indecomposable but reducible, as tabulated in Tables 2 and 3 below.*
(iii) *2 ρ for any choice of irreducible ρ_0 (there are infinitely many such choices).*

Additionally, there are 12 isomorphism classes of ρ where $\rho\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is not unitarizable, with representatives given by (12). □

By studying the dual representations of the above ρ, one is easily led to the following result which covers the remaining cases.

Theorem 1.2. *Suppose that $\dim(\rho_0) = 1$ in (1). Then ρ is equivalent to one of the representations in (i) or (ii) of Theorem 1.1 if and only if ρ admits a two-dimensional invariant subspace. Otherwise, the quotient ρ_1 is irreducible and either $\rho\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is unitarizable and ρ is the dual of one of the representations in (iii) of Theorem 1.1, or $\rho\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$*

is not unitarizable and ρ is one of the 12 representations given by (13) (these are the dual representations of those given by (12)). □

In Section 3, the generating weights for the associated spaces of holomorphic vector-valued modular forms are worked out. A key observation here is that, even though ρ is a non-trivial extension, it is still possible that the corresponding module of modular forms $M(\rho)$ splits as the direct sum $M(\rho_0) \oplus M(\rho_1)$. In this case we say that ρ is *M-split*, and the generating weights of ρ are just the union of the generating weights of ρ_0 and ρ_1. To decide when ρ is M-split we develop sheaf-theoretic cohomological tools based on the geometric theory of vector-valued modular forms [CF16]. Applying these geometric methods, we obtain a list of representations ρ that are M-split in Theorems 4.2, 4.5, 4.6, 4.7, and 4.8 below. We summarize these results as follows.

Theorem 1.3. *Let $\rho : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{GL}_3(\mathbb{C})$ be as in (1). Then*

(i) *If ρ_0 is two-dimensional and completely reducible, there are 2 isomorphism classes of ρ that are not M-split, and all the other classes are M-split.*
(ii) *If ρ_0 is two-dimensional, indecomposable but reducible, there is 1 isomorphism class of ρ that is not M-split, and all the other classes are M-split.*
(iii) *For each ρ_0 two-dimensional, irreducible of finite image with generating weights $k_1 = 8, k_2 = 10$, there is one isomorphism class of ρ that is not M-split. For all other ρ_0 irreducible of finite image, ρ is M-split.* □

We then treat separately each of the cases where ρ is not M-split to compute the generating weights. Note that the same cohomological tools can be applied to study the case of general ρ_0, not necessarily factoring through a finite group, and even to study the case of higher-dimensional representations ρ. The classification when ρ_0 one-dimensional is similarly obtained.

In Section 5, we provide the details for the above discussion regarding the use of these representations in the computation of modular curve periods (Theorem 5.1 below), and establish the following result.

Theorem 1.4. *Let $G \leq \Gamma$ be a normal subgroup of finite index and genus g. Let $\{f_1, \ldots, f_g\}$ be a basis of the space $S_2(G)$ of weight two cusp forms for G, and let ρ_0 be the representation given by the $|_2$ action of Γ on $S_2(G)$. Then*

$$F(\tau) = \left(\int_{i\infty}^{\tau} f_1(z)\, dz, \ldots, \int_{i\infty}^{\tau} f_g(z)\, dz, 1 \right)^t$$

is a holomorphic vector-valued modular function for an indecomposable representation ρ of dimension $g+1$ of the form

$$0 \to \rho_0 \to \rho \to 1 \to 0.$$

□

As ρ_0 factors through the finite group Γ/G, when $g = 1, 2$ we can use our classification together with the classification given in [MM10, Sec 4] to determine ρ. We demonstrate this in Section 5 by computing explicitly the periods for three modular curves of low genus. We also make the following observation regarding the algebraicity of periods of modular curves (Theorem 5.2 below):

Theorem 1.5. *Let $G \subseteq \mathrm{PSL}_2(\mathbb{Z})$ be a subgroup containing a finite index normal subgroup $G' \lhd \mathrm{PSL}_2(\mathbb{Z})$ of genus one or two. Then the period matrix of X_G has entries in $\overline{\mathbb{Q}}$.*

It is easy to show (see Remark 5.4 below) that the above theorem requires G' to have genus one or two, and that it cannot be extended to the case of G' having arbitrary genus. Our method of computing periods does however extend to groups of higher genus g, provided a classification of $\mathrm{PSL}_2(\mathbb{Z})$-representations of dimension $g+1$ is given. We leave the details of these computations to further exploration.

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Notation. In this article we generate the modular group $SL_2(\mathbb{Z})$ by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, subject to the relations $S^2 = (ST)^3$ and $S^4 = 1$. We will continue to use the notation S and T to refer to the images of these matrices in the quotient group $\Gamma = PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm 1\}$. Thus a function $\rho : \Gamma \to GL_n(\mathbb{C})$ defines a matrix representation of Γ iff $\rho(S)^2 = \rho(ST)^3 = 1$. We recall here that the commutator quotient Γ/Γ' of Γ is cyclic of order 6, so that any *character* (i.e. 1-dimensional representation) of Γ must have order dividing 6. In particular, since $\det \rho : \Gamma \to \mathbb{C}^\times$ is a character of Γ, it must be true that $\det \rho(T) = \mathbf{e}\left(\frac{x}{6}\right)$ for some integer x; here and throughout, for $r \in \mathbb{R}$ we set $\mathbf{e}(r) = e^{2\pi i r}$.

2. Three-dimensional indecomposable representations

Assume that $\rho : \Gamma \to GL_3(\mathbb{C})$ is reducible but indecomposable. Since ρ is reducible, by definition it has a nontrivial invariant subspace $W \leq \mathbb{C}^3$, which we assume for now is two-dimensional. Then conjugation with the appropriate element of $GL_3(\mathbb{C})$ allows us to assume that the first two standard basis vectors for $\mathbb{C}^3$ are a basis for W. This implies that

$$\rho(S) = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & \sigma \end{pmatrix} \tag{2}$$

for some complex numbers a, b, c, d, x, y, σ. We split the classification into two subcases, according to whether $\rho(T)$ is, or is not, unitarizable, and consider the former case first. As unitary matrices are diagonalizable, we may assume that

$$\rho(T) = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\} \tag{3}$$

with $\lambda_j = \mathbf{e}(r_j)$, $0 \leq r_j < 1$. We also note that conjugation by the invertible matrix $U = \mathrm{diag}\{u, v, w\}$ leaves (3) invariant but takes (2) to

$$U\rho(S)U^{-1} = \begin{pmatrix} a & buv^{-1} & xuw^{-1} \\ cvu^{-1} & d & yvw^{-1} \\ 0 & 0 & \sigma \end{pmatrix}. \tag{4}$$

In particular, this shows that the property of either of x, y vanishing is not affected by conjugation with a diagonal matrix as above.

Continuing with our assumption that $\rho(S)$ is as in (2), we note that the upper left 2×2 block of ρ defines a subrepresentation

$$\rho_0 : \Gamma \to GL_2(\mathbb{C})$$

with

$$\rho_0(T) = \mathrm{diag}\,\{\lambda_1, \lambda_2\}\,, \quad \rho_0(S) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5}$$

and the lower right entry of ρ defines a character $\chi : \Gamma \to \mathbb{C}^\times$ with

$$\chi(T) = \lambda_3, \quad \chi(S) = \sigma.$$

This already shows that λ_3 must be a sixth root of unity, and each of σ and λ_3 determines the other uniquely. We now proceed by cases, based on the nature of the subrepresentation ρ_0.

Suppose first that (5) is reducible, so that ρ can be made upper triangular. Then we may assume that $c = 0$ in (2), and we have

$$\rho(S)^2 = \begin{pmatrix} a^2 & b(a+d) & x(a+\sigma)+by \\ 0 & d^2 & y(d+\sigma) \\ 0 & 0 & \sigma^2 \end{pmatrix}. \tag{6}$$

Setting this equal to the identity matrix shows, among other things, that $a^2 = d^2 = \sigma^2 = 1$. We consider first the subcase where ρ_0 is completely reducible, which allows us to assume that $b = 0$ in (2). Note that in this case we must have $xy \neq 0$, since otherwise ρ is the direct sum of a one- and a two-dimensional subrepresentation ($x = 0, y \neq 0$ or $x \neq 0, y = 0$) or three one-dimensional subrepresentations ($x = y = 0$). Now (4) shows that we may take $x = y = 1$, and setting (6) equal to the identity matrix (and using $b = 0$) shows that $a = d = -\sigma$. Thus

$$\rho(S) = \begin{pmatrix} -\sigma & 0 & 1 \\ 0 & -\sigma & 1 \\ 0 & 0 & \sigma \end{pmatrix}$$

and

$$\rho(ST) = \begin{pmatrix} -\sigma\lambda_1 & 0 & \lambda_3 \\ 0 & -\sigma\lambda_2 & \lambda_3 \\ 0 & 0 & \sigma\lambda_3 \end{pmatrix}.$$

One then computes and finds (using $\sigma^2 = 1$) that

$$\rho(ST)^3 = \begin{pmatrix} -\sigma\lambda_1^3 & 0 & \lambda_3(\lambda_1^2+\lambda_3^2-\lambda_1\lambda_3) \\ 0 & -\sigma\lambda_2^3 & \lambda_3(\lambda_2^2+\lambda_3^2-\lambda_2\lambda_3) \\ 0 & 0 & \sigma\lambda_3^3 \end{pmatrix}.$$

