Harmonic weak Maass forms of integral weight: a geometric approach

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Abstract

The purpose of the present work is to provide a geometric framework for the study of the Fourier coefficients of *harmonic weak Maass forms*, a space of smooth modular forms first introduced by Bruinier and Funke in the context of singular theta lifts. In this geometric framework harmonic weak Maass forms arise from the construction of differentials whose classes are exact in certain de Rham cohomology groups attached to modular forms. We show how this new interpretation naturally leads to strengthenings of the theorems of Bruinier, Ono and Rhoades, by answering in the affirmative conjectures about the field of definitions of Fourier coefficients of harmonic weak Maass forms. Moreover, as part of our geometric framework, we describe a geometric interpretation for the Shimura-Maass lowering operator analogous to the description of the Shimura-Maass raising operator given by Katz. We also produce Eichler-Shimura-style isomorphisms for the de Rham cohomology attached to modular forms, generalizing results of Bringmann, Guerzhoy, Kent and Ono to any level and field of definition.

1 Introduction

We begin by briefly recalling the basic theory of harmonic weak Maass forms. The reader is referred to the survey article [14] for a more thorough introduction.

Classical modular forms are often defined as *holomorphic* functions $f : \mathfrak{h} \to \mathbb{C}$ on the complex upper half-plane satisfying certain modular transformation properties. If we replace the requirement of f being holomorphic with that of being *smooth* (i.e. infinitely differentiable) several differential operators become available which have no counterparts in the holomorphic case. In particular, if $k \in \mathbb{Z}$ and $\tau = u + iv$ denotes a variable on $\mathfrak{h}$, we have:

- The *Shimura-Maass raising operator*:

$$R_k := \frac{1}{2\pi i}\left(\frac{\partial}{\partial\tau} + \frac{k}{2iv}\right), \tag{1.1}$$

which raises the weight from k to $k+2$.

- The *Shimura-Maass lowering operator*:

$$L_k := -8\pi i\, v^2 \cdot \frac{\partial}{\partial\overline{\tau}}, \tag{1.2}$$

which lowers the weight from k to $k-2$.

- The *Hyperbolic Laplacian:*

$$\Delta_k := -R_{k-2}L_k = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Smooth modular forms of weight k which are in the kernel of Δ_k are called *harmonic*. For example classical modular forms, which are holomorphic on $\mathfrak{h}$, are clearly harmonic. But the kernel of Δ_k also contains the larger space of *harmonic weak Maass forms*, defined as follows:

Definition 1.1. A *harmonic weak Maass form* of weight $k \in \mathbb{Z}$ on $\Gamma_0(N)$ with character χ is a smooth function $F : \mathfrak{h} \to \mathbb{C}$ satisfying:

(i) $\Delta_k F(\tau) = 0$ for all $\tau \in \mathfrak{h}$.

(ii) $F(\gamma\tau) = \chi(d)(c\tau + d)^k F(\tau)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

(iii) For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z})$ there exists a positive integer h, a polynomial $P_{F,\gamma} \in \mathbb{C}[e^{-2\pi i\tau/h}]$ and $\epsilon \in \mathbb{R}_{>0}$ such that:

$$\left|\chi(d)^{-1}(c\tau + d)^{-k}F(\gamma\tau) - P_{F,\gamma}\right| \in O(e^{-\epsilon v})$$

as $v \to \infty$.

Denote by $\mathcal{H}_k(\Gamma_0(N), \chi)$ the vector space of all such harmonic weak Maass forms.

In the definition, $\Gamma_0(N)$ is the congruence subgroup of $\mathbf{SL}_2(\mathbb{Z})$ of matrices that become upper-triangular modulo N and χ is a Dirichlet character modulo N. Note that condition (iii) in Definition 1.1 is a requirement of growth at the cusps of $\Gamma_0(N)$. It requires the smooth modular form F to behave like a classical modular form which is meromorphic at the cusps or, loosely speaking, to have 'at worst poles at the cusps'. Familiar functions such as the j-function are special cases of harmonic weak Maass forms with the extra property of being holomorphic on $\mathfrak{h}$: these are known as *weakly holomorphic modular forms*. The *cusp forms* are the weakly holomorphic modular forms which vanish at the cusps (i.e. the $P_{F,\gamma}$ appearing in (iii) are all zero).

In [2] it is shown that a harmonic weak Maass form $F \in \mathcal{H}_k(\Gamma_0(N), \chi)$ has a Fourier expansion of the form:

$$F = F^+ + F^- = \sum_{n \gg -\infty} c^+(n)\, q^n + \sum_{n<0} c^-(|n|, v)\, q^n, \qquad q = e^{2\pi i \tau},$$

where F^- is anti-holomorphic and F^+ is meromorphic. The series F^+ is called the *holomorphic part* of F. In general, this is is not the Fourier expansion of a modular form: the transformation properties of F^+ have been studied extensively in connection with Ramanujan's *mock modular forms* ([14]).

Let $M_k^!(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) be the space of weakly holomorphic modular forms (resp. cusp forms) of weight k, level $\Gamma_0(N)$ and character χ. It is proven in [2] that there is an exact sequence:

$$0 \longrightarrow M_k^!(\Gamma_0(N), \chi) \longrightarrow \mathcal{H}_k(\Gamma_0(N), \chi) \xrightarrow{\xi_k} S_{2-k}(\Gamma_0(N), \overline{\chi}) \longrightarrow 0,$$

where the first arrow is inclusion and the second is given by the differential operator:

$$\xi_k := v^{k-2}\overline{L_k} = 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}}}. \tag{1.3}$$

This exact sequence is a key tool in the theory of harmonic weak Maass forms: given a cusp form $f \in S_{2-k}(\Gamma_0(N), \overline{\chi})$, we can always find a harmonic weak Maass form $F \in \mathcal{H}_k(\Gamma_0(N), \chi)$ such that $\xi_k(F) = f$. The hope then is that the Fourier coefficients of F would reveal additional structure in the Fourier coefficients of f. The kernel of ξ_k however is a large infinite dimensional space and not *every* harmonic weak Maass form F with $\xi_k(F) = f$ yields useful information about f. Instead, we concentrate our efforts on the notion of a *K-rational harmonic weak Maass form*, which is inspired by the notion of a *good lift* in [3] and [4]:

Definition 1.2. Let $K \subset \mathbb{C}$ be a subfield. A *K-rational harmonic weak Maass form* is a harmonic weak Maass form $F \in \mathcal{H}_k(\Gamma_0(N), \chi)$ such that:

- $P_{F,s_i} \in K$ for all cusps s_i except ∞.

- $P_{F,\infty} \in K[q^{-1}]$.

The significance of this notion is highlighted by the results of [4], which will be expanded upon in this paper, and which we now summarize. Let $f = \sum b(n)q^n \in S_{2-k}(\Gamma_0(N), \overline{\chi})$ be the q-expansion at ∞ of a normalized newform, whose coefficients $b(n)$ all lie in the number field K_f:

Theorem 1.3 ([4]). *Let $F \in \mathcal{H}_k(\Gamma_0(N), \chi)$ be a K_f-rational harmonic weak Maass form such that $\xi_k(F) = f/(f, f)$, where (f, f) is the Petersson norm of f. Denote by $F^+ = \sum_{n\gg-\infty} c^+(n)q^n$ the holomorphic part of F. If $b(n) = 0$, then the coefficient $c^+(n)$ belongs to $\overline{\mathbb{Q}}$.*

Numerical experiments led the authors of [4] to ask whether the field of definition of $c^+(n)$ should in fact be K_f ([4] Remark 3.6) . The present work answers the question in the affirmative:

Theorem 1.4. *Under the hypotheses of Theorem 1.3, if $b(n) = 0$ then $c^+(n)$ belongs to K_f.*

The proof of Theorem 1.4 follows directly once the geometric framework of harmonic weak Maass forms is set in place. This geometric framework can be generalized to any level, including non-congruence modular forms, by working with the de Rham cohomology theory of [15]. On the other hand our geometric theory is limited to the integral weight case, hence it cannot be applied directly to the work of Bruinier and Ono on the vanishing of derivatives of L-series ([3]), which employs vector-valued, half-integral weight harmonic weak Maass forms. A geometric theory of these latter objects would need to be founded upon a geometric theory of vector-valued modular forms, which is being currently developed in the author's Ph.D. thesis.

We begin in Section 2 by briefly reviewing the algebraic theory of modular forms and q-expansions along the lines of [11], mainly to fix ideas and notations. In Section 3 we build upon the results of [15] to construct an algebraic differential operator which appears prominently in the Eichler-Shimura theory of harmonic weak Maass forms ([1]) and whose classical incarnation gives rise to 'Bol's Identity' (as stated in [14] Lemma 7.7, for example). Then, in Section 4 we present a geometric interpretation of the raising and lowering operators (1.1) and (1.2) acting on smooth modular forms. In the case of the raising operator, this is essentially due to Katz ([11] A1.4.2), but a similar interpretation for the lowering operator could not be found in the literature.

Sections 5 and 6 contain the geometric construction of harmonic weak Maass forms. The key elements of this construction are the comparison isomorphisms of Section 5, which give a way to construct differentials whose classes are exact in the de Rham cohomology groups attached to modular forms. The connection with the theory of harmonic weak Maass forms is spelled out in Section 6, where we explicitly compute in local coordinates and give a proof of Theorem 1.4. We also describe how the same proof can be modified to tackle the case of modular forms with complex multiplication.

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2 Notations and background

Fix an integer $N \geq 5$ and consider the level N congruence subgroup:

$$\Gamma = \Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \text{ and } a, d \equiv 1 \mod N \right\}.$$

The group Γ acts on the complex upper half-plane $\mathfrak{h}$ by linear fractional transformations and the quotient $\mathfrak{h}/\Gamma$ can be realized as the analytic space $Y_1(N)^{\mathrm{an}}$ associated to a smooth affine algebraic curve $Y_1(N)$. This algebraic curve represents the functor on $\mathbb{C}$-schemes given by:

$$F_1^\circ(N): \ T/\mathbb{C} \rightsquigarrow \left\{ \begin{array}{c} \text{isomorphism classes of pairs } (E, \iota) \\ \text{of elliptic curves } E/T \text{ endowed with an} \\ \text{injection } \iota : \mu_N \hookrightarrow E[N] \end{array} \right\}, \tag{2.1}$$

where $E[N] \subseteq E$ is the closed subscheme of N-torsion points of E and μ_N is the group scheme of N-th roots of unity. The functor $F_1^\circ(N)$ also makes sense over the category of $\mathbb{Z}$-schemes, and as such it is represented by a smooth affine curve $M_1^\circ(N)$ over $\mathbb{Z}$ with geometrically irreducible fibers ([12], II.2.5).

Remark 2.1. The injection $\iota : \mu_N \hookrightarrow E[N]$ in (2.1) is known as an 'arithmetic' Γ-level structure on E. Over $\mathbb{Z}$, it differs from the more common 'naive' Γ-level structure given by 'a point of order N on E'. The two agree as soon as we extend the base to $\mathbb{Z}[1/N][\zeta_N]$, for $\zeta_N \in \mu_N$ a primitive root of unity. A choice of arithmetic level structure ensures that the cusp ∞ (see (2.3)) is defined over $\mathbb{Z}$, and not over the larger ring $\mathbb{Z}[\zeta_N]$.

REMARK 2.2. The assumption $N \geq 5$ ensures that the fine moduli stack $M_1^\circ(N)$ exists as a scheme over $\mathbb{Z}$. This assumption is not essential, and it is only needed to simplify the arguments. For lower levels there are ad-hoc constructions which circumvent the issue of representability (as in [11] 1.9 and 1.10).

