Math 140     Lecture 13
Recall: Since $\log_b x$ and $b^x$ are inverses of each other:
- $\log_b x = y$ iff $x = b^y$
- $\log_b b^x = x$
- $b^{\log_b x} = x$

$log_0 1 = 0, \ln(1) = 0 \quad b^0 = 1$

$log_b 1 = 1, \ln e = 1 \quad b^1 = b$

Combine into a single logarithm.

$log_b \sqrt[3]{y^3 + a} = \log_b \left( \frac{2y}{y^3 + a} \right) = \frac{1}{4} \log_b \left( \frac{2y}{y^3 + a} \right) = \frac{1}{4} \left[ \log_b (2y) - \log_b (y^3 + a) \right]$

$log \frac{1}{\sqrt{x^2 + y^2}} = \ln (x^2 + y^2)^{-1/2} = -\frac{1}{2} \ln (x^2 + y^2)$

Note: $\log_b (x \cdot y) = \log_b x + \log_b y \neq \log_b (x + y)$. The last term cannot be broken into simpler pieces.

Solving for $x$.
- Get terms involving $x$ on the left, the rest on the right.
- Combine into a single logarithm.
- Exponentiate both sides to the base of the logarithm.
- Solve.
- Delete solutions that give undefined logarithms (logarithms with arguments $\leq 0$).

$log_2 4x - log_2 3 = log_2 (x + 2)$

$log_2 4x - log_2 (x + 2) = log_2 3$

$log_2 \frac{4x}{x + 2} = log_2 3$

$2^{\log_2 \frac{4x}{x + 2}} = 2^{\log_2 3}$

$\frac{4x}{x + 2} = 3, \quad 4x = 3x + 6, \quad x = 6$

Validity: for $x = 6$, both $\log_2 4x = \log_2 24$ and $\log_2 (x + 2) = \log_2 8$ are defined, thus the solution is valid.

- $\ln (x + 1) = 2 - \ln (x - 1)$
- $\ln [(x + 1)(x - 1)] = 2$
- $\ln (x^2 - 1) = 2$
- $e^{\ln (x^2 - 1)} = e^2$
- $x^2 - 1 = e^2$
- $x^2 = 1 + e^2$
- $x = \pm \sqrt{1 + e^2}$

Validity: $x = -\sqrt{1 + e^2}$ is not valid since $\ln (x - 1) = \ln (-\sqrt{1 + e^2} - 1) = \text{undefined}$

But $\ln (\sqrt{1 + e^2} - 1)$ is defined since $\sqrt{1 + e^2} - 1 > 0$.

Answer: The only valid solution is $x = \sqrt{1 + e^2}$.

Change of base formula. $\log_a x = \frac{\log_b x}{\log_b a}$

Proof: $\log_b x = \frac{\log_b (a^{\log_a x})}{\log_b a} = \frac{\log_a (a) \log_b a}{\log_b a} = \log_a (x)$.

Express in terms of logs: $\ln 2 = \log_{\log_2 e}\frac{\log_2 2}{\log_2 e}$

Express in terms of natural logarithms $\ln$: $\log_{\ln 5} t = \frac{\ln t}{\ln 5}$

Most properties of logarithms are the same as for exponentiation but with

<table>
<thead>
<tr>
<th>the symbols</th>
<th>+</th>
<th>-</th>
<th>$n(_)$</th>
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<tbody>
<tr>
<td>exchanged with</td>
<td>$\times$</td>
<td>$\div$</td>
<td>$(_)^n$</td>
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Assume $b > 0, b \neq 1$. On the log side, assume $x, y > 0$.

**LOG PROPERTIES**

**EXPONENT PROPERTIES**

$log_b xy = log_b x + log_b y \quad b^{xy} = b^x b^y$

$log_b x/y = log_b x - log_b y \quad b^{x/y} = b^x / b^y$

$log_b x^n = n log_b x \quad b^{x^n} = (b^x)^n$

Proofs. (In these proofs, the “iff”s are required, -1 if missing.)

$log_b xy = log_b x + log_b y$ if $b^{log_b x} = b^{log_b x + log_b y} \iff xy = b^{log_b x} b^{log_b y}$ \iff $xy = xy$ if true.

Likewise for $log_b \frac{x}{y} = log_b x - log_b y$.

$log_b x^n = n log_b x$ if $b^{log_b x^n} = b^{n log_b x}$ \iff $x^n = b^{n log_b x}$ \iff $x^n = [b^{log_b x}]^n$ \iff $x^n = x^n$ if true.

Combine into a single logarithm.

- $2 log_{10} x + log_{10} y$
  \quad = log_{10} x^2 + log_{10} y = log_{10} x^2 y$

- $log_2 x - 4 log_2 y$
  \quad = log_2 x - log_2 2^4 = log_2 (x / 16)$

- $ln(x^2 + y^2) - ln(y^3 + a)$ = $ln (x^2 + y^2) / (y^3 + a)$