A GEOMETRIC PERSPECTIVE ON p-ADIC PROPERTIES OF MOCK MODULAR FORMS

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Abstract. In [BGK12], Bringmann, Guerzhoy and Kane have shown how to correct mock modular forms by a certain linear combination of the Eichler integral of their shadows in order to obtain p-adic modular forms in the sense of Serre. In this paper, we give a new proof of their results (for good primes p) by employing the geometric theory of harmonic Maass forms developed by the first author [Can14] and the theory of overconvergent modular forms due to Katz and Coleman.

1. Introduction

Over the past decade, there has been a renewed interest in Ramanujan's *mock modular forms* and related objects, such as *harmonic Maass forms*, whose Fourier coefficients seem to encode interesting arithmetic data not elsewhere found in the classical theory of modular forms. In this article, we offer a new perspective on the p-adic properties of the Fourier coefficients of mock modular forms, based on the algebro-geometric theory of p-adic modular forms of Katz-Coleman ([Kat73], [Col96]). Such p-adic properties were originally discovered in [BGK12], [GKO10], but we believe our perspective simplifies some of the arguments and provides a theoretical platform for further exploration.

In order to state our results precisely, let $\tau = u + iv \in \mathfrak{h}$, let $\Gamma_0(N)$ be the congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of matrices that become upper-triangular modulo N, and let χ be a Dirichlet character modulo N. Denote by $\mathcal{H}_k(\Gamma_0(N), \chi)$ the vector space of all weight k harmonic Maass forms on $\Gamma_0(N)$ and character χ (see e.g. [BGK12, §2] for definitions). Any harmonic Maass form F has a decomposition

$$F = F^+ + F^-$$

into a holomorphic part F^+ (with poles supported at the cusps) and an anti-holomorphic part F^-. The function $F^+ : \mathfrak{h} \to \mathbb{C}$ is what is called a *mock* modular form, since it does not transform like a modular form, but its Fourier coefficients resemble those of a modular form. Harmonic Maass forms map into spaces of classical modular forms via differential operators. In particular, let $M_k^!(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$) be the space of weakly holomorphic modular forms (resp. cusp forms) of weight k, level $\Gamma_0(N)$ and character χ. If we let

$$\xi_k := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}}}, \tag{1}$$

then $\xi_{2-k}(F) = f \in S_k(\Gamma_0(N), \chi)$ for all $F \in \mathcal{H}_{2-k}(\Gamma_0(N), \overline{\chi})$, and the resulting cusp form is called the *shadow* of F. A fundamental question in the subject is to relate the

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coefficients of a mock modular form F^+ to the coefficients of its shadow f. In order to obtain results in this direction, we first have to restrict to normalized newforms f and then slightly refine the definition of a harmonic Maass form, as follows. Let $K \subseteq \mathbb{C}$ be a subfield. For $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ a congruence subgroup, we denote by $S_k(\Gamma, K)$ the space of cusp forms of weight k and level Γ whose q-expansion coefficients all lie in the field K. Let also $M_k^!(\Gamma, K)$ be the space of weakly holomorphic modular forms with coefficients in K.

Definition 1.1. Let $f \in S_k(\Gamma_1(N), K)$ be a newform defined over K. A harmonic Maass form $F \in \mathcal{H}_{2-k}(\Gamma_1(N))$ is *good* for f if

(i) The principal parts of F all lie in K.
(ii) $\xi_{2-k}(F) = f/\|f\|^2$, where $\|f\|$ is the Petersson norm of f.

Suppose that $f = \sum_{n=1}^{\infty} a_n q^n$ is a (normalized) newform as above, let F be a harmonic Maass form that is good for f, and write $F = F^+ + F^-$ for its holomorphic and anti-holomorphic parts, with

$$F^+ = \sum_{n \gg -\infty} c^+(n) q^n.$$

Let $E_f = \sum_{n=1}^{\infty} n^{1-k} a_n q^n$ be the Eichler integral of f, so that $D^{k-1}(E_f) = f$, where D^{k-1} is the differential operator on modular forms acting as $(qd/dq)^{k-1}$ on q-expansions. It is shown in [GKO10] (and also in Theorem 4.1 of this paper, by different methods) that for any $\alpha \in \mathbb{C}$ such that $\alpha - c^+(1) \in K$, the coefficients of

$$\mathcal{F}_\alpha := F^+ - \alpha E_f$$

all lie in K, so it makes sense to study their p-adic properties. To this end, let $p \nmid N$ be a prime, fix once and for all a choice of complex and p-adic embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, and fix a valuation v_p on $\mathbb{C}_p$ extending the p-adic valuation of $\mathbb{Q}$. Suppose the Hecke polynomial $T^2 - a_p T + \chi(p) p^{k-1}$ has roots β, β' with $v_p(\beta) \leqslant v_p(\beta')$, and let V be the operator acting as $q \mapsto q^p$ on q-expansions. The *p-stabilizations* of f are the p-adic modular forms

$$f_\beta := f - \beta' V(f), \quad f_{\beta'} := f - \beta V(f),$$

which are easily seen to be eigenvectors for U with eigenvalues β and β', respectively. Here, U is defined by $U(\sum_n a_n q^n) = \sum_n a_{pn} q^n$, and our first main result shows that, for most values of α, the p-stabilized shadow $f_{\beta'}$ can be recovered p-adically from the corrected mock modular form $\mathcal{F}_\alpha$ by an iterated application of the U-operator.

Theorem 1.2 ([GKO10], Theorem 1.2(i))**.** *Assume that $v_p(\beta) \neq v_p(\beta')$ and assume that $v_p(\beta') \neq k-1$. Then for all but at most one choice of α with $\alpha - c^+(1) \in K$, we have*

$$\lim_{w \to +\infty} \frac{U^w D^{k-1}(\mathcal{F}_\alpha)}{c_\alpha(p^w)} = f_\beta.$$

In Section 4, we give a new proof of this result by viewing f_β and $f_{\beta'}$ as *overconvergent modular forms*, in the sense of [Col96]. Based on ideas of [BDP13], we prove (Theorem 3.5 below) that these two modular forms are p-adic representatives of cohomology classes in the f-isotypical component of a certain parabolic cohomology group attached to the modular curve $X_1(N)$. Under the assumptions of Theorem 1.2, the classes $f_\beta, f_{\beta'}$ form a basis for this space, and so the modular form $D^{k-1}(\mathcal{F}_\alpha)$ (which gives a class in the same space, as shown in [Can14]) can be expressed as a linear combination f_β and $f_{\beta'}$. Our proof of Theorem 1.2 then follows by analyzing the action of U in cohomology.