Setting this equal to the identity matrix, we see that

$$-\lambda_1^3 = -\lambda_2^3 = \lambda_3^3 = \sigma, \tag{7}$$

so each of the λ_j are sixth roots of unity. The off-diagonal entries show that the ratios $\frac{\lambda_3}{\lambda_1}$ and $\frac{\lambda_3}{\lambda_2}$ are primitive cube roots of -1, since each is a solution of $x^2 - x + 1 = 0$. This is also implied by (7), so any sixth roots of unity λ_j satisfying (7) give a representation in this setting and, furthermore, the λ_j define all the entries of $\rho(S)$ explicitly. Note, however, that the choice of ordering of λ_1 and λ_2 is irrelevant, since e.g. conjugation by

$U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ takes

$$\rho(T) = \mathrm{diag}\left\{\lambda_1, \lambda_2, \lambda_3\right\}, \quad \rho(S) = \begin{pmatrix} a & 0 & x \\ 0 & b & y \\ 0 & 0 & \sigma \end{pmatrix}$$

to

$$U\rho(T)U^{-1} = \mathrm{diag}\left\{\lambda_2, \lambda_1, \lambda_3\right\}, \quad U\rho(S)U^{-1} = \begin{pmatrix} b & 0 & y \\ 0 & a & x \\ 0 & 0 & \sigma \end{pmatrix}.$$

With this in mind, and writing $\lambda_j = \mathbf{e}\left(\frac{x_j}{6}\right)$ for integers $0 \le x_j \le 5$, we obtain eighteen possibilities as follows:

TABLE 1. ρ_0 completely reducible

x_1	x_2	x_3
1	1	0
1	5	0
5	5	0
2	2	1
2	0	1
0	0	1
3	3	2
3	1	2
1	1	2
4	4	3
4	2	3
2	2	3
5	5	4
5	3	4
3	3	4
0	0	5
0	4	5
4	4	5

This establishes part i) of Theorem 1.1.

We now consider the case where the subrepresentation (5) is reducible but indecomposable. This allows us to assume that $c = 0$ but $b \neq 0$ in (2). Note that (6) implies that $x \neq 0$, since otherwise we would have $x = y = 0$ and ρ would be completely reducible. It may, however, occur that $y = 0$, so we assume this first. We find from (4) that conjugation by a diagonal matrix allows us to assume that $b = x = 1$, and (6) shows that $a = -\sigma = -d$ with $\sigma^2 = 1$, so we have

$$\rho(S) = \begin{pmatrix} -\sigma & 1 & 1 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{pmatrix}. \tag{8}$$

We compute and find (using $\sigma^2 = 1$) that

$$\rho(ST)^3 = \begin{pmatrix} \sigma\lambda_1^3 & \lambda_2(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2) & \lambda_3(\lambda_1^2 - \lambda_1\lambda_3 + \lambda_3^2) \\ 0 & -\sigma\lambda_2^3 & 0 \\ 0 & 0 & -\sigma\lambda_3^3 \end{pmatrix}.$$

Setting this equal to the identity matrix shows that $\sigma = \lambda_1^3 = -\lambda_2^3 = -\lambda_3^3$, and the ratios $\frac{\lambda_2}{\lambda_1}$ and $\frac{\lambda_3}{\lambda_1}$ are primitive cube roots of -1. In this case, one sees that conjugation by $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ leaves (8) invariant but takes (3) to $U\rho(T)U^{-1} = \mathrm{diag}\,\{\lambda_1, \lambda_3, \lambda_2\}$, so the relabeling of λ_2 and λ_3 does not give a different representation. Writing $\lambda_j = \mathbf{e}\left(\frac{x_j}{6}\right)$ for integers $0 \le x_j \le 5$, we then have the following 18 possibilities:

TABLE 2. ρ_0 reducible, indecomposable and $y = 0$

x_1	x_2	x_3
1	0	0
5	0	0
1	2	0
5	4	0
2	1	1
0	1	1
2	3	1
0	5	1
1	2	2
3	2	2
3	4	2
2	3	3
4	3	3
4	5	3
5	4	4
3	4	4
0	5	5
4	5	5

This gives 18 of the 30 equivalence classes in statement ii) of Theorem 1.1.

The final case for upper triangular representations occurs when $c = 0$ but $bxy \neq 0$ in (2). In this case we can assume (using (4)) that $x = y = 1$, and setting (6) equal to the identity matrix shows that

$$\rho(S) = \begin{pmatrix} \sigma & -2\sigma & 1 \\ 0 & -\sigma & 1 \\ 0 & 0 & \sigma \end{pmatrix}$$

with $\sigma^2 = 1$. Using this identity, we now compute and find that

$$\rho(ST)^3 = \begin{pmatrix} \sigma\lambda_1^3 & -2\sigma\lambda_2(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2) & \lambda_3(\lambda_1^2 - 2\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2) \\ 0 & -\sigma\lambda_2^3 & \lambda_3(\lambda_2^2 - \lambda_2\lambda_3 + \lambda_3^2) \\ 0 & 0 & \sigma\lambda_3^3 \end{pmatrix}.$$

The diagonal entries show that $\sigma = \lambda_1^3 = -\lambda_2^3 = \lambda_3^3$, so $\frac{\lambda_2}{\lambda_1}$ and $\frac{\lambda_2}{\lambda_3}$ are each cube roots of -1, and the super diagonal entries show that these roots are primitive, since they each solve the equation $x^2 - x + 1 = 0$. Using the identities defined by these entries, the final nonzero entry yields the additional constraint $\lambda_1\lambda_3 - \lambda_1\lambda_2 - \lambda_2\lambda_3 = 0$, which in terms of primitive roots is the relationship $\frac{\lambda_2}{\lambda_1} + \frac{\lambda_2}{\lambda_3} = 1$. This shows either $\frac{\lambda_2}{\lambda_1} = \mathbf{e}\left(\frac{1}{6}\right)$ and $\frac{\lambda_2}{\lambda_3} = \mathbf{e}\left(\frac{5}{6}\right)$, or vice versa. Writing $\lambda_j = \mathbf{e}\left(\frac{x_j}{6}\right)$ with $0 \le x_j \le 5$, we obtain the following 12 representations:

TABLE 3. ρ_0 reducible, indecomposable and $y \neq 0$

x_1	x_2	x_3
5	0	1
1	0	5
0	1	2
2	1	0
1	2	3
3	2	1
2	3	4
4	3	2
3	4	5
5	4	3
4	5	0
0	5	4

This completes the proof of statement ii) in Theorem 1.1.

Now we deal with the case where the subrepresentation (5) is irreducible. This allows us to assume that $\rho(T)$ is as in (3) and $\rho(S)$ has the form (2) with $abcd \neq 0$. We also note that for any such ρ_0 it is known that $\lambda_1 \neq \lambda_2$ (*cf.* [Mas08, Thm 3.1]). In this case we obtain

$$\rho(S)^2 = \begin{pmatrix} a^2 + bc & b(a+d) & by + x(a+\sigma) \\ c(a+d) & d^2 + bc & cx + y(d+\sigma) \\ 0 & 0 & \sigma^2 \end{pmatrix}$$

and this shows that $xy \neq 0$ (or ρ would be completely reducible). Again employing (4) we may assume that $x = y = 1$, and setting $\rho(S)^2$ equal to the identity matrix now shows that $b = -a - \sigma$, $c = a - \sigma$ and we have

$$\rho(S) = \begin{pmatrix} a & -a-\sigma & 1 \\ a-\sigma & -a & 1 \\ 0 & 0 & \sigma \end{pmatrix}. \tag{9}$$

Note that $a \neq \pm\sigma$ since ρ is not triangularizable. Using this information, we compute and find that $\rho(ST)^3 = (a_{ij})$ with

$$a_{11} = a\lambda_1(a^2\lambda_1^2 + 2\lambda_1\lambda_2 - 2a^2\lambda_1\lambda_2 - \lambda_2^2 + a^2\lambda_2^2)$$
$$a_{12} = -(a+\sigma)\lambda_2[a^2\lambda_1^2 + \lambda_1\lambda_2 - 2a^2\lambda_1\lambda_2 + a^2\lambda_2^2]$$
$$a_{13} = \lambda_3[a^2\lambda_1^2 + \lambda_1\lambda_2 - 2a^2\lambda_1\lambda_2 + a^2\lambda_2^2 - \lambda_2\lambda_3 + \lambda_3^2 - a\sigma\lambda_1\lambda_2 + a\sigma\lambda_2^2 + a\sigma\lambda_1\lambda_3 - a\sigma\lambda_2\lambda_3]$$
$$a_{21} = (a-\sigma)\lambda_1[a^2\lambda_1^2 + \lambda_1\lambda_2 - 2a^2\lambda_1\lambda_2 + a^2\lambda_2^2]$$
$$a_{22} = -a\lambda_2[-\lambda_1^2 + a^2\lambda_1^2 + 2\lambda_1\lambda_2 - 2a^2\lambda_1\lambda_2 + a^2\lambda_2^2]$$
$$a_{23} = \lambda_3(a^2\lambda_1^2 + \lambda_1\lambda_2 - 2a^2\lambda_1\lambda_2 + a^2\lambda_2^2 - \lambda_1\lambda_3 + \lambda_3^2 - a\lambda_1^2\sigma + a\lambda_1\lambda_2\sigma + a\lambda_1\lambda_3\sigma - a\lambda_2\lambda_3\sigma),$$
$$a_{31} = a_{32} = 0, \quad a_{33} = \sigma\lambda_3^3.$$