The compactification $\widehat{\mathfrak{h}}/\Gamma$ of $\mathfrak{h}/\Gamma$ is obtained by adding the 'cusps', a set of finitely many points corresponding to the Γ-orbits of $\mathbb{P}^1(\mathbb{Q})$. The quotient $\widehat{\mathfrak{h}}/\Gamma$ can be realized as the analytic space associated to a smooth projective algebraic curve $X_1(N)$. This curve has a smooth proper model $M_1(N)$ over $\mathbb{Z}[1/N]$ which can be viewed as a smooth proper 'compactification' of $M_1^\circ(N)$ over $\mathbb{Z}[1/N]$ ([12], II.2.5).

The 'cusp at ∞' has a moduli interpretation in terms of the generalized Tate elliptic curve $\mathrm{Tate}(q) = \mathbf{G}_m/q^{\mathbb{Z}}$ over $\mathbb{Z}[\![q]\!]$ ([8], VII.1). The canonical inclusion $\mu_N \subseteq \mathbf{G}_m$ gives a (generalized) Γ-level structure on $\mathrm{Tate}(q)$ which is defined over $\mathbb{Z}$. The corresponding classifying map gives a point on $M_1(N)$:

$$\psi : \operatorname{Spec} \mathbb{Z}[1/N][\![q]\!] \longrightarrow M_1(N). \tag{2.2}$$

The singular fiber of $\mathrm{Tate}(q)$ above (q) gives a point:

$$\underline{\infty} : \operatorname{Spec} \mathbb{Z}[1/N] \longrightarrow M_1(N), \tag{2.3}$$

which we call the *cusp at* ∞. Similarly, removing the singular fiber of $\mathrm{Tate}(q)$ gives a point

$$\psi_\circ : \operatorname{Spec} \mathbb{Z}[1/N](\!(q)\!) \longrightarrow M_1^\circ(N) \tag{2.4}$$

on $M_1^\circ(N)$ corresponding to the elliptic curve $(\mathrm{Tate}^{\mathrm{sm}}(q), \mu_N)$.

The sheaf of regular differentials $\Omega^1_{\mathrm{Tate}(q)}$ of the Tate curve $\mathrm{Tate}(q)$ is trivial, generated everywhere by a canonical section $\omega_{\mathrm{can}} = dt/t$, where t is a coordinate on $\mathbb{G}_m$. In particular ([8] VII.1.16.2, [11] A1.2):

$$H^0\left(\mathrm{Tate}(q), \Omega^1_{\mathrm{Tate}(q)}\right) = \mathbb{Z}[\![q]\!]\, \omega_{\mathrm{can}}. \tag{2.5}$$

Over $\mathbb{C}$, the Tate curve $\mathrm{Tate}(q)$ can be defined over an analytic space

$$\mathrm{Tate}(q) \longrightarrow \Delta = \{q \in \mathbb{C} : |q| < 1\}. \tag{2.6}$$

The cusp $\underline{\infty}$ corresponds to the cusp $i\infty$ of $\widehat{\mathfrak{h}}/\Gamma$ and the map ψ gives a coordinate map for the complex analytic manifold $\widehat{\mathfrak{h}}/\Gamma$ around the cusp ∞.

From now on, set $Y := Y_1(N)_{\mathbb{Q}}$ (resp. $X := X_1(N)_{\mathbb{Q}}$) for the generic fiber of the curve $M_1^\circ(N)$ (resp. $M_1(N)$). This is an affine smooth algebraic curve defined over $\mathbb{Q}$ (resp. proper smooth) with the property that $Y^{\mathrm{an}} = \mathfrak{h}/\Gamma$ (resp. $X^{\mathrm{an}} = \widehat{\mathfrak{h}}/\Gamma$). From the moduli interpretation of X ([8], V.4.3) we deduce the existence of a *universal generalized elliptic curve* of level Γ:

$$\pi : \mathcal{E} \longrightarrow X,$$

a flat proper morphism of relative dimension 1. Let $\omega_{\mathcal{E}/X}$ be the relative dualizing sheaf of π. The sheaf

$$\underline{\omega} := \pi_*(\omega_{\mathcal{E}/X}) \tag{2.7}$$

is invertible ([8] II.1.6) and compatible with base change $\mathbb{Q} \to K$ to a any field K of characteristic 0. The Serre dual:

$$\underline{\omega}^{-1} := R^1\pi_*(\mathcal{O}_{\mathcal{E}})$$

is also invertible and compatible with base change, and the same is true for all tensor powers $\underline{\omega}^k$, $k \in \mathbb{Z}$.

Definition 2.3. Let $k \in \mathbb{Z}$. The space of *weakly holomorphic algebraic modular forms* of weight k and level Γ over a field K of characteristic 0 is the K-vector space of sections over Y_K:

$$M_k^!(\Gamma, K) := H^0(Y_K, \underline{\omega}^k).$$

The space of *algebraic cusp forms* of weight k and level Γ over K is the space of sections over X_K:

$$S_k(\Gamma, K) := H^0(X_K, \underline{\omega}^k(-Z_N)),$$

where $Z_N = X - Y$ is the closed subscheme of cusps of X.

When $K = \mathbb{C}$, these definitions capture the classical spaces of modular forms of the introduction. To establish the relationship, consider the *universal elliptic curve over* $\mathfrak{h}$ ([6] Prop. 2.2.2):

$$(\mathbb{C} \times \mathfrak{h})/\mathbb{Z}^2 = \mathcal{E}_{\mathfrak{h}} \xrightarrow{\pi^{\mathfrak{h}}} \mathfrak{h},$$

where $\mathbb{Z}^2$ acts by $(z,\tau) \mapsto (z+m+n\tau, \tau)$. This is a proper, smooth map of complex analytic spaces and the fiber above $\tau \in \mathfrak{h}$ is the torus $\mathbb{C}/\langle \tau, 1\rangle$. The sheaf of holomorphic relative differential 1-forms:

$$\underline{\omega}_{\mathfrak{h}} := \pi_*^{\mathfrak{h}}\left(\Omega^1_{\mathcal{E}_{\mathfrak{h}}/\mathfrak{h}}\right)$$

is trivialized over $\mathfrak{h}$ by the everywhere non-vanishing section $2\pi i\, dz$, and similarly its tensor powers $\underline{\omega}_{\mathfrak{h}}^k$, $k \geq 0$, are trivialized by $(2\pi i\, dz)^k$.

For each $k \in \mathbb{Z}$, the group Γ acts on sections of $\underline{\omega}_{\mathfrak{h}}^k$ by the 'slash operator':

$$f(\tau)|_k\gamma = (c\tau + d)^{-k} f(\gamma(\tau)), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{2.8}$$

This action lies above the action of Γ on $\mathfrak{h}$ given by linear fractional transformations. The (smooth, holomorphic) sections of $\underline{\omega}_{\mathfrak{h}}^k$ which are invariant under this action are precisely the (smooth, holomorphic) classical modular forms (ignoring growth conditions at the cusps), and can be viewed as sections of a sheaf $\underline{\omega}_{\Gamma}^k$ over $\mathfrak{h}/\Gamma$. Now

$$\underline{\omega}_{\Gamma} \simeq \underline{\omega}_{\mathrm{an}}, \tag{2.9}$$

where $\underline{\omega}_{\mathrm{an}}$ is the analytification of the sheaf (2.7) over Y.

In order to establish growth conditions at the cusps, extend $\underline{\omega}_{\Gamma}$ to an invertible sheaf on the compact Riemann surface $\widehat{\mathfrak{h}}/\Gamma$, which can be done canonically since the cusps $Z_N = \widehat{\mathfrak{h}}/\Gamma - \mathfrak{h}/\Gamma$ of $\Gamma = \Gamma_1(N)$ are all regular for $N \geq 5$. We then have canonical identifications:

$$\begin{aligned} S_k(\Gamma) &= H^0\left(\widehat{\mathfrak{h}}/\Gamma, \underline{\omega}_{\Gamma}^k(-Z_N)\right) \\ M_k^!(\Gamma) &= H^0\left(\widehat{\mathfrak{h}}/\Gamma, \underline{\omega}_{\Gamma}^k(*Z_N)\right), \end{aligned}$$

where by $*Z_N$ we mean that sections are allowed to have poles of arbitrary order on Z_N. Note in particular that $S_k(\Gamma)$ is finite dimensional and $M_k^!(\Gamma)$ is a direct limit of finite dimensional vector spaces.

By (2.9) we can also write these spaces in terms of the *analytic* curve $\pi^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to X^{\mathrm{an}}$:

$$\begin{aligned} S_k(\Gamma) &= H^0\left(X^{\mathrm{an}}, \underline{\omega}_{\mathrm{an}}^k(-Z_N)\right) \\ M_k^!(\Gamma) &= H^0\left(X^{\mathrm{an}}, \underline{\omega}_{\mathrm{an}}^k(*Z_N)\right). \end{aligned}$$

By GAGA we then recover Definition 2.3, given in terms of the *algebraic* curve $\pi : \mathcal{E} \to X_{\mathbb{C}}$:

$$\begin{aligned} S_k(\Gamma) &= H^0\left(X_{\mathbb{C}}, \underline{\omega}^k(-Z_N)\right) \\ M_k^!(\Gamma) &= H^0\left(Y_{\mathbb{C}}, \underline{\omega}^k\right). \end{aligned}$$

The notion of 'q-expansions' of modular forms also has a geometric interpretation. By [11], A.1.3 and (2.5) we have:

$$\psi^*(\underline{\omega}) = \Omega^1_{\mathrm{Tate}(q)/\mathbb{Q}\otimes\mathbb{Z}[1/N][\![q]\!]} = \mathbb{Q} \otimes_{\mathbb{Z}[1/N]} \mathbb{Z}[1/N][\![q]\!]\,\omega_{\mathrm{can}}$$

where ψ is the *q-expansion map* (2.2) extended to $\mathbb{Q}$.

Definition 2.4. Let $f \in M_k^!(\Gamma, K)$ be an algebraic weakly holomorphic modular form of weight k over K. The *q-expansion of f* is the element $\widetilde{f} \in K \otimes \mathbb{Z}[1/N]((q))$ defined by:

$$\psi_\circ^*(f) = \widetilde{f}\,\omega_{\mathrm{can}}^k,$$

where $\psi_\circ$ is the map (2.4) and ω_{can} is the differential (2.5) on $\mathrm{Tate}^{\mathrm{sm}}(q)$ extended to K.

Similarly, the q-expansion of a cusp form $f \in S_k(\Gamma, K)$ is the element $\widetilde{f} \in K \otimes \mathbb{Z}[1/N][\![q]\!]$ defined by:

$$\psi^*(f) = \widetilde{f}\,\omega_{\mathrm{can}}^k,$$

where ψ is the map (2.2) extended to K.

By the 'q-expansion principle' ([12], II.2.2.7), if the q-expansion of f has coefficients in K, the modular form f must be defined over K.

Remark 2.5. We have only defined q-expansions at ∞, but the q-expansions at the other cusps can be obtained in the same way by changing the choice of arithmetic level structure on the Tate curve. These q-expansions, however, will be defined over the possibly larger field $K(\zeta_N)$.