This new proof-template can be applied to similar questions in the theory of mock modular forms. For example, in Section 5 we interpret the exceptional value of α in Theorem 1.2 as giving the precise value for which $\mathcal{F}_\alpha$ can be p-adically 'completed' to obtain a p-adic modular form. This was initially discovered by Bringmann, Guerzhoy and Kane, and we reprove here their results [BGK12] using our p-adic analytic/geometric methods. Finally, in Section 6 we discuss the case of when f has CM (also considered [BGK12] and [GKO10]), which requires a different treatment due to the failure of the assumptions in Theorem 1.2.

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2. Harmonic Maass forms: the geometric point of view

We begin by quickly recalling the geometric interpretation of harmonic Maass forms given in [Can14], which will be needed in later sections. For $N > 4$, the moduli functor $\mathscr{M}_1(N)$ of generalized elliptic curves with a point of order N is represented by a smooth and proper scheme over $\mathbb{Z}[1/N]$. Let $\mathcal{E}^{\mathrm{gen}} \to \mathscr{M}_1(N)$ be the universal generalized elliptic curve, and let $\underline{\omega}$ be its relative dualizing (invertible) sheaf. Let $X := \mathscr{M}_1(N) \times_{\mathbb{Z}[1/N]} \mathbb{Q}$ and $Y := X \smallsetminus C$, where C is the cuspidal subscheme, whose ideal sheaf we denote by $\mathcal{I}_C$. For any subfield $K \subseteq \mathbb{C}$, we denote by X_K, Y_K the base-change to K. We have well-known canonical isomorphisms

$$M_k^!(\Gamma_1(N), K) \simeq H^0(Y_K, \underline{\omega}^k), \quad S_k(\Gamma_1(N), K) \simeq H^0(X_K, \underline{\omega}^k \otimes \mathcal{I}_C),$$

where a modular form f of weight k is identified with the differential $f(dq/q)^k$. Let $\pi : \mathcal{E} \to Y$ be the universal elliptic curve with $\Gamma_1(N)$-level structure. The relative de Rham cohomology of $\pi : \mathcal{E} \to Y$ canonically extends to a rank 2 vector bundle $\mathcal{H}^1_{\mathrm{dR}}$ over X. For any $r \geqslant 0$ let

$$\mathcal{H}_r := \mathrm{Sym}^r(\mathcal{H}^1_{\mathrm{dR}}),$$

which is a vector bundle of rank $r+1$ over X. The Gauss-Manin connection of $\pi : \mathcal{E} \to Y$ extends to a connection with logarithmic poles $\nabla : \mathcal{H}^1_{\mathrm{dR}} \to \mathcal{H}^1_{\mathrm{dR}} \otimes \Omega^1_X(\log C)$ over X, and the r-th symmetric power of ∇ is a connection with logarithmic poles

$$\nabla_r : \mathcal{H}_r \longrightarrow \mathcal{H}_r \otimes \Omega^1_X(\log C).$$

Define

$$\mathbb{H}^1_{\mathrm{par}}(X, \nabla_r) := \mathbb{H}^1(\mathcal{H}_r \otimes \mathcal{I}_C \xrightarrow{\nabla_r} \mathcal{H}_r \otimes \Omega^1_X),$$

where $\mathbb{H}^\bullet$ denotes hypercohomology. Over $\mathbb{C}$, this group is canonically isomorphic to the classical weight r parabolic cohomology obtained by taking periods of cusp forms. The formation of this cohomology group is compatible under base-change by a field extension $K \supseteq \mathbb{Q}$ and for all such K and $k \geqslant 2$ there is a filtration ([Sch85, Thm. 2.7.(i)])

$$0 \longrightarrow H^0(X_K, \underline{\omega}^k \otimes \mathcal{I}_C) \longrightarrow \mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2}) \longrightarrow H^1(X_K, \underline{\omega}^{2-k}) \longrightarrow 0$$

of K-vector spaces, so that $S_k(\Gamma_1(N), K)$ is naturally a subspace of parabolic cohomology. More generally, all parabolic cohomology classes can be represented in terms of classical modular forms. To state this result, recall that for $k \geqslant 2$ there is an algebraic differential operator of order $k-1$:

$$D^{k-1} : M^!_{2-k}(\Gamma_1(N), K) \longrightarrow M^!_k(\Gamma_1(N), K)$$

which acts as $(q\, d/dq)^{k-1} = \left(\frac{1}{2\pi i}\frac{\partial}{\partial \tau}\right)^{k-1}$.

Theorem 2.1 ([Can14, Thm. 6]). *Let $K \subseteq \mathbb{C}$ be a subfield and let $S^!_k(\Gamma_1(N), K)$ be the subspace of those modular forms in $M^!_k(\Gamma_1(N), K)$ with vanishing constant coefficient in their q-expansions at the cusps. Then there is a canonical isomorphism:*

$$\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2}) \simeq \frac{S^!_k(\Gamma_1(N), K)}{D^{k-1} M^!_{2-k}(\Gamma_1(N), K)}.$$

Let now $f \in S_k(\Gamma_1(N), K)$ be a newform. Let $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$ be the f-isotypical component, and let

$$[\phi] \in \mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$$

be a class represented by an element $\phi \in S^!_k(\Gamma_1(N), K)$. By the Shimura isomorphism $\mathbb{H}^1_{\mathrm{par}}(X_{\mathbb{C}}, \nabla_{k-2})_f \simeq \mathbb{C}\, f \oplus \mathbb{C}\, \bar{f}$, we may write

$$[\phi] = s_1[f] + s_2[\bar{f}] \tag{2}$$

for some $s_1, s_2 \in \mathbb{C}$. Let now C^∞_Y (resp. $\mathcal{A}^1_Y$) be the sheaf of smooth functions (resp. smooth differential forms) on $Y_{\mathbb{C}}$. The differential $\phi - s_1 f - s_2 \bar{f}$ is smooth over $Y_{\mathbb{C}}$, and it defines a class in

$$\mathbb{H}^1(\mathcal{H}_{k-2} \otimes C^\infty_Y \xrightarrow{\nabla_{k-2}} \mathcal{H}_{k-2} \otimes \mathcal{A}^1_Y) = \frac{H^0(Y_{\mathbb{C}}, \mathcal{H}_{k-2} \otimes \mathcal{A}^1_Y)}{\nabla_{k-2} H^0(Y_{\mathbb{C}}, \mathcal{H}_{k-2} \otimes C^\infty_Y)}.$$