Setting a_{12} equal to zero gives $a^2 = -\frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}$, and using this in the identity $a_{11} = 1$ yields $a = \frac{1}{\lambda_1\lambda_2(\lambda_1-\lambda_2)}$. Since ρ_0 and χ are representations, there is no need to check any of the remaining entries except for a_{13} and a_{23}. Substituting our values for a, a^2, and σ, we now obtain

$$a_{13} = \frac{(\lambda_1\lambda_2+\lambda_3^2)[(\lambda_1\lambda_2)^2-\lambda_1\lambda_2^2\lambda_3+\lambda_3^4]}{\lambda_1^2\lambda_2^2\lambda_3^5},$$
$$a_{23} = \frac{(\lambda_1\lambda_2+\lambda_3^2)[(\lambda_1\lambda_2)^2-\lambda_1^2\lambda_2\lambda_3+\lambda_3^4]}{\lambda_1^2\lambda_2^2\lambda_3^5}.$$

Note that if $\lambda_1\lambda_2 \neq -\lambda_3^2$ then setting $a_{13} = a_{23} = 0$ forces $\lambda_1 = \lambda_2$, which is false. Therefore $\lambda_3^2 = -\lambda_1\lambda_2 = -\det\rho_0(T)$. Either of the two possible choices of λ_3 gives a representation, and there is no restriction on the irreducible ρ_0 we started with. Thus we have the following

Theorem 2.1. *Suppose the representation ρ_0 in (5) is irreducible. Then up to equivalence there are exactly two indecomposable $\rho : \Gamma \to GL_3(\mathbb{C})$ with $\rho(T)$, $\rho(S)$ as in (3), (2) respectively. These representations are uniquely determined by the choice of $\lambda_3 = \sqrt{-\det\rho_0(T)}$.* □

This implies statement iii) of Theorem 1.1.

We now wish to consider the case where ρ has a one-dimensional invariant subspace but no two-dimensional invariant subspace. Note that such a representation may be obtained by taking the dual of one of the representations classified above, and a ρ of the above form has dual ρ^* (where $\rho^*(\gamma) = \rho(\gamma^{-1})^t$ for each $\gamma \in \Gamma$) with

$$\rho^*(T) = \operatorname{diag}\left\{\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1}\right\}, \quad \rho^*(S) = \begin{pmatrix} a & c & 0 \\ b & d & 0 \\ x & y & \sigma \end{pmatrix}.$$

Conjugation by $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ allows us to assume that the first standard basis vector spans the resulting one-dimensional invariant subspace, and this gives the representation

$$\rho'(T) = \operatorname{diag}\left\{\lambda_3^{-1}, \lambda_2^{-1}, \lambda_1^{-1}\right\}, \quad \rho'(S) = \begin{pmatrix} \sigma & y & x \\ 0 & d & b \\ 0 & c & a \end{pmatrix}. \tag{10}$$

Thus the classification for the T-unitarizable case will be complete once we determine which of the above ρ' are not conjugate to any ρ as in (2), (3). We may assume via conjugation that $c = 0$ if and only if (5) is reducible, and in this case it is evident that ρ' is one of the representations already classified. This establishes the first statement in Theorem 1.2. On the other hand, if (5) is irreducible then ρ cannot have a one-dimensional invariant subspace (or ρ would be completely reducible), so the dual ρ' is not one of the representations we have already classified. This implies the second statement of Theorem 1.2.

To complete the proofs of Theorems 1.1 and 1.2, we need to classify those indecomposable but reducible $\rho : \Gamma \to GL_3(\mathbb{C})$ for which $\rho(T)$ is not unitarizable. We begin by assuming that ρ has a 2-dimensional invariant subspace and that $\rho(S)$ again has the form (2), with the subrepresentation ρ_0 in (5) in the upper left-hand corner. Note that $\rho_0(T)$ is not diagonalizable, or $\rho(T)$ would be too. In particular, this implies that ρ_0 is irreducible and we may assume there is an integer k such that

$$\rho_0(T) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}, \ \rho_0(S) = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \tag{11}$$

with $\lambda = \mathbf{e}\left(\frac{2k+1}{12}\right)$, $\epsilon = \mathbf{e}\left(-\frac{2k+1}{4}\right) = \pm i$. Thus we are now assuming that

$$\rho(T) = \begin{pmatrix} \lambda & \lambda & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \rho(S) = \begin{pmatrix} 0 & -\epsilon & x \\ \epsilon & 0 & y \\ 0 & 0 & \sigma \end{pmatrix}.$$

One finds that

$$\rho(S^2) = \begin{pmatrix} 1 & 0 & x\sigma - y\epsilon \\ 0 & 1 & x\epsilon + y\sigma \\ 0 & 0 & \sigma^2 \end{pmatrix},$$

and setting this equal to the identity matrix shows that $\sigma^2 = 1$ and $x = 0$ if and only if $y = 0$. Since ρ is indecomposable this forces $xy \neq 0$, and conjugation with a diagonal matrix shows that we may assume that

$$\rho(T) = \begin{pmatrix} \lambda & a & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \rho(S) = \begin{pmatrix} 0 & b & 1 \\ c & 0 & 1 \\ 0 & 0 & \sigma \end{pmatrix}$$

for some $a, b, c \in \mathbb{C}^\times$. Now we have

$$\rho(S)^2 = \begin{pmatrix} bc & 0 & b+\sigma \\ 0 & bc & c+\sigma \\ 0 & 0 & \sigma^2 \end{pmatrix},$$

so we must have $b = c = -\sigma$. This gives

$$\rho(ST)^3 = \begin{pmatrix} -a\lambda^2\sigma & -\lambda(a^2+\lambda^2)\sigma & \lambda_3(a\lambda + \lambda^2 - \lambda\lambda_3 + \lambda_3^2) \\ -\lambda(a^2+\lambda^2)\sigma & -a(a^2+2\lambda^2)\sigma & \lambda_3(a^2 + a\lambda + \lambda^2 - a\lambda_3 - \lambda\lambda_3 + \lambda_3^2) \\ 0 & 0 & \lambda_3^3\sigma \end{pmatrix}.$$

Now the term in parenthesis in the $(1,3)$ entry must equal zero, and making this substitution in the parenthetical term of the $(2,3)$ entry and setting it equal to zero shows that $a = \lambda_3$, so the equivalence class of ρ is completely determined by λ and λ_3. The $(1,2)$, $(1,3)$ and $(2,1)$ entries now all vanish if and only if $a^2 = \lambda_3^2 = -\lambda^2$. One now checks easily that either choice of $\lambda_3 = \sqrt{-\lambda^2}$ does define a representation of Γ.

As in the T-unitarizable setting, since (11) is irreducible the above ρ do not admit a one-dimensional invariant subspace. This means that when we take the dual representations of these ρ we obtain 12 additional inequivalent representations that are not T-unitarizable, and none of these are equivalent to the above 12. We summarize the non-T-unitarizable cases as follows.

Theorem 2.2. *Up to equivalence, there are 24 indecomposable but reducible representations $\rho : \Gamma \to GL_3(\mathbb{C})$ for which $\rho(T)$ is not unitarizable. In 12 of the corresponding equivalence classes there is a representative of the form*

$$\rho(T) = \begin{pmatrix} \lambda & \lambda & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \rho(S) = \begin{pmatrix} 0 & -\lambda_3^3 & 1 \\ -\lambda_3^3 & 0 & 1 \\ 0 & 0 & \lambda_3^3 \end{pmatrix} \tag{12}$$

where $\lambda = \boldsymbol{e}\left(\frac{2k+1}{12}\right)$ with $0 \leq k \leq 5$ and $\lambda_3 = \sqrt{-\lambda^2} = \pm\boldsymbol{e}\left(\frac{k}{6}\right)$. The other 12 equivalence classes are obtained by taking the dual representation of (12), and each contains a representative

$$\rho'(T) = \begin{pmatrix} \lambda_3^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & -\lambda^{-1} \\ 0 & 0 & \lambda^{-1} \end{pmatrix}, \quad \rho'(S) = \begin{pmatrix} \lambda_3^3 & 1 & 1 \\ 0 & 0 & -\lambda_3^3 \\ 0 & -\lambda_3^3 & 0 \end{pmatrix} \tag{13}$$

with λ and λ_3 as above. ☐

This result completes the proofs of Theorems 1.1 and 1.2, and with it the classification of three-dimensional indecomposable but reducible representations of Γ.

3. Generating Weights and the M-functor

Let $\rho : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be a finite-dimensional complex representation. For any $k \in \mathbb{Z}$, let $M_k(\rho)$ be the (finite-dimensional) complex vector space of *holomorphic* ρ-valued modular forms ([CF16],[MM10]) of weight k, and let

$$M(\rho) := \bigoplus_{k\in\mathbb{Z}} M_k(\rho)$$

be the $\mathbb{Z}$-graded module of ρ-valued modular forms over the ring $R = M(1) = \mathbb{C}[E_4, E_6]$ of modular forms of level 1. By [MM10, Thm 1] this module is free of rank $d = \dim\rho$, thus any choice of homogeneous generators gives an isomorphism

$$M(\rho) \simeq \bigoplus_{i=1}^{d=\dim\rho} R[-k_i],$$

where by $R[a]$ we denote the rank one graded module over R obtained by shifting the grading by a. The d-tuple of integers $(k_1, \ldots, k_d)$ does not depend on the choice of generators, and the k_i's are called the *generating weights* of ρ. The goal of this section is to elucidate the relationship between the generating weights of indecomposable, reducible representations $0 \to \rho_0 \to \rho \to \rho_1 \to 0$ and the generating weights of ρ_0, ρ_1. For simplicity we almost always assume that the generating weights of ρ_0 and ρ_1 all lie in $\{0, \ldots, 11\}$ (this is always the case for unitarizable, or even *positive* representations [CF16], §6), but the same methods apply in full generality.