When $K = \mathbb{C}$, we recover the usual notion of q-expansions. The assignment

$$(z, \tau) \longmapsto (t = e^{2\pi i z}, q = e^{2\pi i \tau})$$

induces an isomorphism $\mathcal{E}_{\mathfrak{h}} \to \mathrm{Tate}^{\mathrm{sm}}(q)$ with the smooth locus $\mathrm{Tate}^{\mathrm{sm}}(q) \to \Delta^*$ of the analytic Tate curve (2.6). By pulling back this isomorphism on $\Omega^1_{\mathrm{Tate}^{\mathrm{sm}}(q)}$ we get:

$$\frac{dt}{t} = 2\pi i\,dz$$

and the q-expansion map is given by:

$$f(\tau)\,(2\pi i\,dz)^r \longmapsto \widetilde{f}(q)\,\left(\frac{dt}{t}\right)^r.$$

In particular, $\widetilde{f}(q)$ is a well-defined function on the punctured open unit disk Δ^*. When f is a weakly holomorphic modular form (resp. cusp form), this function extends to a meromorphic function on Δ (resp. holomorphic function on Δ vanishing at $q = 0$).

3 Shimura-Maass operators: algebraic theory

In this section and the next we study several differential operators which are essential to the construction of harmonic weak Maass forms. We first define a differential operator

(Proposition 3.1) on algebraic modular forms by building upon a result of Scholl ([15], Theorem 2.7.2).

Let

$$\pi : \mathcal{E} \longrightarrow Y$$

be the universal elliptic curve over $\mathbb{Q}$ arising from the functor (2.1). Consider the *relative de Rham cohomology sheaf* over Y:

$$\mathcal{L}^1 := \mathbb{R}^1\pi_*(0 \to \mathcal{O}_{\mathcal{E}} \overset{d_{\mathcal{E}/Y}}{\to} \Omega^1_{\mathcal{E}/Y} \overset{d_{\mathcal{E}/Y}}{\to} 0).$$

This is a locally free sheaf of rank 2, whose geometric fibers are the algebraic de Rham cohomology of the fibers of π (see [9] for generalities about algebraic de Rham cohomology). We have a filtration on $\mathcal{L}^1$:

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathcal{L}^1 \longrightarrow \underline{\omega}^{-1} \longrightarrow 0, \tag{3.1}$$

whose specialization to each geometric fiber is the Hodge filtration on the de Rham cohomology of the fiber. This filtration is self-dual with respect to the pairing

$$\langle \cdot, \cdot \rangle : \mathcal{L}^1 \times \mathcal{L}^1 \longrightarrow \mathcal{O}_Y \tag{3.2}$$

induced by the cup-product on each fiber. The sheaf $\mathcal{L}^1$ is endowed with the *Gauss-Manin connection*

$$\nabla : \mathcal{L}^1 \longrightarrow \mathcal{L}^1 \otimes \Omega^1_Y,$$

a canonical, integrable connection on $\mathcal{L}^1$ ([10], 1.4). The *Kodaira-Spencer map* is defined as the composite

$$\sigma : \underline{\omega} \hookrightarrow \mathcal{L}^1 \overset{\nabla}{\longrightarrow} \mathcal{L}^1 \otimes \Omega^1_Y \twoheadrightarrow \underline{\omega}^{-1} \otimes \Omega^1_Y,$$

which gives rise to an isomorphism

$$\rho : \underline{\omega}^2 \longrightarrow \Omega^1_Y.$$

In particular, the isomorphism ρ induces a canonical isomorphism:

$$M^!_{r+2}(\Gamma, K) = H^0\left(Y_K, \underline{\omega}^r \otimes \Omega^1_Y\right) \tag{3.3}$$

for any field K of characteristic 0.

By [11], A1.3.18 or [7], ∇ extends over X to a connection with logarithmic poles:

$$\nabla : \mathcal{L}^1 \longrightarrow \mathcal{L}^1 \otimes \Omega^1_X\left(\log Z_N\right),$$

and ρ extends to an isomorphism:

$$\rho : \underline{\omega}^2 \longrightarrow \Omega^1_X (\log Z_N) . \tag{3.4}$$

On the Tate curve, the isomorphism ρ pulls back via (2.2) by:

$$\rho(\omega^2_{\mathrm{can}}) = \frac{dq}{q} \in \Omega^1_{\mathrm{Spec}\, \mathbb{Q}\otimes\mathbb{Z}[\![q]\!]} (\log\{q = 0\})$$

which gives a canonical isomorphism:

$$S_{r+2}(\Gamma, K) = H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X) \tag{3.5}$$

for any field K of characteristic 0.

We now work over the curve $\pi : \mathcal{E} \to X$ using the extensions of ∇ and ρ described above. Let

$$\mathcal{L}^r := \mathrm{Sym}^r \mathcal{L}^1.$$

These are locally free sheaves endowed with a pairing

$$\langle \cdot, \cdot \rangle_r : \mathcal{L}^r \times \mathcal{L}^r \longrightarrow \mathcal{O}_X \tag{3.6}$$

given in terms of the pairing (3.2) by:

$$\langle \omega_1 \cdots \omega_r, \eta_1 \cdots \eta_r \rangle_r = \frac{1}{r!} \sum_{\sigma \in S_r} \langle \omega_1, \eta_{\sigma(1)} \rangle \cdots \langle \omega_r, \eta_{\sigma(r)} \rangle . \tag{3.7}$$

The relative Hodge filtration (3.1) extended to X induces a decreasing filtration F on $\mathcal{L}^r$. To describe F explicitly, note that over any affine subset $U \subseteq X$ we can choose a splitting $\mathcal{L}^1|_U \simeq (\underline{\omega} \oplus \underline{\omega}^{-1})|_U$ of (3.1), which induces a splitting:

$$\mathcal{L}^r|_U \simeq (\underline{\omega}^r \oplus \underline{\omega}^{r-2} \oplus \ldots \oplus \underline{\omega}^{-r})|_U.$$

Globally, we get a filtration:

$$\mathcal{L}^r = F^0(\mathcal{L}^r) \supseteq F^1(\mathcal{L}^r) \supseteq \cdots \supseteq F^{r+1}(\mathcal{L}^r) = 0, \tag{3.8}$$

whose graded pieces for $i = 0, \ldots, r$ are given by:

$$\mathrm{gr}^i_F(\mathcal{L}^r) = F^i(\mathcal{L}^r)/F^{i+1}(\mathcal{L}^r) \simeq \underline{\omega}^{2i-r}.$$

The symmetric r-fold product of the Gauss-Manin connection gives a connection

$$\nabla : \mathcal{L}^r \longrightarrow \mathcal{L}^r \otimes \Omega^1_X (\log Z_N) \tag{3.9}$$

via the recursive formula:

$$\nabla(\omega_1 \otimes \omega_2) = \nabla(\omega_1) \otimes \omega_2 + \omega_1 \otimes \nabla(\omega_2).$$

The connection (3.9) and the filtration (3.8) are not quite compatible, but we have:

$$\nabla(F^i(\mathcal{L}^r)) \subseteq F^{i-1}(\mathcal{L}^r) \otimes \Omega^1_X (\log Z_N)\,,$$

that is, ∇ shifts the filtration up by one level ([10], 1.4.1.6). The map induced by ∇ on the graded pieces:

$$\underline{\omega}^{2i-r} \xrightarrow{\simeq} \underline{\omega}^{2i-r-2} \otimes \Omega^1_X (\log Z_N) \tag{3.10}$$

is an isomorphism for $i = 1, \ldots, r$ induced by the Kodaira-Spencer isomorphism (3.4) ([15], 2.7.4). As in ([5], Lemma 4.2) these isomorphisms at the level of graded pieces induce an isomorphism:

$$F^1(\mathcal{L}^r) \simeq \frac{\mathcal{L}^r \otimes \Omega^1_X (\log Z_N)}{\underline{\omega}^r \otimes \Omega^1_X (\log Z_N)}, \tag{3.11}$$

from which we derive the following operator.

Proposition 3.1. *The connection* (3.9) *induces a map of abelian sheaves:*

$$\partial^{r+1} : \underline{\omega}^{-r} \longrightarrow \underline{\omega}^r \otimes \Omega^1_X (\log Z_N)\,. \tag{3.12}$$

Proof. Let s be a section of $\underline{\omega}^{-r}$. We want to show that there exists a unique section $\widetilde{s}$ of $\mathcal{L}^r$ lifting s under the natural map $\mathcal{L}^r \to \underline{\omega}^{-r}$ with the property that $\nabla\widetilde{s}$ is a section of $\underline{\omega}^r \otimes \Omega^1_X (\log Z_N)$. Pick any section $\widetilde{w}$ of $\mathcal{L}^r$ lifting s under $\mathcal{L}^r \to \underline{\omega}^{-r}$. Consider the class

$$\nabla\widetilde{w} \mod \underline{\omega}^r \otimes \Omega^1_X (\log Z_N)$$

in the quotient $F^0/F^r(\mathcal{L}^r) \otimes \Omega^1_X (\log Z_N)$. By the isomorphism (3.11) there exists a unique section $\widetilde{z}$ of $F^1(\mathcal{L}^r)$ such that

$$\nabla\widetilde{z} \equiv \nabla\widetilde{w} \mod \underline{\omega}^r \otimes \Omega^1_X (\log Z_N)\,.$$

The difference $\widetilde{s} := \widetilde{w} - \widetilde{z}$ is the required section of $\mathcal{L}^r$. □ □

The map ∂^{r+1} has a simple expression in terms of q-expansions. To compute it, let

$$\partial^{r+1} : \psi_\circ^* \underline{\omega}^{-r} \longrightarrow \psi_\circ^* \left(\underline{\omega}^r \otimes \Omega^1_Y\right)$$

be the map obtained from (3.12) by pulling back under the q-expansion map (2.4). Then

$$\partial^{r+1}(\widetilde{f}) = \frac{(-1)^r}{r!} \left(q\frac{d}{dq}\right)^{r+1} (\widetilde{f}) \tag{3.13}$$

for any section $\widetilde{f}$ of $\psi_\circ^* \underline{\omega}^{-r}$ ([15], 2.7.2).

4 Shimura-Mass operators: transcendental theory

In this section we describe geometric interpretations of the raising and lowering operators (1.1) and (1.2). In the case of the raising operator, this interpretation is due to Katz ([11], A1.4.2). We adapt Katz's approach to smooth modular forms and obtain a similar interpretation for the lowering operator.

To work explicitly with the analytic space

$$\pi^{\text{an}} : \mathcal{E}^{\text{an}} \longrightarrow Y^{\text{an}}$$

we consider the constructions of the previous sections (Hodge bundles, de Rham cohomology, Gauss-Manin connection ...) on the universal elliptic curve over $\mathfrak{h}$

$$\pi^{\mathfrak{h}} : \mathcal{E}_{\mathfrak{h}} \to \mathfrak{h}.$$

Over $\mathfrak{h}$, all these constructions can be trivialized and computed explicitly. To descend back under $\text{pr} : \mathfrak{h} \to \mathfrak{h}/\Gamma$ corresponds to taking Γ-invariants. In particular, recall that our choice of coordinates for the curve $\pi^{\mathfrak{h}} : \mathcal{E}_{\mathfrak{h}} \to \mathfrak{h}$ is:

$$(z, \tau) \in (\mathbb{C} \times \mathfrak{h}) / \mathbb{Z}^2, \tag{4.1}$$

with $\tau = u + iv \in \mathfrak{h}$ and $z \in E_\tau = \mathbb{C}/\langle \tau, 1 \rangle$ a coordinate on the elliptic curve over τ. The Hodge bundle of $\pi^{\mathfrak{h}}$ can be trivialized as:

$$\underline{\omega}_{\mathfrak{h}} \simeq \mathcal{O}_{\mathfrak{h}} \, 2\pi i \, dz$$

and the sheaf of Γ-invariant sections gives a sheaf $\underline{\omega}_\Gamma$ over $\mathfrak{h}/\Gamma$ which is isomorphic to $\underline{\omega}_{\text{an}}$.