This class is trivial by construction, and so there exists a smooth $\mathcal{H}_{k-2}$-valued modular form $\mathbf{F}$ such that $\nabla_{k-2}(\mathbf{F}) = \phi - s_1 f - s_2 \bar{f}$. The vector bundle $\mathcal{H}_{k-2}$ decomposes into line bundles as $\mathcal{H}_{k-2} \simeq \underline{\omega}^{2-k} \oplus \underline{\omega}^{4-k} \oplus \ldots \oplus \underline{\omega}^{k-2}$, and we let $F := F_{2-k}$ be the component of $\mathbf{F}$ of weight $2-k$. As shown in [Can14, Prop. 4], F is a harmonic Maass form. If we write $F = F^+ + F^-$ for the holomorphic and anti-holomorphic parts of F, then

$$D^{k-1}(F^+) = \phi - s_1\, f, \quad \frac{2i\, v^{2-k}}{(-4\pi)^{k-1}} \frac{\partial}{\partial \bar{\tau}}(F^-) = s_2\, \bar{f}.$$

To obtain a 'true' harmonic Maass form we should insist that $\phi \notin S_k(\Gamma_1(N), K)$, i.e. $s_2 \neq 0$ in Equation (2). Then we may rescale ϕ so that $\langle \phi, f \rangle = 1$ (cup-product), which amounts to letting $s_2 = 1/\langle \bar{f}, f \rangle = 1/(-4\pi)^{k-1} \|f\|^2$. With this choice, it is clear from the above that $\xi_{2-k}(F) = f/\|f\|^2$, so that F is good for f in the sense of Definition 1.1.

3. Overconvergent modular forms

Let $p \geqslant 5$ be a prime and let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. We fix a valuation v_p on $\mathbb{C}_p$ such that $v_p(p) = 1$ and an absolute value $|\cdot|$ on $\mathbb{C}_p$ which is compatible with v_p. Let $K_p \subseteq \mathbb{C}_p$ be a complete discretely-valued subfield and let R_p be its ring of integers. Suppose $(p, N) = 1$, and let $\mathscr{X} := \mathscr{M}_1(N) \times_{\mathbb{Z}[1/N]} R_p$ be the base-change to R_p. Let $E_{p-1} \in H^0(\mathscr{X} \times_{R_p} K_p, \underline{\omega}^{p-1})$ be the global section given by the Eisenstein series of weight $p-1$ and level 1, normalized so that its constant coefficient is 1. As shown in [Col96, §1], for any $\epsilon \in |R_p|$ there is a unique rigid analytic space X_ϵ with the property that

$$X_\epsilon^{\mathrm{cl}} = \{x \in (\mathscr{X} \times_{R_p} K_p)^{\mathrm{cl}} : |E_{p-1}(x)| \geqslant \epsilon\},$$

where by the superscript 'cl' we have denoted the set of closed points, and also a unique rigid analytic space $X_{(\epsilon)}$ with the property that

$$X_{(\epsilon)}^{\mathrm{cl}} = \{x \in (\mathscr{X} \times_{R_p} K_p)^{\mathrm{cl}} : |E_{p-1}(x)| > \epsilon\}.$$

When $\epsilon = 1$, the rigid analytic space $X^{\mathrm{ord}} := X_1$ is called the *ordinary locus* of X, since every geometric point of X^{ord} reduces mod p to a point classifying an ordinary elliptic curve. The rigid analytic spaces $X_{(\epsilon)}$, for $0 < \epsilon < 1$, can be viewed as 'complements of closed disks' and are called *open neighborhoods* of X^{ord}. They are examples of *wide open spaces*. For all $\epsilon \in |R_p|$, we have inclusions $X_1 \subseteq X_\epsilon \subseteq X_{(\epsilon)} \subseteq X$. The invertible sheaves $\underline{\omega}^k$, for $k \in \mathbb{Z}$, restrict to rigid analytic line bundles over X_ϵ.

Definition 3.1. Let $\epsilon \in |R_p|$. An *overconvergent modular form* of weight $k \in \mathbb{Z}$ is a rigid analytic section $f \in H^0(X_{(\epsilon)}, \underline{\omega}^k)$, for $\epsilon < 1$.

Note that for $\epsilon = 1$ the sections of $\underline{\omega}^k$ over X^{ord} are Serre's p-adic modular forms of integral weight k. Overconvergent modular forms can thus be viewed as p-adic modular forms which converge not just over X^{ord} but on a slightly larger neighborhood of it.

Since $|E_{p-1}(c)| = 1$ at all cusps $c \in C$, we have that $C \subseteq X_{(\epsilon)}$ for all $\epsilon \in |R_p|$. Let

$$Y^{\mathrm{ord}} := X^{\mathrm{ord}} \smallsetminus C, \quad Y_\epsilon := X_\epsilon \smallsetminus C, \quad Y_{(\epsilon)} := X_{(\epsilon)} \smallsetminus C$$

be the rigid analytic spaces obtained by removing the cusps.

Remark 3.2. For $\epsilon = 1$, sections of $H^0(Y_\epsilon, \underline{\omega}^k)$ correspond to the p-adic modular forms of integral weight considered in [BGK12]. As explained in [*loc.cit.*, p. 2394], these can be directly related to the p-adic modular forms introduced by Serre [Ser73].

Let $W_1 = X_{(p^{-p/p+1})}$ and $W_2 = X_{(p^{-1/p+1})}$, both open neighborhoods of X^{ord} with $W_2 \subseteq W_1$. Let

$$U : H^0(W_2, \underline{\omega}^k) \longrightarrow H^0(W_1, \underline{\omega}^k) \subseteq H^0(W_2, \underline{\omega}^k)$$
$$V : H^0(W_1, \underline{\omega}^k) \longrightarrow H^0(W_2, \underline{\omega}^k)$$

be the operators defined in the introduction. Let $f \in S_k(\Gamma_1(N), K)$ be a newform defined over a number field K, and consider f as an element of $H^0(W_1, \underline{\omega}^k)$ by restriction. Then

$$T_p(f) = U(f) + \chi(p)p^{k-1}V(f) \in H^0(W_2, \underline{\omega}^k),$$

where T_p is the p-th Hecke operator. In particular, if f is an eigenform of level $\Gamma_0(N)$ and character χ with T_p-eigenvalue equal to a_p then

$$a_p f = U(f) + \chi(p)p^{k-1}V(f) \in H^0(W_2, \underline{\omega}^k).$$

Proposition 3.3. *Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), K)$ be a newform and let*

$$T^2 - a_p T + \chi(p)p^{k-1} = (T - \beta)(T - \beta')$$

be the p-th Hecke polynomial of f. Then the overconvergent modular forms

$$f_\beta := f - \beta' V(f), \quad f_{\beta'} := f - \beta V(f)$$

in $H^0(W_2, \underline{\omega}^k)$ are U-eigenvectors with eigenvalues β and β', respectively.