Let $\mathrm{Rep}(\mathrm{SL}_2(\mathbb{Z}))$ be the category of finite-dimensional complex representations of $\mathrm{SL}_2(\mathbb{Z})$. In [MM10], it is shown that the functor

$$\begin{aligned} M : \mathrm{Rep}(\mathrm{SL}_2(\mathbb{Z})) &\longrightarrow \mathrm{grMod}_R \\ \rho &\longmapsto M(\rho) \end{aligned}$$

to the category of finitely generated graded R-modules is faithful and left exact, but not right exact. In particular, given an exact sequence of representations

$$0 \to \rho_0 \to \rho \to \rho_1 \to 0, \tag{14}$$

applying the M-functor we obtain a long exact sequence of graded vector spaces

$$0 \to M(\rho_0) \to M(\rho) \to M(\rho_1) \xrightarrow{\delta} K,$$

where K is, in general, non-zero. To obtain control over this 'error term', recall the construction of [CF16], which assigns to each $\rho \in \mathrm{Rep}(\mathrm{SL}_2(\mathbb{Z}))$ a vector bundle $\mathcal{V}(\rho)$ over the compact complex orbifold $\overline{\mathcal{M}} := \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h} \cup \{\infty\}$, which is just the *canonical extension* to $\overline{\mathcal{M}}$ of the local system over $SL_2(\mathbb{Z})\backslash\mathfrak{h}$ determined by ρ. The relationship between $M(\rho)$ and $\mathcal{V}(\rho)$ can be given as follows. Let $\mathcal{L}_k$ be the line bundle of holomorphic modular forms on $\overline{\mathcal{M}}$, characterized by

$$H^0(\overline{\mathcal{M}}, \mathcal{L}_k) = M_k(1) = \text{holomorphic modular forms of weight } k.$$

Then $\mathcal{V}(\rho)$ is a vector bundle whose defining property is that the global sections of $\mathcal{V}_k(\rho) := \mathcal{V}(\rho) \otimes \mathcal{L}_k$ are precisely the holomorphic ρ-valued modular forms of weight k, i.e.

$$H^0(\overline{\mathcal{M}}, \mathcal{V}_k(\rho)) = M_k(\rho) = \text{holomorphic } \rho\text{-valued modular forms of weight } k.$$

Using the basic properties of sheaf cohomology, we may thus express K as

$$K = \bigoplus_{k\in\mathbb{Z}} H^1(\overline{\mathcal{M}}, \mathcal{V}_k(\rho_0)).$$

By standard vanishing theorems in algebraic geometry, K is a finite-dimensional (graded) vector space. The linear map $\delta = \oplus_k \delta_k$ is also graded, and by the above it is non-zero only for finitely many $k \in \mathbb{Z}$.

Definition 3.1. *The representation ρ is said to be M*-split *if $\delta = 0$.*

Note that if ρ is M-split then there is an R-module isomorphism

$$M(\rho) \simeq M(\rho_0) \oplus M(\rho_1),$$

and therefore the generating weights of ρ are just the union of the generating weights of ρ_0 and ρ_1. We thus determine a criterion for when ρ is M-split:

Theorem 3.2. *Suppose the generating weights of ρ_0, ρ_1 lie in $[0, 11] \cap \mathbb{Z}$. Then ρ is M-split unless at least one of the following occurs:*

(i) *$k = 0$ is a generating weight of ρ_1 and $k = 10$ is a generating weight of ρ_0*
(ii) *$k = 1$ is a generating weight of ρ_1 and $k = 11$ is a generating weight of ρ_0*

Proof. Suppose $k_1, \ldots, k_n$ are the generating weights of ρ_0, Then $K = K_0 \oplus K_1$ is only graded in degrees $k = 0, 1$, since

$$K = \bigoplus_{k\geq 0} H^1(\overline{\mathcal{M}}, \oplus_{i=1}^n \mathcal{L}_{k-k_i}),$$

and $H^1(\overline{\mathcal{M}}, \mathcal{L}_k) = 0$ for all $k = 0, -1, \ldots, -9, -11$ and it is a 1-dimensional vector space of $k = -10$. More precisely, if we denote by $m_0, \ldots, m_{11}$ the multiplicity of each weight of ρ_0, we have

$$K_0 = H^1(\overline{\mathcal{M}}, \mathcal{L}_{-10}^{\oplus m_{10}}), \quad K_1 = H^1(\overline{\mathcal{M}}, \mathcal{L}_{-10}^{\oplus m_{11}}).$$

Therefore if $m_{10} = 0$ and $m_{11} = 0$ then $K = 0$ and ρ is M-split. Suppose then that $m_{10} \neq 0$. Applying the M-functor we get an exact sequence of R-modules

$$0 \to M(\rho_0) \to M(\rho) \to M(\rho_1) \xrightarrow{\delta} K_0$$

so that $\delta = \delta_0$ can only be non-zero in degree 0, where it is given by

$$M_0(\rho_1) \xrightarrow{\delta_0} K_0.$$

But if $k = 0$ is not a generating weight of ρ_1, then $M_0(\rho_1) = 0$ and thus $\delta = 0$, i.e. ρ is M-split. The case when $m_{11} \neq 0$ is obtained similarly by looking at degree 1. □

Remark 3.3. *The hypotheses of Theorem 3.2 always hold whenever ρ_0, ρ_1 are unitarizable, or even* positive *([CF16], §6).*

Remark 3.4. *Note that Theorem 3.2 can be viewed as a generalization to higher dimensions of [MM10], Theorem 4, which covers the case of ρ being 2-dimensional. The result of [MM10] is obtained by different means using the modular derivative D.*

3.1. **2-dimensional indecomposable representations.** We now employ Theorem 3.2 to compute the generating weights for reducible, indecomposable 2-dimensional representations of $\mathrm{SL}_2(\mathbb{Z})$. As is well-known, there are exactly 12 one-dimensional representations χ^a, $a = 0, \ldots, 11$ of $\mathrm{SL}_2(\mathbb{Z})$, characterized by

$$\chi^a(T) = \mathbf{e}\left(\tfrac{a}{12}\right).$$

As shown in [CF16], in this case we have

$$\mathcal{V}(\chi^a) \simeq \mathcal{L}_{-a}$$

and therefore the unique generating weight of χ^a is just the integer $a \in [0, 11]$.

Next, suppose that we have a 2-dimensional representation ρ which is an extension

$$1 \to \chi^a \to \rho \to \chi^b \to 1, \tag{15}$$

and suppose that ρ is indecomposable. By [MM10, Lem 4.3] there are precisely 24 isomorphism classes of such representations. These isomorphism classes are in bijection with ordered pairs $(a, b) \in \{0, \ldots, 11\}^2$ such that $a-b \equiv \pm 2 \mod 12$. A representative $\rho_{(a,b)}$ in each isomorphism class can be chosen to have

$$\rho_{(a,b)}(T) = \begin{pmatrix} \mathbf{e}\left(\frac{a}{12}\right) & 0 \\ 0 & \mathbf{e}\left(\frac{b}{12}\right) \end{pmatrix},$$

so that the pair (a, b) corresponds to the presentation of ρ as in (15).

Theorem 3.5.

(1) *If* $(a, b) \neq (10, 0), (11, 1)$, *then the generating weights of* $\rho_{(a,b)}$ *are* $(k_1, k_2) = (a, b)$.

(2) *If* $(a, b) = (10, 0), (11, 1)$ *the generating weights of* $\rho_{(a,b)}$ *are* $(4, 6)$, $(5, 7)$, *respectively.*

Proof. Part (1) follows from Theorem 3.2, since in this case ρ is M-split, and the generating weights of ρ are just (a, b). In part (2), we do not know a priori whether ρ is M-spit. We want to show that in fact ρ is not M-split. Suppose first $(a, b) = (10, 0)$. By applying the global section functor we obtain an exact sequence of vector spaces

$$0 \to H^0(\mathcal{V}(\rho)) \to H^0(\mathcal{V}(1)) \xrightarrow{\delta} H^1(\mathcal{V}(\chi^{10})) \to H^1(\mathcal{V}(\rho)) \to 0,$$

where both terms $H^0(\mathcal{V}(1)), H^1(\mathcal{V}(\chi^{10}))$ are 1-dimensional since $\mathcal{V}(1) = \mathcal{L}_0$ and $\mathcal{V}(\chi^{10}) = \mathcal{L}_{-10}$. If $\delta = 0$ (i.e. ρ is M-split), then the generating weights of ρ are 0,10. In particular, there is a modular form $F \in M_0(\rho)$ of minimal weight 0. This modular form must satisfy $F' = 0$, since $F' \in M_2(\rho) = 0$. In other words, $F \in V$ is constant, and it gives a ρ-invariant vector. Now since ρ contains the 1-dimensional subrepresentation χ^{10} (with quotient the trivial representation) we may find another vector $v \in V$ such that

$$\rho(\gamma)v = \chi^{10}(\gamma)v, \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Clearly v and F are linearly independent, since they are both eigenvectors of $\rho(T)$ with different eigenvalues, thus they form a basis for ρ. But this is impossible, since then $\rho \sim \chi^{10} \oplus 1$, which would contradict the fact that ρ is indecomposable. Therefore δ is an isomorphism, so that the minimal weight is $k_1 = 4$ and the generating weights are $(4, 6)$. For the case $(a, b) = (11, 1)$, note that multiplication by η^2 gives an isomorphism of graded R-modules $M(\rho_{(10,0)}) \simeq M(\rho_{(11,1)})$ shifting the degree by 1. But we just proved that $M(\rho_{(10,0)}) \simeq R[-4] \oplus R[-6]$ and therefore $M(\rho_{(11,1)}) \simeq R[-5] \oplus R[-7]$, i.e. the generating weights of $\rho_{(11,1)}$ are $(5, 7)$. □