Similarly, the relative de Rham cohomology sheaf $\mathcal{L}^1_{\mathfrak{h}}$ of $\pi^{\mathfrak{h}} : \mathcal{E}_{\mathfrak{h}} \to \mathfrak{h}$, whose fiber above τ is $H^1_{\text{dR}}(\mathbb{C}/\langle \tau, 1 \rangle)$, is trivial of rank 2 over $\mathfrak{h}$. It is equipped with a pairing:

$$\langle \cdot, \cdot \rangle : \mathcal{L}^1_{\mathfrak{h}} \times \mathcal{L}^1_{\mathfrak{h}} \longrightarrow C^\infty_{\mathfrak{h}}$$

coming from cup-product on the fibers, given by:

$$\langle \omega_1, \omega_2 \rangle (\tau) = \frac{1}{2\pi i} \int_{E_\tau} \omega_1 \wedge \omega_2. \tag{4.2}$$

This pairing clearly descends to $\mathcal{L}^1$ over Y^{an}, i.e. to Γ-invariant sections of $\mathcal{L}^1_{\mathfrak{h}}$.

The Hodge filtration gives an exact sequence:

$$0 \longrightarrow \underline{\omega}_{\mathfrak{h}} \longrightarrow \mathcal{L}^1_{\mathfrak{h}} \longrightarrow \underline{\omega}_{\mathfrak{h}}^{-1} \longrightarrow 0, \tag{4.3}$$

whose specialization at each fiber is the Hodge filtration on $H^1_{\mathrm{dR}}(E_\tau)$. The Hodge decomposition on the fibers gives rise to a canonical splitting of the exact sequence (4.3):

$$\mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} = C^\infty_{\mathfrak{h}}\, 2\pi i\, dz \oplus C^\infty_{\mathfrak{h}}\, (-2\pi i)\, d\bar{z}. \tag{4.4}$$

This splitting however does not descend to a splitting of the relative Hodge filtration over Y^{an}. Instead, we modify it slightly to obtain a C^∞-splitting Φ_{Hodge} determined by:

$$\Phi_{\mathrm{Hodge}}(\underline{\omega}_{\mathfrak{h}}^{-1} \otimes C^\infty_{\mathfrak{h}}) = C^\infty_{\mathfrak{h}} \left(\frac{-d\bar{z}}{2iv} \right), \tag{4.5}$$

which does descend and it is adapted to the pairing (4.2), i.e. $\left\langle 2\pi i\, dz, \frac{-d\bar{z}}{2iv} \right\rangle = 1$.

Next, consider the *relative Betti cohomology* sheaf:

$$\mathcal{B}^1_{\mathfrak{h}} := R^1\pi^{\mathfrak{h}}_*(\mathbb{C}_{\mathcal{E}^{\mathfrak{h}}}).$$

This is a rank 2 constant sheaf of $\mathbb{C}$-vector spaces over $\mathfrak{h}$ whose fibers are the first Betti cohomology $H^1(E_\tau, \mathbb{C})$ of the fibers of $\pi^{\mathfrak{h}}$. Its smooth sections are equipped with a canonical integrable connection, the *Gauss-Manin connection* ([17], Def. 9.13):

$$\nabla : \mathcal{B}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \longrightarrow \mathcal{B}^1_{\mathfrak{h}} \otimes \mathcal{A}^1_{\mathfrak{h}},$$

where $\mathcal{A}^1_{\mathfrak{h}}$ is the sheaf of smooth 1-forms on $\mathfrak{h}$. The periods of E_τ trivialize $\mathcal{B}^1_{\mathfrak{h}}$, and with respect to this trivialization the connection ∇ is just $d \oplus d$. The Betti-de Rham comparison isomorphism on the fibers induces a canonical isomorphism:

$$\mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \simeq \mathcal{B}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}, \tag{4.6}$$

from which we can endow $\mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}$ with a canonical integrable connection:

$$\nabla : \mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \longrightarrow \mathcal{L}^1_{\mathfrak{h}} \otimes \mathcal{A}^1_{\mathfrak{h}}, \tag{4.7}$$

also referred to as the *Gauss-Manin connection*. Furthermore, from the decomposition $\mathcal{A}^1_{\mathfrak{h}} = \mathcal{A}^{1,0}_{\mathfrak{h}} \oplus \mathcal{A}^{0,1}_{\mathfrak{h}}$ into $(1,0)$- and $(0,1)$-forms we obtain an integrable ∂-connection:

$$\nabla^{1,0} : \mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \longrightarrow \mathcal{L}^1_{\mathfrak{h}} \otimes \mathcal{A}^{1,0}_{\mathfrak{h}}, \tag{4.8}$$

and a 'conjugate' integrable $\overline{\partial}$-connection:

$$\nabla^{0,1} : \mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \longrightarrow \mathcal{L}^1_{\mathfrak{h}} \otimes \mathcal{A}^{0,1}_{\mathfrak{h}}. \tag{4.9}$$

With respect to the trivialization (4.4) given by the Hodge decomposition, these connections can be computed as follows.

Lemma 4.1. *Let* $\nabla^{1,0}$ *(resp.* $\nabla^{0,1}$*) be the* ∂*- (resp.* $\overline{\partial}$*-) connection on* $\mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}$ *given by* (4.8) *(resp.* (4.9)*). Then in terms of the decomposition* (4.4) *we have:*

(a)

$$\nabla^{1,0}(dz) = \frac{dz - d\bar{z}}{2iv}\, d\tau, \qquad \nabla^{1,0}(d\bar{z}) = 0.$$

(b)

$$\nabla^{0,1}(dz) = 0, \qquad \nabla^{0,1}(d\bar{z}) = \frac{dz - d\bar{z}}{2iv}\, d\bar{\tau}$$

Proof. To compute, write:

$$\mathcal{P} : \mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \longrightarrow \mathcal{B}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}$$

for the isomorphism (4.6), which is given in local coordinates by $\omega \mapsto (\gamma \mapsto \int_\gamma \omega)$. Let γ_1,γ_2 be the basis for $\mathcal{B}^1_{\mathfrak{h}}$ dual to the loops $0 \to 1$ and $0 \to \tau$ on E_τ, respectively.

(a) By the definition of $\nabla^{1,0}$ we have:

$$\nabla^{1,0}(dz) = \mathcal{P}^{-1} \begin{pmatrix} \frac{\partial}{\partial \tau} \int_0^1 dz \\ \frac{\partial}{\partial \tau} \int_0^\tau dz \end{pmatrix} d\tau = \mathcal{P}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\tau$$

with respect to the basis γ_1, γ_2. One readily checks that the differential form on the left-hand side of (a) is the unique one with periods 0 and 1. It is also clear that $\nabla^{1,0}(d\bar{z})$ must vanish.

(b) Similar to part (a), we have:

$$\nabla^{0,1}(d\bar{z}) = \mathcal{P}^{-1} \begin{pmatrix} \frac{\partial}{\partial \bar{\tau}} \int_0^1 d\bar{z} \\ \frac{\partial}{\partial \bar{\tau}} \int_0^\tau d\bar{z} \end{pmatrix} d\bar{\tau} = \mathcal{P}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\bar{\tau}.$$

The differential on the right-hand side of (b) has the required periods, and it is clear that $\nabla^{0,1}(dz) = 0$.

□ □

Next, consider the *Kodaira-Spencer map* ([17], Def. 9.6 and [18], Th. 5.7):

$$\sigma : \underline{\omega}_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \longrightarrow \mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \xrightarrow{\nabla} \mathcal{L}^1_{\mathfrak{h}} \otimes \mathcal{A}^1_{\mathfrak{h}} \longrightarrow \underline{\omega}_{\mathfrak{h}}^{-1} \otimes \mathcal{A}^1_{\mathfrak{h}}, \tag{4.10}$$

where the first and last maps are induced by the filtration (4.3). The map σ gives rise to two maps (which are both non-zero by Lemma 4.1):

$$\sigma^{1,0} : \underline{\omega}_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \longrightarrow \mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \xrightarrow{\nabla^{1,0}} \mathcal{L}^1_{\mathfrak{h}} \otimes \mathcal{A}^{1,0}_{\mathfrak{h}} \longrightarrow \underline{\omega}_{\mathfrak{h}}^{-1} \otimes \mathcal{A}^{1,0}_{\mathfrak{h}}$$
$$\sigma^{0,1} : \underline{\omega}_{\mathfrak{h}}^{-1} \otimes C^\infty_{\mathfrak{h}} \xrightarrow{\Phi_{\mathrm{Hodge}}} \mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \xrightarrow{\nabla^{0,1}} \mathcal{L}^1_{\mathfrak{h}} \otimes \mathcal{A}^{0,1}_{\mathfrak{h}} \xrightarrow{\Psi_{\mathrm{Hodge}}} \underline{\omega}_{\mathfrak{h}} \otimes \mathcal{A}^{0,1}_{\mathfrak{h}},$$

where

$$\Psi_{\text{Hodge}} : \mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \longrightarrow \underline{\omega}_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}$$

is the projection induced by the Hodge splitting Φ_{Hodge} of (4.5). The maps $\sigma^{1,0}$ and $\sigma^{0,1}$ can be computed as follows.

LEMMA 4.2. *Let $\rho^{1,0} : \mathcal{A}^{1,0}_{\mathfrak{h}} \overset{\simeq}{\to} \underline{\omega}^2_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}$ and $\rho^{0,1} : \mathcal{A}^{0,1}_{\mathfrak{h}} \overset{\simeq}{\to} \underline{\omega}^{-2}_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}$ be the isomorphisms obtained from $\sigma^{1,0}$ and $\sigma^{0,1}$. Then in terms of the splitting* (4.5) *we have:*

(a)

$$\rho^{1,0}\,(2\pi i\, d\tau) = (2\pi i\, dz)^2$$

(b)

$$\rho^{0,1}\left(\frac{d\bar{\tau}}{-2iv^2 \cdot 2\pi i \cdot 2i}\right) = \left(\frac{d\bar{z}}{-2iv}\right)^2.$$

Proof.

(a) Note that $\nabla^{1,0}(2\pi i\, dz) = 2\pi i\, \frac{dz - d\bar{z}}{2iv}\, d\tau$ by Lemma 4.1 part (a). Using the splitting (4.5) we can use $-d\bar{z}/2iv$ as a generator for $\underline{\omega}^{-1}_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}$ inside $\mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}$. The result follows by projecting onto the line spanned by $-d\bar{z}/v$ and dualizing with respect to the pairing (4.2).

(b) Similar to part (a), note that $\nabla^{0,1}(-d\bar{z}/2iv) = \frac{d\bar{z} - dz}{2i \cdot 2iv^2}\, d\bar{\tau}$ using Lemma 4.1 part (b). The result follows by projecting onto the line spanned by dz and dualizing.

□ □

Following the construction of [11] of the Shimura-Maass operator, we extend the connections (4.8), (4.9) and the Hodge splitting maps $\Phi_{\text{Hodge}}, \Psi_{\text{Hodge}}$ to the r-symmetric powers of $\mathcal{L}^1_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}$ and compose with the Kodaira-Spencer isomorphisms $\rho^{1,0}$ and $\rho^{0,1}$ of Lemma 4.2.