Proof. This follows from a straightforward calculation. Indeed, viewing f and $V(f)$ as sections in $H^0(W_2, \underline{\omega}^k)$, we see that

$$\begin{aligned} U f_\beta = Uf - \beta' UV(f) &= Uf - \beta' f \\ &= T_p f - \chi(p)p^{k-1}V(f) - \beta' f \\ &= (a_p - \beta')f - \chi(p)p^{k-1}V(f) \\ &= \beta f_\beta, \end{aligned}$$

using the relations $a_p = \beta + \beta'$ and $\chi(p)p^{k-1} = \beta\beta'$ for the last equality. The proof for $f_{\beta'}$ is obviously the same. □

Let now $W = X_{(\epsilon)}$, with $0 < \epsilon < 1$, be an open neighborhood of X^{ord}, and for any $r \geqslant 0$ consider the space

$$\mathbb{H}^1(W, \nabla_r) := \mathbb{H}^1(\mathcal{H}_r|_W \xrightarrow{\nabla_r} \mathcal{H}_r|_W \otimes \Omega^1_W(\log C)).$$

Theorem 3.4 (See [Col96], §5)**.**

(i) *There is a canonical isomorphism*

$$\mathbb{H}^1(W, \nabla_r) \simeq \frac{H^0(W, \underline{\omega}^{r+2})}{\theta^{r+1} H^0(W, \underline{\omega}^{-r})}.$$

(ii) *For any two open neighborhoods W, W' of X^{ord}, there is a canonical isomorphism*

$$\mathbb{H}^1(W, \nabla_r) = \mathbb{H}^1(W', \nabla_r).$$

By restriction, there is an injection

$$\mathbb{H}^1_{\mathrm{par}}(X, \nabla_r) \hookrightarrow \mathbb{H}^1(W \smallsetminus C, \nabla_r)$$

for any choice of open neighborhood W of X^{ord}. The image of this map can be characterized by p-adic residues ([BDP13, Prop. 3.9]). In particular, if $f \in S_k(\Gamma_1(N), K)$ is a newform of weight $k \geqslant 2$, the cohomology classes

$$\{[f_\beta], [f_{\beta'}]\} \subseteq \mathbb{H}^1(W_2 \smallsetminus C, \nabla_{k-2})$$

naturally lie in $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})$, and more precisely they lie in the f-isotypical component $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$, which is a two-dimensional K-vector space.

Theorem 3.5. *Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), K)$ be a newform of weight $k \geqslant 2$, and let β and β' be the roots of $T^2 - a_p T + \chi(p)p^{k-1}$, ordered so that $v_p(\beta) \leqslant v_p(\beta')$. Assume that the following two conditions hold:*

(i) $\beta \neq \beta'$.
(ii) $v_p(\beta') \neq k-1$.

Then $\{[f], [V(f)]\}$ is a basis for $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$.

Proof. Since $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$ is two-dimensional, it suffices to show that $[f]$ and $[V(f)]$ are linearly independent. By Proposition 3.3 and [Col96, Lem. 6.3], condition (ii) guarantees that $[f_{\beta'}] \neq 0$, and therefore it is an eigenvector of Ver (see [*loc.cit.*, Thm. 5.4]) acting on parabolic cohomology with eigenvalue β'. In the same manner, the class $[f_\beta]$ is non-trivial, and it is an eigenvector of Ver with eigenvalue β. Thus by condition (i), the classes $[f_\beta]$ and $[f_{\beta'}]$ are linearly independent, and so must be $[f]$ and $[V(f)]$, since f_β and $f_{\beta'}$ are linear combinations of f and $V(f)$. □

Remark 3.6. As clear from the proof, condition (ii) in Theorem 3.5 could be weakened to the following:

(ii') either $v_p(\beta') \neq k-1$ or $[f_{\beta'}] \neq 0$.

By [Col96, Prop. 7.1], condition (ii') fails if f has CM by an imaginary quadratic field in which p splits; conjecturally (see e.g. [Eme]), these are the *only* cases in which condition (ii') fails, but this is not known in general.

4. Recovering the shadow

Let $f = \sum_{n=1}^{\infty} a(n)q^n$ be a newform satisfying the hypotheses of Theorem 3.5, and let F be a harmonic Maass form which is good for f, in the sense of Definition 1.1. The shadow f can be recovered from F by

$$\xi_{2-k}(F) = \frac{f}{\|f\|^2}$$

where $\|f\|$ is the Petersson norm of f. By the results in Section 2, the harmonic Maass form F has a holomorphic part F^+ with the property that

$$D^{k-1}(F^+) = \phi - s_1 f \tag{3}$$

for some $\phi \in S_k^!(\Gamma_1(N), K)$ and some $s_1 \in \mathbb{C}$.

In [GKO10, Thm. 1.2], Bringmann, Guerzhoy and Kane prove that one of the two p-stabilizations of f can be recovered p-adically from an iterated application of U to a certain 'correction' of $D^{k-1}(F^+)$. In this section, we deduce their result from the p-adic techniques developed above. We begin by giving a new proof of [GKO10, Thm. 1.1].

Theorem 4.1. *Let $\alpha \in \mathbb{C}$ be such that $\alpha - c^+(1) \in K$. Then the coefficients of*

$$\mathcal{F}_\alpha := F^+ - \alpha E_f := \sum_{n \gg -\infty} c^+(n)q^n - \alpha \sum_{n=1}^{\infty} a(n)n^{1-k}q^n$$

are all in K.

Proof. Write $\phi = \sum_{n \gg -\infty} d(n) q^n$, with $d(n) \in K$. By (3), we have the formula

$$c^+(n) = \left(\frac{d(n) - s_1 a(n)}{n^{k-1}} \right) \tag{4}$$

where $a(n) := 0$ for $n \leqslant 0$. The result is thus clear for $n \leqslant 0$. Now let $n \geqslant 1$, and write $\alpha = c^+(1) + \gamma$ with $\gamma \in K$, or equivalently, $\alpha = d(1) - s_1 + \gamma$. Using (4), an immediate calculation then reveals that the coefficient of q^n in $\mathcal{F}_\alpha$ is given by

$$\frac{d(n) - d(1) - \gamma}{n^{k-1}}$$

and the result follows. □

Since one can always take $\alpha = c^+(1)$ in Theorem 4.1, the coefficients of $\mathcal{F}_{c^+(1)}$ are all in K, and so they may be viewed in $\mathbb{C}_p$ via our fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.