Remark 3.6. *Note that the contents of Theorem 3.5 can be essentially extracted from the proof of* [MM10]*, Theorem 4, where they are proved by different means using the modular derivative D.*

4. Generating weights in dimension three

We now turn to computing the generating weights of indecomposable, reducible representation of dimension 3. For simplicity, we restrict here to representations that factor through $\mathrm{PSL}_2(\mathbb{Z})$ and such that ρ_0, ρ_1 have generating weights lying in $\{0, \ldots, 11\}$, but the same methods can be applied in full generality. First, we consider those representations ρ that have a 2-dimensional subrepresentation, i.e. they are of the form

$$0 \to \rho_0 \to \rho \to \chi^{2a} \to 0, \tag{16}$$

with ρ a 2-dimensional representation and $a = 0, \ldots, 5$. Then Theorem 3.2 in this case reduces to

Corollary 4.1. *Suppose ρ_0 has generating weights given by $(k_1, k_2) \in \{0, \ldots, 11\}^2$. Then ρ is M-split unless $a = 0$ and either k_1 or k_2 is equal to 10.*

We use this criterion to find the generating weights of ρ. We divide the computations in three cases, according to whether ρ_0 is irreducible, reducible but indecomposable, or completely reducible.

4.1. ρ_0 **is irreducible.** According to Theorem 2.1, for each isomorphism class of a 2-dimensional, irreducible ρ_0, there are precisely two isomorphism classes of representations ρ, characterized by

$$\lambda_3 = \mathbf{e}\left(\tfrac{a}{6}\right) = \sqrt{-\det \rho_0(T)}.$$

For such representations, we have:

Theorem 4.2. *Suppose ρ_0 is irreducible, with generating weights equal to $(k_1, k_2) \in \{0, \ldots, 11\}^2$. Then*

(i) *if $(k_1, k_2) \neq (8, 10)$, or $(k_1, k_2) = (8, 10)$ and $a = 3$, then ρ is M-split.*
(ii) *if $(k_1, k_2) = (8, 10)$ and $a = 0$, then ρ is not M-split and the generating weights are $(4, 6, 8)$.*

Proof. Let (k_1, k_2) be the generating weights of ρ_0. Then

$$-\det \rho_0(T) = \mathbf{e}\left(\tfrac{k_1+k_2+6}{12}\right)$$

and thus the two choices for λ_3 (hence for a) are completely determined by the generating weights of ρ_0. Since ρ has finite image, its generating weights lie in the interval $[0, 11]$. The weights must be even, since all $\mathrm{PSL}_2(\mathbb{Z})$-representations have even weights, and since ρ is irreducible we must have that $k_2 = k_1 + 2$. This leaves only 5 possibilities for the pair (k_1, k_2). For each possible pair, we also compute in the table below both choices of a such that ρ is indecomposable:

(k_1, k_2)	$2a$
(0,2)	4,10
(2,4)	0,6
(4,6)	2,8
(6,8)	4,10
(8,10)	0,6

By part (i) of Corollary 4.1, all ρ_0 with pairs of generating weights different than $(8, 10)$ give rise to M-split representations ρ. For the pair $(8, 10)$, there are two choices of a given by $a = 0, 3$. When $a = 3$, we again conclude by part (i) of Corollary 4.1 that ρ is M-split. It remains to consider the case of $(k_1, k_2) = (8, 10)$ and $a = 0$. In this case, suppose that ρ is M-split, so that its generating weights are $(0, 8, 10)$. Let $F \in M_0(\rho)$ be a modular form of minimal weight. Then $F' \in M_2(\rho)$ must be equal to 0, since there are no modular forms of weight 2 for ρ, thus $F \in V$ is a constant ρ-invariant vector. Choose a basis v_1, v_2 for the subrepresentation $\rho_0 \subseteq \rho$. Then $\{v_1, v_2, F\}$ must be linearly independent, since they are all eigenvectors of $\rho(T)$ corresponding to different eigenvalues. Indeed, the eigenvalue $\lambda = 1$ corresponding to F cannot be an eigenvalue of $\rho_0(T)$: if we let L be the unique matrix such that $e^{2\pi i L} = \rho_0(T)$ and such that the real parts of the eigenvalues of L lie in $[0, 1)$ (the *standard choice of exponents*) then we know that L must satisfy $12\,\mathrm{Tr}(L) = k_1 + k_2 = 18$ ([CF16]). But if one of the eigenvalues of $\rho_0(T)$ is 1, then one of the eigenvalues of L is 0, which means the other one must be equal to $18/12$, contradicting the choice of logarithm. Therefore $\{v_1, v_2, F\}$ must be linearly independent, hence they must form a basis for V, which is impossible since then $\rho \sim \rho_0 \oplus 1$. Thus in this case ρ is not M-split. □

To compute the generating weights of ρ, write

$$\rho(T) = \mathrm{diag}\,\{\lambda_1, \lambda_2, \lambda_3\}$$

with respect to a basis such that $\rho_0(T) = \mathrm{diag}\,\{\lambda_1, \lambda_2\}$. Write $\lambda_j = \mathbf{e}\,(r_j)$, with $0 \leq r_j < 1$.

Corollary 4.3. *Suppose ρ_0 is as in Theorem 4.2.*

(i) *If $r_1 + r_2 \neq 3/2$, or if $r_1 + r_2 = 3/2$ and $a = 3$, then the generating weights of ρ are $(6(r_1 + r_2) - 1, 6(r_1 + r_2) + 1, 2a)$.*

(ii) *if $(r_1 + r_2) = 3/2$ and $a = 0$, then the generating weights of ρ are $(4, 6, 8)$.*

Proof. Let k_1, k_2 be the generating weight of ρ_0. Then $k_1 = 6(r_1 + r_2) - 1$ and $k_2 = 6(r_1 + r_2) + 1$, so that part (i) follows from Theorem 4.2. For part (ii), note that we are in the case $(k_1, k_2) = (8, 10)$ and $a = 0$, so that ρ is not M-split by Theorem 4.2. Thus $M_0(\rho) = \ker \delta_0 = 0$, and $M_k(\rho) = M_k(\rho_0) \oplus M_k(1)$ for $k \geq 2$, so that the generating weights are $(4, 6, 8)$. □

Remark 4.4. *A representation of finite image is necessarily unitarizable. Mason* [Mas08, Thm 3.7] *has classified all irreducible two-dimensional representations of $SL_2(\mathbb{Z})$ with finite image, finding 54 equivalence classes of these. Thus there are 108 equivalence classes of 3-dimensional indecomposable representations ρ with such a subrepresentation ρ_0. Mason (loc. cit.) has also calculated the generating weights for each ρ_0, thus Theorem 4.2 explicitly gives the generating weights for all 108 isomorphism classes of 3-dimensional indecomposable ρ with an irreducible 2-dimensional factor ρ_0 of finite image.*

4.2. **ρ_0 is reducible but indecomposable.** In this case, the λ_j are all 6-th roots of unity and we may write $\lambda_j = \mathbf{e}\left(\frac{x_j}{6}\right)$, for some $x_j = 0, \ldots, 5$. The isomorphism class of ρ is entirely determined by the triple (x_1, x_2, x_3) and there are 30 such possible triples, listed in Tables 2 and 3.

Theorem 4.5. *Suppose ρ_0 is reducible but indecomposable, and write $\rho_{(x_1,x_2,x_3)}$ for the unique (up to equivalence) representation determined by one of the triples (x_1, x_2, x_3) of Tables 2 and 3.*

(i) *If* $(x_1,x_2,x_3) \neq (5,0,0),(5,4,0),(5,0,1),(4,5,0)$*, then* $\rho_{(x_1,x_2,x_3)}$ *is* M*-split and its generating weights are* $(2x_1,2x_2,2x_3)$*.*
(ii) *If* $(x_1,x_2,x_3) = (5,0,0),(5,0,1)$*, then* $\rho_{(x_1,x_2,x_3)}$ *is also* M*-split and its generating weights are* $(0,4,6),(2,4,6)$*, respectively.*
(iii) *If* $(x_1,x_2,x_3) = (5,4,0),(4,5,0)$*, then* $\rho_{(x_1,x_2,x_3)}$ *is not* M*-split and its generating weights are* $(4,6,8)$*.*

Proof. By Corollary 4.1, the only possible $\rho_{(x_1,x_2,x_3)}$'s that are not M-split must have $x_3 = 0$ and one of the generating weights of ρ_0 must be equal to 10. By Theorem 3.5, we know that the generating weights of ρ_0 are $2x_1, 2x_2$, unless $x_1 = 5$ and $x_2 = 0$, in which case the generating weights of ρ_0 are $(4,6)$. By going through Tables 2 and 3 we get part (i) and (ii). For part (iii), we use the same argument as in Thm. 4.2, part (ii). Namely, if ρ were M-split then the generating weights would have to be $(0,8,10)$, which would imply that there exists an invariant vector $F \in V^\rho$. But ρ_0 is upper-triangular with χ^8, χ^{10} on the diagonal, so F is linearly independent from ρ_0 and we could form a basis $\{v_1, v_2, F\}$ giving a splitting $\rho \sim \rho_0 \oplus 1$, which is impossible. Therefore both representations $\rho_{(5,4,0)}$ and $\rho_{(4,5,0)}$ are not M-split, and their generating weights can be computed by noting that in each case $M_0(\rho) = 0$ and $M_k(\rho) = M_k(\rho_0) \oplus M_k(1)$ for all $k \geq 2$. □

4.3. ρ_0 **is completely reducible.** In this case $\rho_{(x_1,x_2,x_3)}$ is determined (up to equivalence) by one of the 18 triples of Table 1.