DEFINITION 4.3. The *Shimura-Maass raising operator* is the differential operator R_r obtained as the composite:

$$R_r :\ \underline{\omega}^r_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \longrightarrow \mathcal{L}^r_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \xrightarrow{\nabla^{1,0}} \mathcal{L}^r_{\mathfrak{h}} \otimes \mathcal{A}^{1,0}_{\mathfrak{h}} \xrightarrow{\Psi^r_{\text{Hodge}} \otimes \rho^{1,0}} \underline{\omega}^{r+2}_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}. \tag{4.11}$$

The *Shimura-Maass lowering operator* is the differential operator L_r obtained as the composite:

$$L_r :\ \underline{\omega}^{-r}_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \xrightarrow{\Phi^r_{\text{Hodge}}} \mathcal{L}^r_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}} \xrightarrow{\nabla^{0,1}} \mathcal{L}^r_{\mathfrak{h}} \otimes \mathcal{A}^{0,1}_{\mathfrak{h}} \xrightarrow{\text{proj}^{\text{r}} \otimes \rho^{0,1}} \underline{\omega}^{-r-2}_{\mathfrak{h}} \otimes C^\infty_{\mathfrak{h}}, \tag{4.12}$$

where proj^{r} is the canonical projection induced by the relative Hodge filtration (4.3).

Our definitions are justified by the following computation (compare to (1.1) and (1.2)).

LEMMA 4.4. *Let* $R_r : \underline{\omega}_{\mathfrak{h}}^{r} \otimes C_{\mathfrak{h}}^{\infty} \to \underline{\omega}_{\mathfrak{h}}^{r+2} \otimes C_{\mathfrak{h}}^{\infty}$ *and* $L_r : \underline{\omega}_{\mathfrak{h}}^{-r} \otimes C_{\mathfrak{h}}^{\infty} \to \underline{\omega}_{\mathfrak{h}}^{-r-2} \otimes C_{\mathfrak{h}}^{\infty}$ *be the differential operators* (4.11) *and* (4.12). *Then in terms of the splitting* (4.5) *we have:*

(a)

$$R_r f(\tau) = \frac{1}{2\pi i}\left(\frac{\partial f}{\partial \tau} + \frac{r}{v}\, f\right),$$

(b)

$$L_r f(\tau) = -2\pi i \cdot 2i \cdot 2iv^2\, \frac{\partial f}{\partial \bar{\tau}}.$$

Proof.

(a) This is essentially ([11], A1.4.2). Let $f\,(2\pi i\, dz)^r$ be a section of $\underline{\omega}_{\mathfrak{h}}^{r} \otimes C_{\mathfrak{h}}^{\infty}$. Then by Lemma 4.1 part (a) we have:

$$\begin{aligned}
&\nabla^{1,0}\left(f\,(2\pi i\, dz)^r\right) = (2\pi i)^r \left(\partial f\, dz^r + f\, \nabla^{1,0}\left(dz^r\right)\right) \\
&= (2\pi i)^{r-1}\left(\frac{\partial f}{\partial \tau}\, dz^r + f\, r\, dz^{r-1}\left(\frac{dz - d\bar{z}}{2iv}\right)\right) 2\pi i\, d\tau \\
&= \frac{1}{2\pi i}\left(\frac{\partial f}{\partial \tau} + \frac{r\, f}{2iv}\right)(2\pi i\, dz)^r\,(2\pi i\, d\tau) + r\, f\,(2\pi i\, dz)^{r-1}\left(\frac{-d\bar{z}}{2iv}\right) 2\pi i\, d\tau.
\end{aligned}$$

The result follows by projecting onto $\underline{\omega}_{\mathfrak{h}}^{r} \otimes C_{\mathfrak{h}}^{\infty}$ and by applying Lemma 4.2 part (a).

(b) Using the splitting (4.5) we can write $f\,\left(-d\bar{z}/2iv\right)^r$ for a section of $\Phi^r_{\mathrm{Hodge}}\left(\underline{\omega}_{\mathfrak{h}}^{-r} \otimes C_{\mathfrak{h}}^{\infty}\right)$. Then by Lemma 4.1 part (b) we have:

$$\begin{aligned}
&\nabla^{0,1}\left(\frac{-f}{(2iv)^r}\, d\bar{z}^r\right) = \bar{\partial}\left(\frac{-f}{(2iv)^r}\right) d\bar{z}^r + \frac{-f}{(2iv)^r}\,\nabla^{0,1}\left(d\bar{z}^r\right) \\
&= \left(\frac{-1}{(2iv)^r}\,\frac{\partial f}{\partial \bar{\tau}}\, d\bar{z}^r + \frac{-r}{(2iv)^{r+1}}\, f\, d\bar{z}^r + r\left(\frac{-f}{(2iv)^r}\right) d\bar{z}^{r-1}\left(\frac{dz - d\bar{z}}{2iv}\right)\right) d\bar{\tau} \\
&= \frac{\partial f}{\partial \bar{\tau}}\left(\frac{-d\bar{z}}{2iv}\right)^r d\bar{\tau} + \frac{r\, f}{(2iv)^2}\left(\frac{-d\bar{z}}{2iv}\right)^{r-1} dz\, d\bar{\tau}.
\end{aligned}$$

The result follows by projecting onto $\underline{\omega}_{\mathfrak{h}}^{-r} \otimes C_{\mathfrak{h}}^{\infty}$ and applying Lemma 4.2 part (b).

□ □

All the previous constructions descend to Y^{an}. The main point is that they are all defined intrinsically: the Gauss-Manin connection (4.7), when applied to Γ-invariant sections of $\mathcal{L}^1_{\mathfrak{h}}$, is the Gauss-Manin connection

$$\nabla : \mathcal{L}^1_{\text{an}} \otimes C^\infty_{Y^{\text{an}}} \longrightarrow \mathcal{L}^1_{\text{an}} \otimes \mathcal{A}^1_{Y^{\text{an}}}$$

of the curve

$$\pi^{\text{an}} : \mathcal{E}^{\text{an}} \longrightarrow Y^{\text{an}},$$

where $\mathcal{L}^1_{\text{an}}$ is the relative de Rham cohomology sheaf of π^{an}. Similarly, the Kodaira-Spencer map (4.10) applied to Γ-invariant sections is the Kodaira-Spencer map

$$\sigma : \underline{\omega}_{\text{an}} \otimes C^\infty_{Y^{\text{an}}} \longrightarrow \mathcal{L}^1_{\text{an}} \otimes C^\infty_{Y^{\text{an}}} \xrightarrow{\nabla} \mathcal{L}^1_{\text{an}} \otimes \mathcal{A}^1_{Y^{\text{an}}} \longrightarrow \underline{\omega}^{-1}_{\text{an}} \otimes \mathcal{A}^1_{Y^{\text{an}}},$$

obtained as in (4.10). The splitting (4.5) was chosen so that it descends to a splitting of the relative Hodge filtration of π^{an}:

$$0 \longrightarrow \underline{\omega}_{\text{an}} \longrightarrow \mathcal{L}^1_{\text{an}} \longrightarrow \underline{\omega}^{-1}_{\text{an}} \longrightarrow 0.$$

In summary:

Proposition 4.5. *The Shimura-Maass raising operator R_r (4.11) descends to $\mathfrak{h}/\Gamma$, i.e. there is a commutative diagram of sheaves:*

$$\begin{array}{ccccccc}
\underline{\omega}^r_{\text{an}} \otimes C^\infty_{Y^{\text{an}}} & \longrightarrow & \mathcal{L}^r_{\text{an}} \otimes C^\infty_{Y^{\text{an}}} & \xrightarrow{\nabla^{1,0}} & \mathcal{L}^r_{\text{an}} \otimes \mathcal{A}^{1,0}_{Y^{\text{an}}} & \xrightarrow{\Phi^r_{\text{Hodge}} \otimes \rho^{1,0}} & \underline{\omega}^{r+2}_{\text{an}} \otimes C^\infty_{Y^{\text{an}}} \\
\updownarrow \simeq & & \updownarrow \simeq & & \updownarrow \simeq & & \updownarrow \simeq \\
\underline{\omega}^r_{\Gamma} \otimes C^\infty_{\mathfrak{h}/\Gamma} & \longrightarrow & \mathcal{L}^r_{\Gamma} \otimes C^\infty_{\mathfrak{h}/\Gamma} & \xrightarrow{\nabla^{1,0}} & \mathcal{L}^r_{\Gamma} \otimes \mathcal{A}^{1,0}_{\mathfrak{h}/\Gamma} & \xrightarrow{\Phi^r_{\text{Hodge}} \otimes \rho^{1,0}} & \underline{\omega}^{r+2}_{\Gamma} \otimes C^\infty_{\mathfrak{h}/\Gamma}
\end{array}$$

where $\mathcal{L}_\Gamma$ is the sheaf over $\mathfrak{h}/\Gamma$ of Γ-invariant sections of $\mathcal{L}_{\mathfrak{h}}$. A similar statement holds for the Shimura-Maass lowering operator L_r (4.12).

The sections of $\underline{\omega}^r_\Gamma \otimes C^\infty_{\mathfrak{h}/\Gamma}$ are the classical smooth modular forms. Therefore it follows by Proposition 4.5 that the operator R_r (resp. L_r) preserves the spaces of smooth modular forms by raising (resp. lowering) the weight by 2.

As for the algebraic differential operator ∂^{r+1} (3.12) of Section 3, note that its construction carries over step-by-step in the transcendental setting, i.e. over the universal elliptic curve $\pi^{\mathfrak{h}} : \mathcal{E}_{\mathfrak{h}} \to \mathfrak{h}$. By taking Γ-invariant sections as above we obtain:

Proposition 4.6. *Let*

$$\partial^{r+1} : \underline{\omega}_{\Gamma}^{-r} \longrightarrow \underline{\omega}_{\Gamma}^{r} \otimes \Omega^1_{\mathfrak{h}/\Gamma}$$

be the map obtained as in Proposition 4.5 by constructing the map (3.12) *over the curve* $\pi^{\mathfrak{h}} : \mathcal{E}_{\mathfrak{h}} \to \mathfrak{h}$ *and by taking* Γ*-invariant sections. Then with respect to the coordinates* (4.1) *we have:*

$$\partial^{r+1}(f) = \frac{(-1)^r}{r!} D^{r+1}(f), \qquad D := \frac{1}{2\pi i}\frac{\partial}{\partial \tau}, \tag{4.13}$$

for any holomorphic section f *of* $\underline{\omega}_{\Gamma}^{-r} \simeq \underline{\omega}_{\mathrm{an}}^{-r}$.

Proof. It follows from (3.13) by noting that over $\mathfrak{h}$ we have $q\frac{d}{dq} = D$. □ □

By (3.3) and (4.13) it follows that D^{r+1} defines a linear map on global sections:

$$D^{r+1} : M^!_{-r}(\Gamma) \longrightarrow M^!_{r+2}(\Gamma).$$

Note that in general the powers D^t of D do not preserve modularity. The fact that D^{r+1} does preserve modularity is known as Bol's Identity: Proposition 4.6 provides a geometric proof of this identity.

5 De Rham and parabolic cohomology

In this section we study the cohomology of the complexes arising from the Gauss-Manin connection. In particular, Theorem 5.2 (b) and Theorem 5.5 give a description of these cohomology groups in terms of 'differentials of the second kind', obtaining a generalization of the results of [1] to any level and field of definition (see Remark 5.6 below).

Going back to the algebraic setting $X := X_1(N)_{\mathbb{Q}}$ and $Y := Y_1(N)_{\mathbb{Q}}$, we associate (as in [15]) the following cohomology groups to the pair $(\mathcal{L}^r, \nabla)$:

(i) The *de Rham cohomology of* $(\mathcal{L}^r, \nabla)$, defined as:

$$\mathbb{H}^i(X, \mathcal{L}^r, \nabla) := \mathbb{H}^i(X, 0 \longrightarrow \mathcal{L}^r \xrightarrow{\nabla} \mathcal{L}^r \otimes \Omega^1_X(\log Z_N) \longrightarrow 0).$$

(ii) The *parabolic cohomology of* $(\mathcal{L}^r, \nabla)$ ([15] 2.6, [13] Remark 3.2). To define it, let

$$\Omega^1_X(\mathcal{L}^r) := \nabla(\mathcal{L}^r) + \mathcal{L}^r \otimes \Omega^1_X \subseteq \mathcal{L}^r \otimes \Omega^1_X(\log Z_N)$$

and let

$$\mathbb{H}^i_{\mathrm{par}}(X, \mathcal{L}^r, \nabla) := \mathbb{H}^i(X, 0 \longrightarrow \mathcal{L}^r \xrightarrow{\nabla} \Omega^1_X(\mathcal{L}^r) \longrightarrow 0),$$

be the i-th parabolic cohomology group.