The following result is a special case of [GKO10, Thm. 1.2], but the ideas in the proof will allow us to recover their result in its full strength (see Theorem 4.3 below).

Theorem 4.2. *Assume that $v_p(\beta) < v_p(\beta')$ and that $v_p(\beta') \neq k-1$. Then*

$$\lim_{w \to +\infty} \frac{U^w D^{k-1}(\mathcal{F}_{c^+(1)})}{c_{c^+(1)}(p^w)} = f_\beta,$$

where we write $D^{k-1}(\mathcal{F}_{c^+(1)}) = \sum_{n \gg -\infty} c_{c^+(1)}(n) q^n$.

Proof. First note that by equation (3) and (4), we have

$$D^{k-1}(\mathcal{F}_{c^+(1)}) = \phi - d(1) f,$$

which is a weakly holomorphic cusp form of weight k with coefficients in K, defining a class in $\mathbb{H}^1_{\text{par}}(X_K, \nabla_{k-2})_f$. Now our assumptions clearly imply conditions (i) and (ii) of Theorem 3.5, and so (as shown in the proof of that result) the space $\mathbb{H}^1_{\text{par}}(X_K, \nabla_{k-2})_f$ has a basis $\{[f_\beta], [f_{\beta'}]\}$ of eigenvectors for U. In particular, we can write

$$[D^{k-1}(\mathcal{F}_{c^+(1)})] = t_1 [f_\beta] + t_2 [f_{\beta'}]$$

for some constants $t_1, t_2 \in K$. The differential $D^{k-1}(\mathcal{F}_{c^+(1)}) - t_1 f_\beta - t_2 f_{\beta'}$ defines a class in $\mathbb{H}^1(W_2 \smallsetminus C, \nabla_{k-2}) = H^0(W_2 \smallsetminus C, \underline{\omega}^k)/\theta^{k-1} H^0(W_2 \smallsetminus C, \underline{\omega}^{2-k})$. This class is exact, by construction, and thus we may write

$$D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 f_\beta + t_2 f_{\beta'} + \theta^{k-1} h \tag{5}$$

for some $h \in H^0(W_2 \smallsetminus C, \underline{\omega}^{2-k})$. Applying U to both sides of the equation gives

$$U D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 \beta f_\beta + t_2 \beta' f_{\beta'} + U(\theta^{k-1} h)$$

and more generally, for any power $w \geqslant 1$, we obtain

$$U^w D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 \beta^w f_\beta + t_2 \beta'^w f_{\beta'} + U^w(\theta^{k-1} h).$$

Dividing by β^w we get

$$\beta^{-w} U^w D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 f_\beta + t_2 \left(\frac{\beta'}{\beta} \right)^w f_{\beta'} + \beta^{-w} U^w(\theta^{k-1} h)$$

and taking the limit as $w \to +\infty$ gives

$$\lim_{w\to+\infty} \beta^{-w} U^w D^{k-1}(\mathcal{F}_{c^+(1)}) = t_1 f_\beta.$$

This is because $v_p(\beta'/\beta) > 0$ by the hypotheses and the differential $U^w(\theta^{k-1}h)$ has bounded denominators but its coefficients have arbitrarily high valuation as $w \to +\infty$.

To determine the value of the constant t_1, consider the coefficient of q^p in (5), which is given by

$$\begin{aligned} c_{c^+(1)}(p) &= t_1(a_p - \beta') + t_2(a_p - \beta) + O(p^{k-1}) \\ &= t_1\beta + t_2\beta' + O(p^{k-1}). \end{aligned}$$

By applying the multiplicative properties of the Fourier coefficients of newforms we get

$$c_{c^+(1)}(p^w) = t_1\beta^w + t_2\beta'^w + O(p^{w(k-1)})$$

and taking the limit we obtain

$$\lim_{w\to+\infty} \beta^{-w} c_{c^+(1)}(p^w) = t_1$$

which gives the result. □

Now we modify slightly the argument in Theorem 6.1 to recover [GKO10, Thm. 1.2] in its full strength. This refinement will be key for the results relating mock modular forms to p-adic modular forms in the next section.

For any α with $\alpha - c^+(1) \in K$, define

$$\mathcal{F}_\alpha := F^+ - \alpha E_f$$

and let $c_\alpha(n)$ denote the n-th coefficient in the expansion

$$D^{k-1}(\mathcal{F}_\alpha) = \sum_{n \gg -\infty} c_\alpha(n) q^n.$$

Theorem 4.3. *Assume that $v_p(\beta) \neq v_p(\beta')$ and that $v_p(\beta') \neq k-1$. Then for all but at most one choice of α with $\alpha - c^+(1) \in K$, we have*

$$\lim_{w\to+\infty} \frac{U^w D^{k-1}(\mathcal{F}_\alpha)}{c_\alpha(p^w)} = f_\beta.$$

Proof. As in the proof of Theorem 6.1, we can write

$$[D^{k-1}(\mathcal{F}_{c^+(1)})] = t_1[f_\beta] + t_2[f_{\beta'}] \in \mathbb{H}^1_{\text{par}}(X_K, \nabla_{k-2})_f \tag{6}$$

with

$$t_1 = \lim_{w\to+\infty} \frac{c_{c^+(1)}(p^w)}{\beta^w}.$$

Let $\gamma \in K$ be such that $\alpha = c^+(1) + \gamma$, so that $\mathcal{F}_\alpha = \mathcal{F}_{c^+(1)} - \gamma E_f$ by definition. Noting that

$$f = \frac{\beta f_\beta - \beta' f_{\beta'}}{\beta - \beta'},$$

and substituting into the expression corresponding to (6) for $\mathcal{F}_\alpha$ in place of $\mathcal{F}_{c^+(1)}$, we obtain

$$[D^{k-1}(\mathcal{F}_\alpha)] = \left(t_1 - \gamma\frac{\beta}{\beta-\beta'}\right)[f_\beta] + \left(t_2 + \gamma\frac{\beta'}{\beta-\beta'}\right)[f_{\beta'}],$$

and hence we have the equality

$$D^{k-1}(\mathcal{F}_\alpha) = \left(t_1 - \gamma\frac{\beta}{\beta-\beta'}\right) f_\beta + \left(t_2 + \gamma\frac{\beta'}{\beta-\beta'}\right) f_{\beta'} + \theta^{k-1}h \tag{7}$$

as sections in $H^0(W_2 \setminus C, \underline{\omega}^k)$, for some $h \in H^0(W_2 \setminus C, \underline{\omega}^{2-k})$. Applying U^w to both sides of this equation and letting $w \to +\infty$ as in the proof of Theorem 6.1, we deduce that

$$\lim_{w\to+\infty} \frac{U^w D^{k-1}(\mathcal{F}_\alpha)}{\beta^w} = \left(t_1 - \gamma\frac{\beta}{\beta-\beta'}\right) f_\beta. \tag{8}$$