Theorem 4.6. *Suppose* $\rho_0 \sim \chi^{2x_1} \oplus \chi^{2x_2}$ *is completely reducible, and write* $\rho_{(x_1,x_2,x_3)}$ *for the unique (up to equivalence) representation determined by one of the triples* (x_1,x_2,x_3) *of Table 1.*

(i) *If* $(x_1,x_2,x_3) \neq (1,5,0),(5,5,0)$*, then* $\rho_{(x_1,x_2,x_3)}$ *is* M*-split and its generating weights are* $(2x_1,2x_2,2x_3)$*.*
(ii) *If* $(x_1,x_2,x_3) = (1,5,0)$*, then* $\rho_{(1,5,0)}$ *is not* M*-split and its generating weights are* $(2,4,6)$*.*
(iii) *If* $(x_1,x_2,x_3) = (5,5,0)$*, then* $\rho_{(5,5,0)}$ *is not* M*-split and its generating weights are* $(4,6,10)$*.*

Proof.

(i) The generating weights of ρ_0 are simply $(2x_1, 2x_2)$, thus part (i) follows directly from Corollary 4.1 by going through Table 1.
(ii) For $\rho = \rho_{(1,5,0)}$, there are only two possibilities for the generating weights: $(2,4,6)$ (not M-split case) and $(0,2,10)$ (M-split case). Suppose that the latter holds. Then $M_0(\rho)$ is 1-dimensional, say generated by a modular form F_0 of weight 0, and $M_2(\rho)$ is also 1-dimensional, say generated by a modular form F_2 of weight 2. As in the proof of Theorem 4.2 (ii), this F_0 cannot be constant, for otherwise it would give an invariant vector and we could decompose ρ as $1 \oplus \chi^2 \oplus \chi^{10}$. For each $k \in \mathbb{Z}$, consider now the modular derivative D that acts as the weight-two operator
$$D = \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{k}{12}E_2$$
on $M_k(\rho)$, sending an f in this space to $Df \in M_{k+2}(\rho)$; here E_2 denotes the Eisenstein series in weight two. Applying this operator componentwise, we obtain
$$D_0F_0 = \lambda F_2$$

for some $\lambda \in \mathbb{C}$. On the other hand, up to a costant F_2 must be the modular form $(\eta^4, 0, 0)$, where η is Dedekind's eta function, so we can assume $F_2 = \eta^4$. It follows that $D_2 F_2 = 0$, that is

$$D^2 F_0 = 0.$$

As shown in [FM14b, Ex 22], the solution space of the differential operator D^2 is spanned by vector-valued modular forms with respect to a 2-dimensional, indecomposable-but-reducible subrepresentation ρ' of ρ containing either the character 1 or χ^2 and having either χ^2 or 1 as a quotient, respectively. As explained above, ρ' may not contain the character 1 and therefore ρ' is of the form

$$0 \to \chi^2 \to \rho' \to 1 \to 0.$$

Pick now a basis $\{v_1, v_2\}$ for ρ'. We also know that ρ contains the character χ^{10}, so let v be a generator for the line defined by this character. Then v cannot be linearly dependent on $\{v_1, v_2\}$, because it has eigenvalue $\mathbf{e}\left(\frac{10}{12}\right)$ under the action of $\rho(T)$, while ρ' breaks up into two eigenspaces for $\rho(T)$ with eigenvalues 1 and $e^{2\pi i 2/12}$. Thus $\{v_1, v_2, v\}$ is a basis for ρ, but this is impossible since then $\rho \simeq \rho' \oplus \chi^{10}$, contradicting the fact that ρ is indecomposable. Therefore ρ' is not M-split and the generating weights are $(2, 4, 6)$.

(iii) For $\rho = \rho_{(5,5,0)}$, the possibilities for the generating weights are $(4, 6, 10)$ (not M-split) or $(0, 10, 10)$ (M-split). If the latter holds, then again we may deduce as in Theorem 4.2, part (ii) that ρ contains an invariant vector, which is impossible since then $\rho \sim \chi^{10} \oplus \chi^{10} \oplus 1$. Therefore ρ is not M-split.

□

4.4. ρ **has a 1-dimensional subrepresentation.** The only case to consider is that of ρ being of the form

$$1 \to \chi^{2a} \to \rho \to \rho_1 \to 1,$$

where $\rho_1 = \rho_0^*$ is the dual of an irreducible 2-dimensional representation ρ, in which case ρ^* is one of the indecomposable representations of Section 4.1.

Theorem 4.7. *Suppose ρ has a 1-dimensional sub-representation of the form χ^{2a} and a 2-dimensional irreducible quotient ρ_1, with generating weights k_1, k_2. Then ρ is M-split, so that its generating weights are $(2a, k_1, k_2)$.*

Proof. By Corollary 4.1, ρ is always M-split if $2a \neq 10$. If $2a = 10$ then ρ^* is an indecomposable representation with a 2-dimensional irreducible subrepresentation $\rho_0 = \rho_1^*$ and χ^2 as a quotient. By Section 4.1, this can only happen if the generating weights of ρ_0 are $(4, 6)$. If $\rho_0(T)$ does not have 1 as an eigenvalue, then the generating weights of $\rho_1 = \rho_0^*$ are $(6, 8)$, and thus ρ is M-split by Corollary 4.1. If $\rho_0(T)$ has 1 as an eigenvalue, then

$$\rho_0^*(T) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{2\pi i/6} \end{smallmatrix}\right),$$

which is impossible, since $\rho_1 = \rho_0^*$ is irreducible [Mas08, Thm 3.1]. Therefore ρ is always M-split. □

4.5. ρ **is not** T**-unitarizable.** It remains to compute the generating weights for those representations for which $\rho(T)$ is not unitarizable. By Theorem 2.2 there are precisely 12 of them containing a 2-dimensional subfactor ρ_0, characterized by

$$\rho(T) = \begin{pmatrix} \lambda & \lambda & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

where $\lambda = \mathbf{e}\left(\frac{2k+1}{12}\right)$ with $0 \leq k \leq 5$ and $\lambda_3 = \sqrt{-\lambda^2} = \pm \mathbf{e}\left(\frac{k}{6}\right)$

Theorem 4.8. *Suppose ρ is not T-unitarizable and suppose $\rho_0 \subsetneq \rho$ is a 2-dimensional subfactor with $\lambda = \boldsymbol{e}\left(\frac{2k+1}{12}\right)$, $0 \leq k \leq 5$ and $\lambda_3 = \sqrt{-\lambda^2} = \pm \boldsymbol{e}\left(\frac{k}{6}\right)$ $k = 0, \dots 5$. Then ρ is always M-split, with generating weights $(2k, 2k+2, k)$ or $(2k, 2k+2, k')$, where $0 \leq k' \leq 11$ is congruent to $-k$ mod 6, according to whether $\lambda_3 = \sqrt{-\lambda^2} = \pm \boldsymbol{e}\left(\frac{k}{6}\right)$.*

Proof. Let

$$\rho_{st} : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{C})$$

be the *standard* representation, given by inclusion. Then Theorem 2.2 says that

$$\rho_0 \sim \chi^{2k+1} \rho_{st},$$

for some $k = 0, \dots, 5$, and that $\rho/\rho_0 \sim \chi^{2k}$ or $\chi^{2k'}$, where $k' \equiv -k \mod 6$. Now the generating weights of ρ_{st} are $(-1, 1)$ and the generating weights of ρ_0 are thus $(2k, 2k+2)$ for $k = 0, \dots, 5$. For $k = 0, 1, 2, 3$ then 10 is not a generating weight, thus ρ is M-split by Thm. 3.2. For $k = 4$ then the generating weights of ρ_0 are $(8, 10)$, but the quotient is χ^8 or χ^2, both of which are non-trivial, thus ρ is M-split again by Theorem 3.2. Finally, for $k = 5$ the generating weights of ρ_0 are $(10, 12)$, and $H^1(\overline{\mathcal{M}}, \mathcal{V}_0(\rho_0))$, $H^1(\mathcal{V}_2(\rho_0))$ are thus non-zero. But the quotient in this case is χ^{10} or χ^4, and in both cases $H^0(\overline{\mathcal{M}}, \mathcal{V}_0(\chi^k)) = H^0(\overline{\mathcal{M}}, \mathcal{V}_2(\chi^k)) = 0$, so ρ is M-split by the same argument as in Thm. 3.2. □

5. Periods of modular curves

We give in this final section a motivation from Riemann surface theory for classifying indecomposable representations of the modular group. Explicitly, we show how for appropriate representations of this type the entries of the matrices representing certain hyperbolic elements may be interpreted as the periods of a modular curve associated to the representation. We used the first section of [Man72] as a reference for the homology of modular curves, and follow Manin's notation here.