One of the advantages of working with parabolic cohomology over de Rham cohomology is that the former is self-dual with respect to the cup-product ([13] Remark 3.2):

$$\langle \cdot, \cdot \rangle : \mathbb{H}^1_{\mathrm{par}}(X, \mathcal{L}^r, \nabla) \times \mathbb{H}^1_{\mathrm{par}}(X, \mathcal{L}^r, \nabla) \longrightarrow \mathbb{Q} \tag{5.1}$$

induced by the pairing (3.6). Both de Rham and parabolic cohomology are compatible with base change $\mathbb{Q} \hookrightarrow K$ to any field K of characteristic 0, i.e.

$$\mathbb{H}^i(X_K, \mathcal{L}^r, \nabla) = \mathbb{H}^i(X, \mathcal{L}^r, \nabla) \otimes K$$

and similarly for parabolic cohomology ([15], Theorem 2.7 (iii)). Moreover, if we let

$$\mathbb{H}^i(Y, \mathcal{L}^r, \nabla) := \mathbb{H}^i(Y, 0 \longrightarrow \mathcal{L}^r \stackrel{\nabla}{\longrightarrow} \mathcal{L}^r \otimes \Omega^1_Y \longrightarrow 0),$$

then there is a canonical isomorphism ([7], II.3.15 and II.3.16):

$$\mathbb{H}^i(X, \mathcal{L}^r, \nabla) = \mathbb{H}^i(Y, \mathcal{L}^r, \nabla).$$

Let now $\mathcal{R}_X$ be the sheaf of rational functions on X and let

$$\mathcal{PP}_X := \coprod_{x \in X \mathrm{closed}} \mathcal{R}_{X,x}/\mathcal{O}_{X,x}$$

be the sheaf of principal parts of X. The algebraic de Rham cohomology $H^1_{\mathrm{dR}}(X)$ has the two following important features, valid for any smooth proper curve over a field of characteristic zero:

(a) The Hodge filtration:

$$0 \longrightarrow H^0(X, \Omega^1_X) \longrightarrow H^1_{\mathrm{dR}}(X) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow 0.$$

(b) The classical description

$$H^1_{\mathrm{dR}}(X) = \frac{H^0(X, \Omega^1_X \otimes \mathcal{R}_X)^{\mathrm{II-kind}}}{dH^0(X, \mathcal{R}_X)},$$

where by 'form of the second kind' we mean a section η of $\Omega^1_X \otimes \mathcal{R}_X$ such that for each point $x \in X$ the principal part η_x of η at x is of the form dP_x for $P_x \in \mathcal{PP}_{X,x}$.

The following Theorem 5.2 shows that $\mathbb{H}^1_{\mathrm{par}}(X, \mathcal{L}^r, \nabla)$ enjoys properties similar to (a) and (b). To state the theorem we make the following definition first:

Definition 5.1. Let K be a field of characteristic 0. Define a *$\underline{\omega}^r$-valued 1-form of the second kind* to be a section η of $\underline{\omega}^r \otimes \Omega^1_{X_K} \otimes \mathcal{R}_X$ such that for any point $x \in X_K$ the principal part η_x of η at x is of the form $\partial^{r+1} P_x$ for $P_x \in \underline{\omega}^{-r} \otimes \mathcal{PP}_{X,x}$, where ∂^{r+1} is the map (3.12).

THEOREM 5.2. *Let K be a field of characteristic 0.*

(a) There is an exact sequence of K-vector spaces:

$$0 \longrightarrow H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X) \longrightarrow \mathbb{H}^1_{\mathrm{par}}(X_K, \mathcal{L}^r, \nabla) \longrightarrow H^1(X_K, \underline{\omega}^{-r}) \longrightarrow 0. \tag{5.2}$$

(b) There is a canonical isomorphism:

$$\mathbb{H}^1_{\mathrm{par}}(X_K, \mathcal{L}^r, \nabla) = \frac{H^0(X_K, \underline{\omega}^r \otimes \Omega^1_X \otimes \mathcal{R}_X)^{\mathrm{II-kind}}}{\partial^{r+1} H^0(X_K, \underline{\omega}^{-r} \otimes \mathcal{R}_X)},$$

where the top space on the right-hand side is the space of global $\underline{\omega}^r$-valued 1-forms of the second kind.

Proof.

(a) This is [15] Theorem 2.7 part (i).

(b) For the sake of clarity we first present the argument for the case $r = 0$, which is entirely classical, and then generalize to any r. Moreover, to lighten notation, in this proof it is understood that $X = X_K$ and all constructions take place over K.

When $r = 0$ we have $(\mathcal{L}^r, \nabla) = (\mathcal{O}_X, d)$ so that parabolic cohomology is simply $H^i_{\mathrm{dR}}(X)$ in this case. We can compute de Rham cohomology from the total complex associated to the Cousin resolution of the de Rham complex $\Omega^\bullet_X : \mathcal{O}_X \xrightarrow{d} \Omega^1_X$. The Cousin resolution of $\Omega^\bullet_X$ is given for $i = 0, 1$ by:

$$C^0(\Omega^i_X) := \Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{R}_X$$
$$C^1(\Omega^i_X) := \Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{PP}_X$$

where the differential $\delta : C^0(\Omega^i_X) \to C^1(\Omega^i_X)$ is given by 'taking principal parts at x'. The total complex associated to this resolution is given by:

$$\mathrm{Tot}^\bullet : \qquad C^0(\mathcal{O}_X) \xrightarrow{d \oplus \delta} C^0(\Omega^1_X) \oplus C^1(\mathcal{O}_X) \xrightarrow{\delta - d} C^1(\Omega^1_X).$$

Following Grothendieck ([9], Footnote 8), the de Rham cohomology groups $H^i_{\mathrm{dR}}(X)$ can be computed as the cohomology of the complex $\Gamma\,\mathrm{Tot}^\bullet$ obtained from $\mathrm{Tot}^\bullet$ by taking global sections. In particular:

$$\begin{aligned} H^1_{\mathrm{dR}}(X) &= \frac{\ker\Big(H^0(X, \Omega^1_X \otimes \mathcal{R}_X) \oplus H^0(X, \mathcal{PP}_X) \xrightarrow{\delta - d} H^0(X, \Omega^1_X \otimes \mathcal{PP}_X)\Big)}{\mathrm{im}\Big(H^0(X, \mathcal{R}_X) \xrightarrow{d \oplus \delta} H^0(X, \Omega^1_X \otimes \mathcal{R}_X) \oplus H^0(X, \mathcal{PP}_X)\Big)} \\ &= \frac{\{(\eta, \{P_x\}) \in H^0(X, \mathcal{R}_X) \oplus H^0(X, \mathcal{PP}_X) : \delta\eta_x = dP_x,\ \forall x\}}{\{(df, \{\delta f_x\}) : f \in H^0(X, \mathcal{R}_X)\}} \\ &= \frac{H^0(X, \Omega^1_X \otimes \mathcal{R}_X)^{\mathrm{II-kind}}}{dH^0(X, \mathcal{R}_X)} \end{aligned}$$

which proves the case $r = 0$.

For the general case, let $\Omega^\bullet_X(\mathcal{L}^r)$ be the complex:

$$\mathcal{L}^r \xrightarrow{\nabla} \Omega^1_X(\mathcal{L}^r)$$

defining parabolic cohomology. Before applying the Cousin resolution, we first want to show that the hypercohomology of this complex is the same as that of the complex:

$$\underline{\omega}^{-r} \xrightarrow{\partial^{r+1}} \underline{\omega}^r \otimes \Omega^1_X.$$

To see this, note first that $\Omega^1_X(\mathcal{L}^r)$ can be decomposed Zariski-locally over $U \subseteq X$ into graded pieces of the filtration (3.8) as:

$$\Omega^1_X(\mathcal{L}^r)|_U = \left(\underline{\omega}^r \otimes \Omega^1_X\right)|_U \oplus \left(\bigoplus_{i=0}^{r-1} \underline{\omega}^{2i-r} \otimes \Omega^1_X(\log Z_N)\right)|_U.$$

Let then $G \subseteq \Omega^1_X(\mathcal{L}^r)$ be the subsheaf which is given locally by the second component of this direct sum, i.e. G is defined by the exact sequence of sheaves:

$$0 \to G \to \Omega^1_X(\mathcal{L}^r) \to \underline{\omega}^r \otimes \Omega^1_X \to 0.$$

By (3.11) we have an exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^1(\mathcal{L}^r) & \longrightarrow & \mathcal{L}^r & \longrightarrow & \underline{\omega}^{-r} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nabla^1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \partial^{r+1}} & & \\ 0 & \longrightarrow & G & \longrightarrow & \Omega^1_X(\mathcal{L}^r) & \longrightarrow & \underline{\omega}^r \otimes \Omega^1_X & \longrightarrow & 0 \end{array}$$

where ∇^1 is an isomorphism. The corresponding long exact sequence in hypercohomology then gives:

$$\mathbb{H}^i(\mathcal{L}^r \xrightarrow{\nabla} \Omega^1_X(\mathcal{L}^r)) \simeq \mathbb{H}^i(\underline{\omega}^{-r} \xrightarrow{\partial^{r+1}} \underline{\omega}^r \otimes \Omega^1_X)$$

canonically for all i.

We now apply the same technique as in the case $r = 0$ to compute $\mathbb{H}^1 := \mathbb{H}^1(\underline{\omega}^{-r} \xrightarrow{\partial^{r+1}} \underline{\omega}^r \otimes \Omega^1_X)$. In particular, the total complex $\mathrm{Tot}^\bullet$ associated to the Cousin resolution of $\underline{\omega}^{-r} \xrightarrow{\partial^{r+1}} \underline{\omega}^r \otimes \Omega^1_X$ is given by:

$$\underline{\omega}^{-r} \otimes \mathcal{R}_X \xrightarrow{\partial^{r+1} \oplus \delta} \left(\underline{\omega}^r \otimes \Omega^1_X \otimes \mathcal{R}_X\right) \oplus \left(\underline{\omega}^{-r} \otimes \mathcal{PP}_X\right) \xrightarrow{\delta - \partial^{r+1}} \underline{\omega}^r \otimes \Omega^1_X \otimes \mathcal{PP}_X,$$

where the map δ, as before, is given by 'taking principal parts'. Again by [9], Footnote 8, which works with coefficients in any locally free sheaf over X, hypercohomology can be computed as the cohomology of the complex $\Gamma\,\mathrm{Tot}^\bullet$ obtained from $\mathrm{Tot}^\bullet$ by taking global sections. In particular:

$$\begin{aligned}\mathbb{H}^1 &= \frac{\ker\left(H^0(X,\underline{\omega}^r\otimes\Omega^1_X\otimes\mathcal{R}_X)\oplus H^0(X,\underline{\omega}^{-r}\otimes\mathcal{PP}_X)\overset{\delta-\partial^{r+1}}{\to} H^0(X,\underline{\omega}^r\otimes\Omega^1_X\otimes\mathcal{PP}_X)\right)}{\mathrm{im}\left(H^0(X,\underline{\omega}^{-r}\otimes\mathcal{R}_X)\overset{\partial^{r+1}\oplus\delta}{\longrightarrow} H^0(X,\underline{\omega}^r\otimes\Omega^1_X\otimes\mathcal{R}_X)\oplus H^0(X,\underline{\omega}^{-r}\otimes\mathcal{PP}_X)\right)}\\ &= \frac{\{(\eta,\{P_x\})\in H^0(X,\underline{\omega}^r\otimes\Omega^1_X\otimes\mathcal{R}_X)\oplus H^0(X,\underline{\omega}^{-r}\otimes\mathcal{PP}_X):\delta\eta_x=\partial^{r+1}P_x,\ \forall x\}}{\{(\partial^{r+1}f,\{\delta f_x\}):f\in H^0(X,\underline{\omega}^{-r}\otimes\mathcal{R}_X)\}}\\ &= \frac{H^0(X,\underline{\omega}^r\otimes\Omega^1_X\otimes\mathcal{R}_X)^{\mathrm{II-kind}}}{\partial^{r+1}H^0(X,\underline{\omega}^{-r}\otimes\mathcal{R}_X)},\end{aligned}$$

and the theorem is proved. □ □

The modular curves $X_{\mathbb{C}}, Y_{\mathbb{C}}$ each have the structure of a differentiable manifold, of a complex manifold and of an algebraic variety. Each of these structures induces a different description of the de Rham cohomology groups introduced above, giving the following 'comparison theorems'.