On the other hand, arguing again as in Theorem 6.1 we find that the p^w-th coefficient of $D^{k-1}(\mathcal{F}_\alpha)$ is given by

$$c_\alpha(p^w) = \left(t_1 - \gamma\frac{\beta}{\beta-\beta'}\right)\beta^w + \left(t_2 + \gamma\frac{\beta'}{\beta-\beta'}\right)\beta'^w + O(p^{w(k-1)}),$$

and hence

$$\left(t_1 - \gamma\frac{\beta}{\beta-\beta'}\right) = \lim_{w\to+\infty}\frac{c_\alpha(p^w)}{\beta^w}. \tag{9}$$

Therefore, *except* in the case where

$$\gamma = \frac{t_1(\beta-\beta')}{\beta} = (\beta-\beta')\lim_{w\to+\infty}\frac{c_{c^+(1)}(p^w)}{\beta^{w+1}}, \tag{10}$$

combining (8) and (9) we recover f_β from $\mathcal{F}_\alpha$ as in the statement of the theorem. $\square$

5. Mock modular forms as p-adic modular forms

We now let α range over the larger set of values

$$c^+(1) + \mathbb{C}_p := \{c^+(1) + \gamma : \gamma \in \mathbb{C}_p\},$$

and interpret the exceptional value of α in Theorem 4.3 as the only one for which the 'corrected' mock modular form

$$\mathcal{F}_\alpha = F^+ - \alpha E_f$$

gives rise to a p-adic modular form upon p-stabilization. Recall that we let β and β' be the roots of the p-th Hecke polynomial of f, ordered so that $v_p(\beta) \leqslant v_p(\beta')$.

Definition 5.1. For any $\alpha \in c^+(1) + \mathbb{C}_p$, define

$$\mathcal{F}^*_\alpha := \mathcal{F}_\alpha - p^{1-k}\beta'\mathcal{F}_\alpha|V$$

and write

$$D^{k-1}(\mathcal{F}^*_\alpha) = \sum_{n\gg-\infty} c^*_\alpha(n)q^n.$$

Our first result shows that, similarly as in Theorem 4.3 for $\mathcal{F}_\alpha$, the p-stabilization f_β of the shadow of F^+ can be recovered p-adically from $\mathcal{F}^*_\alpha$.

Theorem 5.2. *Assume that $v_p(\beta) \neq v_p(\beta')$ and that $v_p(\beta') \neq k-1$. Then for all but at most one choice of $\alpha \in c^+(1) + \mathbb{C}_p$, we have*

$$\lim_{w \to +\infty} \frac{U^w D^{k-1}(\mathcal{F}_\alpha^*)}{c_\alpha^*(p^w)} = f_\beta.$$

Proof. The proof is quite similar to the proof of Theorem 4.3. Writing $\alpha = c^+(1) + \gamma$ with $\gamma \in \mathbb{C}_p$, an immediate calculation reveals that

$$D^{k-1}(\mathcal{F}_\alpha^*) = D^{k-1}(\mathcal{F}_{c^+(1)})|(1 - \beta' V) - \gamma f_\beta. \tag{11}$$

As in the proof of Theorem 4.2, we write

$$[D^{k-1}(\mathcal{F}_{c^+(1)})] = t_1[f_\beta] + t_2[f_{\beta'}] \in \mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f,$$

with $t_1 = \lim_{w \to +\infty} \beta^{-w} c_{c^+(1)}$. Applying the operator $1 - \beta' V$ to this last equality, and noting that $V = U^{-1}$ on cohomology, we obtain

$$[D^{k-1}(\mathcal{F}_{c^+(1)})|(1 - \beta' V)] = t_1 \frac{(\beta - \beta')}{\beta}[f_\beta],$$

and hence by (11):

$$[D^{k-1}(\mathcal{F}_\alpha^*)] = \left(\frac{t_1(\beta - \beta')}{\beta} - \gamma \right) [f_\beta]. \tag{12}$$

Arguing again as in the proof of Theorem 6.1, we obtain the equalities

$$\lim_{w \to +\infty} \frac{U^w (D^{k-1}(\mathcal{F}_\alpha^*))}{\beta^w} = \left(\frac{t_1(\beta - \beta')}{\beta} - \gamma \right) f_\beta \tag{13}$$

and

$$\frac{t_1(\beta - \beta')}{\beta} - \gamma = \lim_{w \to +\infty} \frac{c_\alpha^*(p^w)}{\beta^w}. \tag{14}$$

Therefore, *except* in the case where

$$\gamma = \frac{t_1(\beta - \beta')}{\beta} = (\beta - \beta') \lim_{w \to +\infty} \frac{c_{c^+(1)}(p^w)}{\beta^{w+1}}, \tag{15}$$

the combination of (8) and (9) recovers f_β from $\mathcal{F}_\alpha^*$ as in the statement of the theorem. □

Considering the exceptional value of α arising in the proof of Theorem 5.2, we recover the result of [BGK12, Thm. 1.1].

Theorem 5.3. *Assume that $v_p(\beta) \neq v_p(\beta')$ and that $v_p(\beta') \neq k-1$. Then among all values of $\alpha \in c^+(1) + \mathbb{C}_p$, the value*

$$\alpha = c^+(1) + (\beta - \beta') \lim_{w \to +\infty} \frac{c_{c^+(1)}(p^w)}{\beta^{w+1}}$$

is the unique one such that $\mathcal{F}_\alpha^*$ is a p-adic modular form of weight $2-k$.