Let G denote a finite index subgroup of Γ, which we assume for simplicity is normal in Γ, and let g denote the genus of the modular curve $X_G(\mathbb{C}) = G \backslash \mathbb{H}^*$ associated to G. Fix the base point $i\infty$ for the homology group $H^1(X_G(\mathbb{C}), \mathbb{Z})$ of closed paths relative to this base point. Then by [Man72, Prop 1.4] there are hyperbolic matrices $g_j \in G$, $1 \leq j \leq 2g$, such that the paths $\{i\infty, g_j(i\infty)\}$ generate $H^1(X_G(\mathbb{C}), \mathbb{Z})$. The periods of $X_G(\mathbb{C})$ are then computed by integrating a basis $\{f_1, \cdots, f_g\}$ for the space $S_2(G)$ of weight two cusp forms for G, over the above paths in $\mathbb{H}^*$. Thus the periods are given by

$$\Omega_{jk} = \int_{i\infty}^{g_j(i\infty)} f_k(z)\, dz.$$

Because G is normal in Γ, the $|_2$-action of Γ gives a linear representation of Γ/G on $S_2(G)$, so that $F_0 = (f_1, \cdots, f_g)^t$ is a weight two vector-valued cusp form for a representation $\rho_0 : \Gamma \to GL_g(\mathbb{C})$ whose kernel contains G. By employing the modular

Wronskian [Mas07] one obtains a g^{th} order *modular differential equation* $L[f] = 0$, in weight two, whose solution space is spanned by the f_j. For each $1 \le k \le g$ let

$$u_k(\tau) = \int_{i\infty}^{\tau} f_k(z)\, dz,$$

so that $\Omega_{jk} = u_k(g_j(i\infty))$, and let $F = (u_1, \ldots, u_g, 1)^t$.

Theorem 5.1. *F is a holomorphic vector-valued modular function for an indecomposable representation $\rho : \Gamma \to \mathrm{GL}_{g+1}(\mathbb{C})$ of the form*

$$0 \to \rho_0 \to \rho \to 1 \to 0.$$

Proof. Consider the modular differential equation

$$L\theta[f] = 0 \tag{17}$$

where $\theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. Since $\theta u_k = f_k$ for each k, the solution space of (17) is spanned by the components of F, so by the covariance of the modular derivative F is a vector-valued modular function for some representation $\rho : \Gamma \to GL_{g+1}(\mathbb{C})$. Note, however, that $\theta F = (F_0, 0)^t$ is a weight two vector-valued cusp form for the same representation, so in fact the upper left $g \times g$ corner of ρ is actually ρ_0 and the quotient ρ/ρ_0 is the trivial representation. Since ρ_0 has finite image we may assume that $\rho_0(T)$ is diagonal and the basis for $S_2(G)$ has the form

$$f_k(\tau) = q^{\frac{x_k}{N}} + \sum_{n \ge 1} a_k(n) q^{\frac{x_k}{N}+n}$$

where N denotes the level of G (in the more general sense of [Woh64]) and the x_k are positive integers. Then

$$u_k(\tau) = \frac{N}{x_k} q^{\frac{x_k}{N}} + \sum_{n \ge 1} \frac{N a_j(n)}{Nn + x_k} q^{\frac{x_k}{N}+n},$$

and this shows that F is in fact holomorphic on $\mathbb{H}^*$. Since F also has weight zero, it must be that $[\Gamma : \ker \rho] = \infty$. This implies that ρ is reducible but indecomposable. $\square$

Now for $1 \le j \le 2g$ we have

$$\begin{aligned}(\Omega_{j1}, \ldots, \Omega_{jk}, 1) &= F(g_j(i\infty)) \\ &= \rho(g_j) F(i\infty) \\ &= \rho(g_j)(0, \ldots, 0, 1)\end{aligned} \tag{18}$$

by Theorem 5.1, and thus the period Ω_{jk} is equal to the $(k, g+1)$-entry of $\rho(g_j)$. This provides what seems to be a novel method for computing periods of modular curves. In particular, this method applies equally well to noncongruence or congruence subgroups, as it does not require the use of Hecke operators. Taking advantage of the classification of low-dimension representations of the modular group, we obtain the following which may be of independent interest.

Theorem 5.2. *Let $G \subseteq \mathrm{PSL}_2(\mathbb{Z})$ be a subgroup containing a finite index normal subgroup $G' \triangleleft \mathrm{PSL}_2(\mathbb{Z})$ of genus one or two. Then the period matrix of X_G has entries in $\overline{\mathbb{Q}}$.*

Proof. Let $\rho_0 : \Gamma \to \mathrm{GL}(S_2(G'))$ be the representation given by the $|_2$-action of Γ on $S_2(G')$. This representation factors through the finite group Γ/G', and therefore we may choose a basis for $S_2(G')$ so that the entries of $\rho_0(T)$ are algebraic. Since ρ_0 is one- or two-dimensional, it follows that all the entries of ρ_0 are algebraic ([Mas08], [MM10] for the two-dimensional case) as well. Let ρ be the extension of the trivial representation by ρ_0 of Thm. 5.1. Then by the results of Section 2 the entries of ρ are also algebraic (this follows from [MM10] if ρ is two-dimensional). But by (18), the periods of X_G with respect to the chosen basis of $S_2(G) \subseteq S_2(G')$ are linear combinations of entries of ρ, and thus they are algebraic. Therefore the entries of the period matrix of X_G, which do not depend on the choice of basis, are always algebraic. $\square$

Corollary 5.3. *Suppose $G \subseteq \mathrm{PSL}_2(\mathbb{Z})$ is of genus one and suppose it contains a normal subgroup $G' \triangleleft \Gamma$ of genus one or two. Then X_G is an elliptic curve with complex multiplication.*

Proof. As is well-known, X_G is defined over $\overline{\mathbb{Q}}$. Moreover, by Theorem 5.2, the period ratio of X_G is algebraic. But an elliptic curve defined over a number field with algebraic period ratio must necessarily have complex multiplication, by Schneider's Theorem. $\square$

Remark 5.4. *Corollary 5.3 relies on special properties of two- and three-dimensional representations of $\mathrm{PSL}_2(\mathbb{Z})$ and it cannot be extended to include normal subgroups G' of arbitrary genus g (that is, ρ of arbitrary dimensions $g+1$). For example $\Gamma_0(11)$ is of genus one, and it contains the normal subgroup $\Gamma(11)$, which is of genus 26. It is well known that $X_0(11)$ does not have complex multiplication (we thank Bill Hoffman for pointing this out to the authors).*

We conclude this section by utilizing Theorem 5.2 and Corollary 5.3 in the following examples.

5.1. Example: a genus one normal subgroup.

According to [CP03], the genus one normal congruence subgroup of smallest index is $G := \Gamma'$, the commutator subgroup of Γ. This is a subgroup of index 6 and level 6. In this case the representation ρ_0 of Γ on $S_2(G)$ is one-dimensional, so any non-zero $f \in S_2(G)$ must be a weight 2 modular form for a character of Γ. Since $M_2(\chi^a) = 0$ unless $a = 1$, we conclude that $\rho_0 = \chi$ and that f is a multiple of η^4. By [MM10], up to isomorphism there is a unique 2-dimensional representation ρ which is a non-trivial extension of the trivial representation by χ. This is given by

$$\rho(T) = \begin{pmatrix} \mathbf{e}(1/6) & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(S) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that by Theorem 3.5 the generating weights for ρ are $k_1 = 0$ and $k_2 = 2$, which indeed implies that there is a non-constant ρ-valued modular function F and a ρ-valued modular form of weight two given by $\theta F = (f, 0)$.

According to SAGE, G can be generated by $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right) = T^2ST$ and $B = \left(\begin{smallmatrix} 3 & -1 \\ 1 & 0 \end{smallmatrix}\right) = T^3S$, both hyperbolic matrices. Evaluating ρ at these two matrices gives

$$\rho(A) = \begin{pmatrix} 1 & \mathbf{e}(1/3) \\ 0 & 1 \end{pmatrix}, \quad \rho(B) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and thus the two periods of X_Γ are $\omega_1 = \mathbf{e}(1/3)$ and $\omega_2 = -1$. The period ratio, which does not depend on a choice of basis for $S_2(G)$, is given by

$$\tau_G = -\mathbf{e}(1/3) = \mathbf{e}(1/6) \in \mathbb{H}.$$

This computation may also be checked numerically. Choosing $\eta^4 \in S_2(G)$ as generator and i as a base-point for the period integral we find that

$$\tau_G = \frac{\int_i^{Bi} \eta^4(z)\, dz}{\int_i^{Ai} \eta^4(z)\, dz} \sim 0.500000... + 0.866025...\, i,$$

by numerically approximating the integrals using the first 4 terms of the series expansion

$$\eta^4(\tau) = q^{1/6} - 4q^{7/6} + 2q^{13/6} + 8q^{19/6} + \dots$$

We note here that Γ' is one of an infinite family of genus one subgroups that are normal in Γ, as described in [New64]. All but four of these are noncongruence, and one may compute the periods for all of these subgroups using the above technique. We leave this computation to a future publication.