THEOREM 5.3. *Let $(\mathcal{L}^r_{\mathrm{an}},\nabla)$ be the analytification of the sheaf $(\mathcal{L}^r,\nabla)$ over $Y_{\mathbb{C}}$. Then there is a canonical isomorphism:*

$$\mathbb{H}^1(Y_{\mathbb{C}},\mathcal{L}^r,\nabla)=\frac{\ker\left(\nabla:\mathcal{L}^r_{\mathrm{an}}\otimes\mathcal{A}^1_Y(Y)\longrightarrow\mathcal{L}^r_{\mathrm{an}}\otimes\mathcal{A}^2_Y(Y)\right)}{\mathrm{im}\left(\nabla:\mathcal{L}^r_{\mathrm{an}}\otimes C^\infty_Y(Y)\longrightarrow\mathcal{L}^r_{\mathrm{an}}\otimes\mathcal{A}^1_Y(Y)\right)}. \tag{5.3}$$

Proof. Let $L^{\nabla=0}$ be the sheaf of horizontal sections of $(\mathcal{L}^r_{\mathrm{an}},\nabla)$. Then:

$$\mathbb{H}^1(Y^{\mathrm{an}},\mathcal{L}^r_{\mathrm{an}},\nabla)=H^1(Y^{\mathrm{an}},L^{\nabla=0})=\mathbb{H}^1(Y^{\mathrm{an}},\mathcal{A}^\bullet(\mathcal{L}^r_{\mathrm{an}}),\nabla).$$

Since the sheaves of smooth differential forms are fine, hence acyclic, the right-hand term of this equation is precisely the right-hand side of (5.3). On the other hand by [7], Th. II.6.2 and Th. II.7.9, we have a GAGA-style isomorphism:

$$\mathbb{H}^1(Y^{\mathrm{an}},\mathcal{L}^r_{\mathrm{an}},\nabla)=\mathbb{H}^1(Y_{\mathbb{C}},\mathcal{L}^r,\nabla)$$

for the left-hand side. □ □

THEOREM 5.4 (Shimura Isomorphism, [16] Th. 8.4.). *When $K=\mathbb{C}$, the exact sequence (5.2) admits a canonical, functorial splitting:*

$$\mathbb{H}^1_{\mathrm{par}}(X_{\mathbb{C}},\mathcal{L}^r,\nabla)=H^0(X_{\mathbb{C}},\underline{\omega}^r\otimes\Omega^1_X)\oplus\overline{H^0(X_{\mathbb{C}},\underline{\omega}^r\otimes\Omega^1_X)}. \tag{5.4}$$

Proof. From the proof of Theorem 5.2, we know that

$$\mathbb{H}^1_{\text{par}}(X_{\mathbb{C}}, \mathcal{L}^r, \nabla) \simeq \mathbb{H}^1(\underline{\omega}^{-r} \stackrel{\partial^{r+1}}{\to} \underline{\omega}^r \otimes \Omega^1_X).$$

The theorem then follows from the Hodge decomposition of the elliptic complex $\underline{\omega}^{-r} \stackrel{\partial^{r+1}}{\to} \underline{\omega}^r \otimes \Omega^1_X$. □

THEOREM 5.5. *Let $M^{!,\infty}_k$ be the space of weakly holomorphic modular forms which are holomorphic everywhere but at the cusp ∞, and let $S^{!,\infty}_k$ be the subspace of those forms in $M^{!,\infty}_k$ with vanishing constant coefficient in their q-expansion at ∞. Then there is a canonical isomorphism:*

$$\mathbb{H}^1_{\text{par}}(X_K, \mathcal{L}^r, \nabla) = \frac{S^{!,\infty}_{r+2}(\Gamma, K)}{\partial^{r+1} M^{!,\infty}_{-r}(\Gamma, K)}.$$

Proof. Given Theorem 5.2 part (b), all we need to show is that given a $\underline{\omega}^r$-valued form η of the second kind, we can always find a rational section f of $\underline{\omega}^{-r}$ such that $\eta - \partial^{r+1} f$ is regular over $X^* := X - \{\infty\}$. Now by definition the principal parts of η over X^* are of the form $\delta\eta_x = \partial^{r+1} P_x$ for all $x \in X^*$, where $\{P_x\}_{x\in X^*} \in H^0(X^*, \underline{\omega}^{-r} \otimes \mathcal{PP}_{X^*})$. Hence we need to find

$$f \in H^0(X^*, \underline{\omega}^{-r} \otimes \mathcal{R}_{X^*})$$

with the prescribed principal parts $\{P_x\}_{x\in X^*}$. But the obstruction to finding such f lies in $H^1(X^*, \underline{\omega}^{-r})$, and this cohomology vanishes since X^* is affine. Now take the rational function f over X^* and extend it to a rational function on X. □ □

REMARK 5.6. The isomorphism of Theorem 5.5 is a generalization to any level and field of definition of the the Eichler-Shimura isomorphism of [1], valid for $K = \mathbb{C}$ and level 1.

6 Construction of harmonic weak Maass forms

We now show that the operator ξ_k defined by (1.3) is surjective, by explicitly constructing a harmonic weak Maass form F with $\xi_k(F) = (-4\pi)^{1-k} f$, for any given cusp form $f \in S_{2-k}(\Gamma)$. We then show how this construction can be refined to K-rational harmonic weak Maass forms to yield a proof of Theorem 1.4.

Let $f \in S_{r+2}(\Gamma)$ be a cusp form, and let $\omega_f \in H^0(X_{\mathbb{C}}, \underline{\omega}^r \otimes \Omega^1_X)$ be the corresponding differential under (3.5). By the Shimura isomorphism (5.4) we know that:

$$\overline{\omega_f} \in \mathbb{H}^1_{\text{par}}(X_{\mathbb{C}}, \mathcal{L}^r, \nabla)$$

represents a class η_f in parabolic cohomology. The same class can also be represented as

$$\eta_f = [\phi],$$

where ϕ is of the second kind, by Theorem 5.2 part (b). Moreover ϕ can be chosen to have poles supported on Z_N, as in the proof of Theorem 5.5. Now the C_Y^∞-differential form

$$\overline{\omega_f} - \phi \in \mathcal{L}^r \otimes \mathcal{A}_Y^1(Y)$$

gives a trivial class in $\mathbb{H}^1(Y_\mathbb{C}, \mathcal{L}^r, \nabla)$ by construction, hence there exists

$$\mathbf{F} \in \mathcal{L}^r \otimes C_Y^\infty(Y)$$

with:

$$\nabla \mathbf{F} = \overline{\omega_f} - \phi \tag{6.1}$$

by (5.3).

To compute with $\mathbf{F}$ explicitly we work over the universal elliptic curve $\mathcal{E}_\mathfrak{h}$ as in Section 4. In particular, the sheaf $\mathcal{L}_\mathfrak{h}^r \otimes C_\mathfrak{h}^\infty$ is free of rank $r+1$ over $\mathfrak{h}$: the $r+1$ nonvanishing sections

$$\left(\frac{-d\bar{z}}{2iv}\right)^r, \dots, (2\pi i)^j \, dz^j \left(\frac{-d\bar{z}}{2iv}\right)^{r-j}, \dots, (2\pi i)^r \, dz^r, \quad j = 1, \dots, r$$

provide a trivialization $\mathcal{L}_\mathfrak{h}^r \otimes C_\mathfrak{h}^\infty \simeq \left(C_\mathfrak{h}^\infty\right)^{r+1}$ over $\mathfrak{h}$. Using this trivialization, we can write down $\mathbf{F}$ (to be precise $\mathrm{pr}^* \, \mathbf{F}$) as

$$\mathbf{F} = F_{-r} \left(\frac{-d\bar{z}}{2iv}\right)^r + \dots + F_{2j-r} \, (2\pi i)^j \, dz^j \left(\frac{-d\bar{z}}{2iv}\right)^{r-j} + \dots + F_r \, (2\pi i)^r \, dz^r \tag{6.2}$$

where the F_{2j-r}'s are C^∞-functions on $\mathfrak{h}$ which transform under Γ like modular forms of weight $2j-r$. Geometrically, we can say that the F_{2j-r}'s descend under $\mathrm{pr} : \mathfrak{h} \to \mathfrak{h}/\Gamma = Y^{\mathrm{an}}$ to sections of $\underline{\omega}_{\mathrm{an}}^{2j-r} \otimes C_Y^\infty$ over Y. This is the reason why we favor the splitting (4.5) over (4.4).

Now the fact that $\mathbf{F}$ satisfies the differential equation (6.1) imposes further structure on the F_{2j-r}'s. In particular, we claim that $F := F_{-r}$ is a harmonic weak Maass form which maps to the cusp form f under ξ_{-r}.

Proposition 6.1. *Let $f \in S_{r+2}(\Gamma)$ be a cusp form and let*

$$F := F_{-r}$$

be defined by (6.2). *Then:*

(a) $\xi_{-r}(F) = (-4\pi)^{r+1} f$*, where* ξ_{-r} *is defined by* (1.3)*.*

(b) $D^{r+1}(F) = \frac{1}{(2\pi i)^{r+1}}\left(\frac{\partial^{r+1}F}{\partial\tau^{r+1}}\right)$ *belongs to* $M^!_{r+2}(\Gamma)$*, and all constant terms of the Fourier expansions of* $D^{r+1}(F)$ *at the cusps vanish.*

(c) F *is a harmonic weak Maass form of weight* $-r$*, i.e.* $F \in \mathcal{H}_{-r}(\Gamma)$*.*

Proof.