Proof. Write $\alpha = c^+(1) + \gamma$ with $\gamma \in \mathbb{C}_p$. Since $[f_\beta] \neq 0$ (see the proof of Theorem 3.5), we deduce from (12) and (15) that the class of $D^{k-1}(\mathcal{F}_\alpha^*)$ in $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})$ vanishes only for the value of α given in the statement. Now, since the natural restriction map

$$\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_k) \longrightarrow \mathbb{H}^1(W_2 \smallsetminus C, \nabla_{k-2}) = \frac{H^0(W_2 \smallsetminus C, \underline{\omega}^k)}{\theta^{k-1} H^0(W_2 \smallsetminus C, \underline{\omega}^{2-k})}$$

is injective, the above value of α is also the unique one such that the class of $D^{k-1}(\mathcal{F}_\alpha^*)$ becomes trivial in $\mathbb{H}^1(W_2 \smallsetminus C, \nabla_{k-2})$, and hence such that $\mathcal{F}_\alpha^* \in H^0(W_2 \smallsetminus C, \underline{\omega}^{2-k})$. □

Next we consider a second modification of $\mathcal{F}_\alpha$.

Definition 5.4. For any $\delta \in \mathbb{C}_p$, define

$$\mathcal{F}_{\alpha,\delta} := \mathcal{F}_\alpha - \delta(E_f - \beta E_{f|V}).$$

Our next result explores the values of α and δ for which $\mathcal{F}_{\alpha,\delta}$ is a p-adic modular form, recovering the content of [BGK12, Thm 1.2(2)].

Theorem 5.5. *Assume that $v_p(\beta) \neq v_p(\beta')$ and that $v_p(\beta') \neq k-1$. Then there exists a unique pair of values (α, δ) for which $\mathcal{F}_{\alpha,\delta}$ is a p-adic modular. In fact, α is as in Theorem 5.3, and*

$$\delta = \lim_{w \to +\infty} \frac{a_{\mathcal{F}_\alpha}(p^w) p^{w(k-1)}}{\beta'^w}.$$

Here, we write $\mathcal{F}_\alpha = \sum_{n \gg -\infty} a_{\mathcal{F}_\alpha}(n) q^n$.

Proof. With the same notations as in the proof of Theorem 4.3, we can write the equality

$$[D^{k-1}(\mathcal{F}_{\alpha,\delta})] = \left(t_1 - \gamma\frac{\beta'}{\beta - \beta'}\right)[f_\beta] + \left(t_2 + \gamma\frac{\beta}{\beta-\beta'} - \delta\right)[f_{\beta'}] \tag{16}$$

in $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$. Since we may check the triviality of these classes upon restriction to $W_2 \smallsetminus C$, it follows that $\mathcal{F}_{\alpha,\delta}$ is a p-adic modular form of weight $2-k$ if and only if the class $[D^{k-1}(\mathcal{F}_{\alpha,\delta})]$ vanishes. As in the proof of Theorem 3.5, the classes $[f_\beta], [f_{\beta'}]$ form a basis for $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$, and hence $\mathcal{F}_{\alpha,\delta}$ is a p-adic modular form if and only if the coefficients in the right-hand side of (16) both vanish. In particular (second coefficient), this shows that the value of γ is given by (10), and therefore the necessary value of $\alpha = c^+(1) + \gamma$ is the same as in Theorem 5.3. To determine the value of δ, we first rewrite Equation (7) for the above value of α (so that the first summand in the right-hand side of that equation vanishes):

$$D^{k-1}(\mathcal{F}_\alpha) = \left(t_2 + \gamma\frac{\beta'}{\beta - \beta'}\right) f_{\beta'} + \theta^{k-1} h.$$

Equating the p^w-th coefficients in this equality, we obtain

$$c_\alpha(p^w) = \left(t_2 + \gamma\frac{\beta'}{\beta-\beta'}\right)\beta'^w + O(p^{w(k-1)})$$

and hence dividing by β'^w and letting $w \to +\infty$ we deduce

$$\lim_{w\to+\infty} \frac{c_\alpha(p^w)}{\beta'^w} = \left(t_2 + \gamma\frac{\beta'}{\beta-\beta'}\right). \tag{17}$$

(Note that the assumption $v_p(\beta') < k-1$ is being used here.) Finally, substituting (17) into (16), we see that the necessary value for δ is given by

$$\delta = \lim_{w\to\infty} \frac{c_\alpha(p^w)}{\beta'^w} = \lim_{w\to\infty} \frac{a_{\mathcal{F}_\alpha}(p^w)p^{w(k-1)}}{\beta'^w},$$

as was to be shown. □

6. The CM case

In this section we treat the case in which f has CM. This case is of special interest, since then one can choose a good harmonic Mass form F for f as in Section 2 with F^+ having algebraic coefficients. Conjecturally, this characterize the CM property of f (see [GKO10, p.6170]). Thus assume that $f = \sum_{n=1}^\infty a_n q^n \in S_k(\Gamma_1(N), K)$ has CM by an imaginary quadratic field M of discriminant prime to p, and let $F = F^+ + F^-$ be a good harmonic Maass form attached to f. We also assume (upon enlarging K, if necessary) that K contains a primitive m-th root of unity, where $m = N \cdot \mathrm{disc}(M)$. Then by [BOR08, Thm. 1.3], F^+ has coefficients in K, and so $D^{k-1}(F^+)$ defines a class in $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$.

We first treat the case in which p is inert in M. In this case $a_p = \beta + \beta' = 0$, and so by the proof of Theorem 3.5, the space $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$ admits a basis given by the classes $[f_\beta]$ and $[f_{\beta'}]$.

Lemma 6.1. *Assume that p is inert in M, and write $[D^{k-1}(F^+)] = t_1[f_\beta] + t_2[f_{\beta'}]$. Then*

$$\lim_{w\to+\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w+1}} = t_1 - t_2.$$

Proof. The proof will be obtained by arguments similar to the proof of Theorem 4.2, but some adjustments are necessary due to the fact the condition $v_p(\beta) \neq v_p(\beta')$ is not satisfied in this case. Instead, we shall exploit the extra symmetry $\beta' = -\beta$.

Upon restriction to $W_2 \smallsetminus C$, we can write

$$D^{k-1}(F^+) = t_1 f_\beta + t_2 f_{\beta'} + \theta^{k-1} h \tag{18}$$

for some $h \in H^0(W_2 \smallsetminus C, \underline{\omega}^{2-k})$. Taking p^{2w+1}-st coefficients in this identity, we immediately obtain

$$\begin{aligned} a_{D^{k-1}(F^+)}(p^{2w+1}) &= t_1\beta^{2w+1} + t_2\beta'^{2w+1} + O(p^{(2w+1)(k-1)}) \\ &= (t_1 - t_2)\beta^{2w+1} + O(p^{(2w+1)(k-1)}), \end{aligned}$$

and hence dividing by β^{2w+1} and letting $w \to +\infty$ the result follows. □

Definition 6.2. For any $\alpha \in \mathbb{C}_p$, define

$$\widetilde{\mathcal{F}}_\alpha := F^+ - \alpha E_{f|V}.$$

Armed with Lemma 6.1, in Corollary 6.4 below we will determine the values of α for which $\widetilde{\mathcal{F}}_\alpha$ is a p-adic modular form, thus recovering [BGK12, Thm. 1.3]. This will be an immediate consequence of the following result.