5.2. **Example: the genus two normal subgroup.** There is only one genus two normal subgroup $G \subseteq \mathrm{PSL}_2(\mathbb{Z})$, a congruence subgroup of index 48 and level 8, denoted by $8A^2$ in [CP03]. In this case the representation ρ_0 of Γ on $S_2(G)$ is two-dimensional. It cannot be reducible, for in this case it would be a direct sum of characters each having level dividing 6. Moreover, if ρ_0 factored through $\Gamma(4)$ then the cusp forms in $S_2(G)$ would give cusp forms in $S_2(\Gamma(4))$, which is impossible since $\Gamma(4)$ has genus zero. Therefore ρ_0 has level 8 and by the tables of [Mas08] we deduce that the only such representation possessing weight two modular forms is the one whose eigenvalues at $\rho_0(T)$ are $\mathbf{e}(1/8)$ and $\mathbf{e}(3/8)$, and whose generating weights are $(2,4)$. According to (9) we must have

$$\rho(T) = \begin{pmatrix} \mathbf{e}(1/8) & 0 & 0 \\ 0 & \mathbf{e}(3/8) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(S) = \begin{pmatrix} -1/\sqrt{2} & -1+1/\sqrt{2} & 1 \\ -1-1/\sqrt{2} & 1/\sqrt{2} & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that by Theorem 2.1 the generating weights of ρ are $(0,2,4)$, so there is a holomorphic non-constant ρ-valued modular function, as expected.

Using SAGE, we find that G can be generated by

$$\begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 15 & -4 \\ 4 & -1 \end{pmatrix}, \begin{pmatrix} 21 & -16 \\ 4 & -3 \end{pmatrix}, \begin{pmatrix} 19 & -24 \\ 4 & -5 \end{pmatrix}, \begin{pmatrix} 25 & -44 \\ 4 & -7 \end{pmatrix},$$
$$\begin{pmatrix} 29 & -80 \\ 4 & -11 \end{pmatrix}, \begin{pmatrix} 27 & -88 \\ 4 & -13 \end{pmatrix}, \begin{pmatrix} 31 & -132 \\ 4 & -17 \end{pmatrix}, \begin{pmatrix} 23 & -52 \\ 4 & -9 \end{pmatrix}. \tag{19}$$

Evaluating ρ at each generator A gives the following list of $\mathbb{R}$-linearly independent periods, appearing in the $(1,3)$- and $(2,3)$- entries of $\rho(A)$ (and rescaled by $c = -2$):

$$\omega_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \omega_2 = \begin{pmatrix} i \\ -i \end{pmatrix}, \quad \omega_3 = \begin{pmatrix} \mathbf{e}(1/8) \\ \mathbf{e}(3/8) \end{pmatrix}, \quad \omega_4 = \begin{pmatrix} \mathbf{e}(3/8) \\ \mathbf{e}(1/8) \end{pmatrix}.$$

To compute the period matrix in Siegel's upper half-space $\mathbb{H}_2$, let

$$P_1 = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad P_2 = \begin{pmatrix} \mathbf{e}(1/8) & \mathbf{e}(3/8) \\ \mathbf{e}(3/8) & \mathbf{e}(1/8) \end{pmatrix}.$$

We may change our basis of cusp forms for $S_2(G)$ so that P_1 is the identity, i.e. the period matrix of X_G is given by

$$P_G = P_2\, P_1^{-1} = \begin{pmatrix} \mathbf{e}(1/8) & 0 \\ 0 & \mathbf{e}(3/8) \end{pmatrix} \in \mathbb{H}_2.$$

Note that his matrix does not depend on the choice of basis for $S_2(G)$. To check the computation of the entries of P_G numerically, note that we may choose

$$f_1 = \eta^4 \left(\frac{1728}{j}\right)^{-\frac{1}{24}} {}_2F_1\left(-\frac{1}{24}, \frac{7}{24}; \frac{3}{4}; \frac{1728}{j}\right) = q^{1/8} - q^{9/8} - 6q^{17/8} + 5q^{25/8} + \dots$$

$$f_2 = \eta^4 \left(\frac{1728}{j}\right)^{\frac{1}{3}} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; \frac{1728}{j}\right) = q^{3/8} - 3q^{11/8} + q^{19/8} + 2q^{27/8} + \dots$$

as a basis for $S_2(G)$ ([FM14a]). If we let

$$P_1' = \begin{pmatrix} \int_i^{A_2 i} f_1(z)\,dz & \int_i^{A_3 i} f_1(z)\,dz \\ \int_i^{A_2 i} f_2(z)\,dz & \int_i^{A_3 i} f_2(z)\,dz \end{pmatrix}, \quad P_2' = \begin{pmatrix} \int_i^{A_4 i} f_1(z)\,dz & \int_i^{A_5 i} f_1(z)\,dz \\ \int_i^{A_4 i} f_2(z)\,dz & \int_i^{A_5 i} f_2(z)\,dz \end{pmatrix},$$

where A_j is the j-th element of the list of generators (19), then

$$P_G = P_2'(P_1')^{-1} \sim \begin{pmatrix} 0.707107... + 0.707107...i & 0 \\ 0 & -0.707107... + 0.707107...i \end{pmatrix}$$

by numerically approximating the integrals using the first 150 terms of the series expansion for f_1, f_2.

5.3. **Example: a genus one non-normal subgroup.** Theorem 5.1 may be applied equally well to modular curves for groups that are not normal in $\mathrm{PSL}_2(\mathbb{Z})$, as we demonstrate in this example. From [CP03], we see that there is a genus one non-normal subgroup $G_0 = 8D^1$ which contains the genus two subgroup $G = 8A^2$ from the previous example as a normal subgroup. The group G_0 is of level 8 and index 24, and can be generated by the matrices

(20)
$$\begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 8 & 21 \\ 3 & 8 \end{pmatrix}, \begin{pmatrix} -12 & -29 \\ 5 & 12 \end{pmatrix}, \begin{pmatrix} -8 & -13 \\ 5 & 8 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now since $S_2(G_0) \subseteq S_2(G)$, we may pick a basis for $S_2(G)$ by first choosing a generator f_1 for the 1-dimensional space $S_2(G_0)$ and extending this choice to a basis $\{f_1, f_2\}$ for $S_2(G)$. Then for all $g_0 \in G_0$ we must have

$$\rho(g_0) \sim \begin{pmatrix} 1 & 0 & \Omega_1(g_0) \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix},$$

where ρ is as in Example 5.2 and $\Omega_1(g_0)$ is the period of f_1 corresponding to $g_0 \in G_0$.

To put ρ into this form, we may change basis via the matrix

$$A = \begin{pmatrix} 3+2\sqrt{2} & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Evaluating $A^t \rho A^{t\,-1}$ at each of the 4 generating hyperbolic matrices for G_0 we get the periods

$$\Omega(g_0) = 2(2+\sqrt{2}),\ 4i(1+\sqrt{2}),\ -2(2+\sqrt{2})$$

which generate the lattice $\langle 2+\sqrt{2},\ 2i(1+\sqrt{2})\rangle \subseteq \mathbb{C}$. The period ratio for the modular curve X_{G_0} is therefore

$$\tau_{G_0} = \sqrt{2}i \in \mathbb{H}.$$

To check this computation numerically, note that

$$f = (1+\sqrt{2})\, f_1 + 2\, f_2 = (1+\sqrt{2})q^{1/8} + 2\, q^{3/8} - (1+\sqrt{2})q^{9/8} + \ldots$$

with f_1, f_2 as in Example 5.2, is a generator for $S_2(G_0)$. We may approximate the period ratio using the first 150 terms of the q-series expansion of f to get

$$\tau_{G_0} = \frac{\int_i^{A_1 i} f(z)\, dz}{\int_i^{A_6 i} f(z)\, dz} \sim 1.41421...i,$$

where A_j denotes the j-th matrix in the list of generators.

References

[CF16] Luca Candelori and Cameron Franc. Vector valued modular forms and the modular orbifold of elliptic curves. *Int. J. of Num. Th.*, 2016.

[CP03] C. J. Cummins and S. Pauli. Congruence subgroups of $\mathrm{PSL}(2,\mathbb{Z})$ of genus less than or equal to 24. *Experiment. Math.*, 12(2):243–255, 2003.

[EZ85] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985.

[FM14a] Cameron Franc and Geoffrey Mason. Fourier coefficients of vector-valued modular forms of dimension 2. *Canad. Math. Bull.*, 57(3):485–494, 2014.

[FM14b] Cameron Franc and Geoffrey Mason. Hypergeometric series, modular linear differential equations and vector-valued modular forms. *The Ramanujan Journal*, pages 1–35, 2014.

[KM03] Marvin Knopp and Geoffrey Mason. Generalized modular forms. *J. Number Theory*, 99(1):1–28, 2003.

[Man72] Ju. I. Manin. Parabolic points and zeta functions of modular curves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 36:19–66, 1972.

[Mas07] Geoffrey Mason. Vector-valued modular forms and linear differential operators. *Int. J. Number Theory*, 3(3):377–390, 2007.

[Mas08] Geoffrey Mason. 2-dimensional vector-valued modular forms. *Ramanujan J.*, 17(3):405–427, 2008.

[MM10] Christopher Marks and Geoffrey Mason. Structure of the module of vector-valued modular forms. *J. Lond. Math. Soc. (2)*, 82(1):32–48, 2010.

[New64] Morris Newman. A complete description of the normal subgroups of genus one of the modular group. *Amer. J. Math.*, 86:17–24, 1964.

[Sch97] A. J. Scholl. On the Hecke algebra of a noncongruence subgroup. *Bull. London Math. Soc.*, 29(4):395–399, 1997.

[Sel65] Atle Selberg. On the estimation of Fourier coefficients of modular forms. In *Proc. Sympos. Pure Math., Vol. VIII*, pages 1–15. Amer. Math. Soc., Providence, R.I., 1965.

[Woh64] Klaus Wohlfahrt. An extension of F. Klein's level concept. *Illinois J. Math.*, 8:529–535, 1964.

[Zhu96] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1), January 1996.

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