(a) Write ω_f in coordinates:

$$\omega_f = f\,(2\pi i\,dz)^{r+2} = f\,(2\pi i\,dz)^r\,2\pi i\,d\tau$$

where the second equality is given by the Kodaira-Spencer isomorphism of Lemma 4.2 part (a). Similarly, write:

$$\phi = \varphi\,(2\pi i\,dz)^r\,2\pi i\,d\tau$$

with ϕ as in (6.1). Using (6.2), we can write down the differential equation (6.1) in coordinates as:

$$\nabla\left(F\left(\frac{-d\bar{z}}{2iv}\right)^r + \dots\right) = \overline{f}\,(-2\pi i\,d\bar{z})^r\,(-2\pi i\,d\bar{\tau}) - \varphi\,(2\pi i\,dz)^r\,2\pi i\,d\tau. \qquad (6.3)$$

By writing $\nabla = \nabla^{1,0} + \nabla^{0,1}$ and by comparing the two sides of equation (6.3) term-by-term we must have:

$$\nabla^{0,1}\left(F\left(\frac{-d\bar{z}}{2iv}\right)^r\right) = \overline{f}\,(-2\pi i)^{r+1} d\bar{z}^r\,d\bar{\tau}.$$

Computing $\nabla^{0,1}$ as in Lemma 4.4 part (b) gives

$$\frac{(-1)^r}{(2iv)^r}\frac{\partial F}{\partial\bar{\tau}} = (-2\pi i)^{r+1}\overline{f},$$

or

$$\frac{1}{(2i\cdot 2\pi i)^{r+1}}\,\frac{-2i}{v^r}\,\frac{\partial F}{\partial\bar{\tau}} = \overline{f}$$

which gives (a).

(b) The $C^\infty_{\mathfrak{h}}$-function F descends to a section of $\underline{\omega}_{\text{an}}^{-r} \otimes C^\infty_Y$ over Y. By construction, $\mathbf{F}$ is then a section of $\mathcal{L}^r \otimes C^\infty_Y$ which lifts F under the natural map $\mathcal{L}^r \to \underline{\omega}^{-r}$. Moreover, by comparing both sides of equation (6.3) we must have

$$\nabla^{1,0}\,\mathbf{F} = -\varphi\,(2\pi i\,dz)^r\,2\pi i\,d\tau \qquad (6.4)$$

and therefore $\nabla^{1,0}\,\mathbf{F}$ descends to a section of $\underline{\omega}_{\mathrm{an}}^{r}\otimes\mathcal{A}_Y^{1,0}$. By the same argument used in Proposition 3.1 we conclude that $\mathbf{F}$ is the unique section of $\mathcal{L}^r$ which lifts F and which maps to $\underline{\omega}_{\mathrm{an}}^{r}\otimes\mathcal{A}_Y^{1,0}$ under $\nabla^{1,0}$. Therefore the expression $\nabla^{1,0}\left(F\left(\frac{-d\bar{z}}{2iv}\right)^r\right)$ coincides with the map:

$$\partial^{r+1}:\underline{\omega}_{\mathrm{an}}^{-r}\otimes C_Y^{\infty}\longrightarrow\underline{\omega}_{\mathrm{an}}^{r}\otimes\mathcal{A}_Y^{1,0}$$

which is the same as that of Proposition 4.6, only applied to smooth sections instead of holomorphic ones. This map can be computed explicitly as in (4.13), so that we can derive the equation

$$\frac{(-1)^r}{r!}D^{r+1}(F)=-\varphi$$

directly from (6.4).

By definition, φ is in $M^!_{r+2}(\Gamma)$. By choice, the residues of the differential ϕ at the cusps are zero, i.e. the constant terms of the Fourier expansions of φ are zero, which proves part (b).

(c) The function F is smooth and modular of weight $-r$ by definition, and it is harmonic since $\xi_{-r}F=f$ implies $\Delta_{-r}F=0$ automatically. Finally, parts (a) and (b) show that the principal parts of the Fourier expansions at the cusps of F are obtained by formally integrating the principal parts of φ a total of $r+1$ times, hence they satisfy the required growth conditions.

□ □

When $f\in S_{r+2}(\Gamma,K)$ is a newform with coefficients in a field K of characteristic 0, the construction of Proposition 6.1 can be refined to construct K-rational harmonic weak Maass forms (Definition 1.2).

As in Proposition 6.1, let ω_f be the differential in $\underline{\omega}^r\otimes\Omega^1_{X_K}$ corresponding to f. Since f is a newform, the parabolic cohomology class represented by ω_f is in fact contained in the 2-dimensional f-isotypical component $\mathbb{H}^1_{\mathrm{par}}(X_K,\mathcal{L}^r,\nabla)_f$ attached to f (this is the f-eigenspace cut out by the Hecke operators, [16] Ch. 8). Let η_f^{alg} be a class such that:

$$\mathbb{H}^1_{\mathrm{par}}(X_K,\mathcal{L}^r,\nabla)_f=K\,\omega_f\oplus K\,\eta_f^{\mathrm{alg}},\qquad\left\langle\omega_f,\eta_f^{\mathrm{alg}}\right\rangle=1,$$

where $\langle\cdot,\cdot\rangle$ is the cup-product (5.1). Now by the isomorphism of Theorem 5.5 the class η_f^{alg} has a representative:

$$\eta^{\mathrm{alg}}=[\phi]$$

such that ϕ is of the second kind with a pole at ∞ and regular everywhere else. On the other hand, once scalars are extended to $\mathbb{C}$ we also have a decomposition

$$\mathbb{H}^1_{\mathrm{par}}(X_{\mathbb{C}}, \mathcal{L}^r, \nabla)_f = \mathbb{C}\,\omega_f \oplus \mathbb{C}\,\overline{\omega_f}$$

of the f-isotypical component, by the Shimura isomorphism (5.4). Hence we can write

$$\eta_f^{\mathrm{alg}} = a\,\omega_f + b\,\overline{\omega_f}$$

for some $a, b \in \mathbb{C}$. By comparing the two expressions for η_f, we obtain a differential:

$$a\,\omega_f + b\,\overline{\omega_f} - \phi \in \mathcal{L}^r \otimes \mathcal{A}^1_Y(Y).$$

The class of this differential is trivial in cohomology and we can proceed as in Proposition 6.1 to find $\mathbf{F} \in \mathcal{L}^r \otimes C^\infty_Y(Y)$ with:

$$\nabla\,\mathbf{F} = a\,\omega_f + b\,\overline{\omega_f} - \phi. \tag{6.5}$$

Computing with local coordinates, we obtain the following.

Proposition 6.2. *Let $f \in S_{r+2}(\Gamma, K)$ be a newform and let $\mathbf{F}$ be as in* (6.5)*. Write*

$$F := F_{-r}$$

using the local coordinates expression (6.2)*. Then:*

(a) $\xi_{-r}(F) = \frac{f}{(f,f)}$, where (f,f) is the Petersson norm of f.

(b) F is a K-rational harmonic weak Maass form in $\mathcal{H}_{-r}(\Gamma)$.

Proof.

(a) Suppose f is given by its q-expansion at ∞:

$$\widetilde{f} = \sum_{n=1}^{\infty} b(n)\,q^n.$$

Over $\mathfrak{h}$, the corresponding differential ω_f is given by:

$$\omega_f = f\,(2\pi i dz)^r\,2\pi i\,d\tau.$$

Proceeding as in Proposition 6.1 but using the refined differential equation (6.5) instead of (6.1), we find that:

$$\xi_{-r}(F) = b\,(-4\pi)^{r+1}\,f.$$

To compute b, use (3.7) and (4.2):

$$\begin{aligned}
\frac{1}{b} &= \langle \omega_f, \overline{\omega}_f \rangle \\
&= \frac{1}{2\pi i} \int_{\mathfrak{h}/\Gamma} f(\tau)\,\overline{f(\tau)} \left\langle (2\pi i\, dz)^r, \overline{(2\pi i\, dz)^r} \right\rangle_r (2\pi i\, d\tau) \wedge \overline{(2\pi i\, d\tau)} \\
&= \frac{(2\pi i)^{r+1} \cdot (-2\pi i)^{r+1}}{2\pi i} \int_{\mathfrak{h}/\Gamma} f(\tau)\,\overline{f(\tau)} \left(\int_{E_\tau} dz \wedge d\bar{z} \right)^r d\tau \wedge d\bar{\tau} \\
&= (2\pi i \cdot 2i)^{r+1} \int_{\mathfrak{h}/\Gamma} f(\tau)\,\overline{f(\tau)}\, v^r\, du \wedge dv \\
&= (-4\pi)^{r+1} (f, f)\,.
\end{aligned}$$

(b) As explained in Proposition 6.1 part (c), the principal parts of the q-expansions of F at the cusps are determined by the principal parts of the differential $\phi = \varphi(2\pi i\, dz)^r 2\pi i\, d\tau$. Since ϕ was chosen to be regular everywhere but at ∞, the principal parts of F are constant at all cusps except ∞, where they have coefficients in K (the cusp ∞ is defined over K, by Remark 2.1).

□ □

Remark 6.3. Recall that there is a decomposition:

$$S_{r+2}(\Gamma) = \bigoplus_{\chi} S_{r+2}(\Gamma_0(N), \chi).$$

If the newform f is chosen to be in $S_{r+2}(\Gamma_0(N), \chi)$, then by direct computation it follows that $F \in \mathcal{H}_{-r}(\Gamma_0(N), \overline{\chi})$.

We are finally ready to prove Theorem 1.4. To adjust our notation to that of [4], let $k \in \mathbb{Z}$ be such that $k = r + 2$. Let $f \in S_k(\Gamma_0(N), \overline{\chi})$ be a newform given by the q-expansion

$$\widetilde{f} = \sum_{n=1}^{\infty} b(n)\, q^n.$$

It is well-known that the coefficients $b(n)$ generate a number field K_f, so by the q-expansion principle we can assume $f \in S_k(\Gamma, K_f)$.

Theorem 1.4. *Let $F \in \mathcal{H}_{2-k}(\Gamma_0(N), \chi)$ be a K_f-rational harmonic weak Maass form such that $\xi_{2-k}(F) = f/(f,f)$. Denote by $F^+ = \sum_{n \gg -\infty} c^+(n) q^n$ the holomorphic part of F. If $b(n) = 0$, then the coefficient $c^+(n)$ belongs to K_f.*

Proof. Given f, construct F' as in Proposition 6.2. Then we can assume $F = F'$. This is because $F - F' = F^+ - F'^+$ is holomorphic on $\mathfrak{h}$ and at all the cusps except ∞, and the principal part of $F - F'$ at ∞ can be eliminated by adding to F' a suitable $(k-1)$-th antiderivative of a K_f-rational weakly holomorphic modular form. But then $D^{k-1}(F - F')$ is a cusp form whose Petersson pairing with any other cusp form is zero ([4], 4.3), so it is zero and $F - F' = 0$ since there are no cusp forms whose $(k-1)$-th derivative is zero. Now using the notation of Proposition 6.2 part (b), write

$$\varphi = \sum_{n \gg -\infty} d(n) q^n$$

for the q-expansion of φ at ∞. Then $d(n) \in K_f$ for all n, since φ is defined over K_f. As in Proposition 6.1 part (b) we must have:

$$\frac{(-1)^{k-2}}{(k-2)!} D^{k-1}(F^+) = a\, f - \varphi.$$

In terms of q-expansions, this formula gives:

$$c^+(n) = (-1)^{k-2}(k-2)! \left(\frac{a \cdot b(n) - d(n)}{n^{k-1}} \right) \tag{6.6}$$

hence if $b(n) = 0$ certainly $c^+(n)$ belongs to K_f. □ □

As a final remark, note that when the cusp form f has complex multiplication then the complex number a appearing in (6.6) can be chosen to be algebraic. This is due to the existence in the CM case of an algebraic analog of the Shimura isomorphism (5.4), i.e. some multiple of $\overline{\omega_f}$ is algebraic. This observation combined with (6.6) proves Theorem 1.3 of [4]. The question of whether cusp forms with CM are the only ones with this property, i.e. possessing an 'algebraic' Shimura isomorphism, is much deeper. When $f \in S_2(\Gamma_0(N), \mathbb{Q})$, for example, the f-isotypical component of parabolic cohomology is isomorphic to the de Rham cohomology of the elliptic curve E_f attached to f, and the question translates into an open problem in transcendence theory ([12], Question 4.0.8).

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