Theorem 6.3. *Assume that $p \nmid N$ is inert in M, and for any $\widetilde{\alpha} \in \mathbb{C}_p$ define*

$$G_{\widetilde{\alpha}} := F^+ - \widetilde{\alpha}(E_f - \beta E_{f|V}).$$

Then there exists a unique value of $\widetilde{\alpha}$ such that $G_{\widetilde{\alpha}}$ is a p-adic modular form of weight $2-k$, and it is given by

$$\widetilde{\alpha} = \lim_{w \to +\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w+1}}.$$

Proof. We will deduce this result by first determining the values of α and δ for which the form $\mathcal{F}_{\alpha,\delta}$ of Definition 5.4 is a p-adic modular form. Note that this case is not covered by Theorem 5.5, since the proof of that result relies crucially on the assumption that $v_p(\beta) < v_p(\beta')$. Instead, we will exploit again the fact that $\beta' = -\beta$. Since $[f_\beta]$ and $[f_{\beta'}]$ form a basis for $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$, Equation (16) for $[D^{k-1}(\mathcal{F}_{\alpha,\delta})]$ still applies, and in this case it reduces (noting that we may set $\gamma = \alpha$ by the algebraicity of $c^+(1)$) to

$$[D^{k-1}(\mathcal{F}_{\alpha,\delta})] = \left(t_1 - \frac{\alpha}{2}\right)[f_\beta] + \left(t_2 - \frac{\alpha}{2} - \delta\right)[f_{\beta'}]. \tag{19}$$

By Theorem 3.5, the classes $[f]$ and $[V(f)]$ form a basis for the space $\mathbb{H}^1_{\mathrm{par}}(X_K, \nabla_{k-2})_f$, and rewriting (19) in terms of them we arrive at

$$[D^{k-1}(\mathcal{F}_{\alpha,\delta})] = (t_1 + t_2 - \alpha - \delta)[f] + \beta(t_1 - t_2 - \alpha - \delta)[V(f)]. \tag{20}$$

Now, $\mathcal{F}_{\alpha,\delta}$ is a p-adic modular form of weight $2-k$ if and only if both coefficients in the right-hand side of Equation (20) vanish; in particular, we need to have

$$\alpha + \delta = t_1 - t_2 = \lim_{w \to +\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w+1}}, \tag{21}$$

where we used Lemma 6.1 for the second equality. The necessary vanishing of (20) also forces the vanishing of t_2 and hence from (19) we deduce that $\delta = -\frac{\alpha}{2}$, or equivalently, $\alpha + \delta = \frac{\alpha}{2}$. Finally, noting that

$$\mathcal{F}_{\alpha,\delta} = F^+ - \frac{\alpha}{2}\left(E_f - \beta E_{f|V}\right) = G_{\frac{\alpha}{2}},$$

we conclude from (21) that $G_{\widetilde{\alpha}}$ is a p-adic modular form if and only if $\widetilde{\alpha}$ is given by the p-adic limit in the statement. ∎

Corollary 6.4. *Assume that $p \nmid N$ is inert in M. Then there exists a unique value of α such that $\widetilde{\mathcal{F}}_\alpha$ is a p-adic modular form of weight $2-k$, and it is given by*

$$\alpha = \lim_{w \to +\infty} \frac{a_{D^{k-1}(F^+)}(p^{2w+1})}{\beta^{2w}}.$$

Proof. Comparing the definitions of $\widetilde{\mathcal{F}}_\alpha$ and $G_{\widetilde{\alpha}}$, we see that

$$G_{\widetilde{\alpha}} = \widetilde{\mathcal{F}}_\alpha - \widetilde{\alpha} E_f,$$

with $\alpha = \widetilde{\alpha}\beta$. Since E_f is easily seen to be a p-adic modular form under our hypotheses (see [BGK12, Prop. 4.2], which remains true in our case $p \nmid N$), the result follows from Theorem 6.3. ∎

We conclude this section by dealing with the case in which f has CM by an imaginary quadratic field M in which p splits, characterizing the values of $\alpha \in \mathbb{C}_p$ for which $\mathcal{F}_\alpha^*$ is a p-adic modular form. As noted in Remark 3.6, the class $[f_{\beta'}]$ vanishes in this case, and so the proofs of Theorem 5.2 and Theorem 5.3 break down. However, based on the observation that (using the algebraicity of $c^+(1)$ to set $\alpha = \gamma$)

$$\mathcal{F}_\alpha^* = (F^+ - \alpha E_f)|(1 - p^{1-k}\beta' V) = \mathcal{F}_0^* - \alpha E_{f_\beta}, \tag{22}$$

we can easily prove the following result (cf. [BGK12, Thm. 1.2]).

Theorem 6.5. *Assume that $p \nmid N$ splits in K. Then among all values of $\alpha \in \mathbb{C}_p$, the value $\alpha = 0$ is the unique one for which $\mathcal{F}_\alpha^*$ is a p-adic modular form of weight $2 - k$.*

Proof. As we have already argued in preceding proofs, $\mathcal{F}_\alpha^*$ is a p-adic modular form of weight $2 - k$ if and only if the class $[D^{k-1}(\mathcal{F}_\alpha^*)]$ vanishes, and from (22) we see that

$$[D^{k-1}(\mathcal{F}_\alpha^*)] = 0 \quad \Longleftrightarrow \quad \alpha[f_\beta] = [D^{k-1}(\mathcal{F}_0^*)]. \tag{23}$$

In particular, this shows that $\mathcal{F}_\alpha^*$ is a p-adic modular form of weight $2-k$ for $\alpha = 0$, and so $[D^{k-1}(\mathcal{F}_0^*)] = 0$. On the other hand, since $[f_\beta] \neq 0$ (see the proof of Theorem 3.5), equivalence (23) shows that $[D^{k-1}(\mathcal{F}_\alpha^*)] \neq 0$ for $\alpha \neq 0$, yielding the result. □

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