**RECALL**

\[
\cos(x - y) = \cos x \cos y + \sin x \sin y \\
\cos(x + y) = \cos x \cos y - \sin x \sin y
\]

Adding these gives

\[
\cos(x - y) + \cos(x + y) = 2 \cos x \cos y
\]

Subtracting gives

\[
\cos(x - y) - \cos(x + y) = 2 \sin x \sin y
\]

Similarly

\[
\sin(x - y) = \sin x \cos y - \cos x \sin y \\
\sin(x + y) = \sin x \cos y + \cos x \sin y
\]

Adding these gives

\[
\sin(x - y) + \sin(x + y) = 2 \sin x \cos y
\]

Dividing by 2 gives

**PRODUCT-TO-SUM FORMULAS**

\[
\sin x \sin y = \frac{1}{2} [\cos(x - y) - \cos(x + y)] \\
\cos x \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)] \\
\sin x \cos y = \frac{1}{2} [\sin(x - y) + \sin(x + y)] = \cos y \sin x
\]
\[
\sin x \sin y = \frac{1}{2} [\cos(x - y) - \cos(x + y)] = \sin y \sin x
\]
\[
\cos x \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)] = \cos y \cos x
\]
\[
\sin x \cos y = \frac{1}{2} [\sin(x - y) + \sin(x + y)] = \cos y \sin x
\]

In the last formula,
\(x\) is the argument of \(\sin\),
\(y\) is the argument of \(\cos\).

- \(\cos a \sin b = ?\)  Careful.

- \(\sin b \sin a = ?\)
\[
\sin x \sin y = \frac{1}{2} [\cos(x - y) - \cos(x + y)] \\
\cos x \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)] \\
\sin x \cos y = \frac{1}{2} [\sin(x - y) + \sin(x + y)] = \cos y \sin x
\]

Write as a sum or difference of trigonometric functions.

- \(\cos \frac{\pi}{6} \sin \frac{\pi}{2}\)
- \(\sin \frac{\pi}{2} \cos \frac{\pi}{4}\)
\[
\sin x \sin y = \frac{1}{2} [\cos(x - y) - \cos(x + y)] \\
\cos x \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)] \\
\sin x \cos y = \frac{1}{2} [\sin(x - y) + \sin(x + y)] = \cos y \sin x
\]

Write as a sum or difference of trigonometric functions.

- \( \cos x \cos 2x \)
- \( \sin(x + y) \sin(x - y) \)
Solving trigonometric equations
\[
\cos x = \cos(-x), \quad \text{if } x \text{ is one solution, } -x \text{ is another solution.}
\]
\[
\sin x = \sin(\pi - x) \quad \text{if } x \text{ is one solution, } \pi - x \text{ is another.}
\]
\[
\sin \theta = \frac{1}{2}. \quad \text{Find all solutions.}
\]

Two simplest solutions:
\[
\theta = \frac{\pi}{6}, \quad \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.
\]
\[
\sin(\theta) \text{ has period } 2\pi, \quad \therefore \text{ adding multiples of } 2\pi \text{ to a solution is also a solution.}
\]
General solution: two sets
\[
\theta = \frac{\pi}{6} + 2\pi n, \quad \theta = \frac{5\pi}{6} + 2\pi n
\]

\[
\cos \theta = \frac{1}{2}. \quad \text{Find all solutions.}
\]

To solve \(\sin(\theta) = \frac{3}{4}\), the simplest solution on the right is \(\theta = \sin^{-1}\left(\frac{3}{4}\right)\), the solution on the left is \(\theta = \pi - \sin^{-1}\left(\frac{3}{4}\right)\).

To solve \(\cos(\theta) = \frac{3}{4}\), the simplest solution above the \(x\)-axis is \(\theta = \cos^{-1}\left(\frac{3}{4}\right)\), the solution below \(\theta = -\cos^{-1}\left(\frac{3}{4}\right)\).
 Convention. $n$ is an arbitrary, possibly negative, integer.
For $\sin$, $\cos$, add $2\pi n$ to the (usually two) simplest solutions.
For $\tan$, add $\pi n$ to the one simplest solution.

Find all solutions.

- $\tan x = \frac{1}{\sqrt{3}} \quad \ldots \pi/6 + \pi n$
- $\tan x = 1$
For $\sin$, $\cos$, add $2\pi n$ to the (usually two) simplest solutions.
For $\tan$, add $\pi n$ to the one simplest solution.

Find all solutions.

- $\cos x = 3$
- $\sin 2x = 1 \quad \cdots \pi/4 + \pi n$
- $\cos 2x = 1$
For \( \sin, \cos \), add \( 2\pi n \) to the (usually two) simplest solutions

For \( \tan \), add \( \pi n \) to the one simplest solution.

\[
\cos \theta = \cos(-\theta), \quad \text{if } \theta \text{ is one solution, } -\theta \text{ is another solution.}
\]

\[
\sin \theta = \sin(\pi - \theta) \quad \text{if } \theta \text{ is one solution, } \pi - \theta \text{ is another.}
\]

\[
\sin(\pi/6) = 1/2
\]
\[
\sin(\pi/4) = 1/\sqrt{2}
\]
\[
\sin(\pi/3) = \sqrt{3}/2
\]
\[
\cos(\pi/6) = \sqrt{3}/2
\]
\[
\cos(\pi/4) = 1/\sqrt{2}
\]
\[
\cos(\pi/3) = 1/2
\]
\[
\tan(\pi/6) = 1/\sqrt{3}
\]
\[
\tan(\pi/4) = 1
\]
\[
\tan(\pi/3) = \sqrt{3}
\]

Remember these values.
Find all solutions. Three sets.

- \(2 \cos^2 x + \cos x = 1\)
- \(2 \sin^2 x + \sin x = 1\)
Find all solutions.  Recall: $\sin^2 x + \cos^2 x = 1$

- $2 \cos^2 x - \sin x - 1 = 0$

  Rewrite entirely in sin or entirely in cos.
Find all solutions. Recall: $\sin^2 x + \cos^2 x = 1$

- $\cos^2 x + \sin x + 1 = 0$
- $\sin^2 x + \cos x + 1 = 0$
\[2 \tan^2 x - 3 \tan x \sec x - 2 \sec^2 x = 0\]
sin, cos and tan are not 1-1. The $x$-axis is a horizontal line which crosses their graphs more than once.

They are 1-1 when restricted to the green intervals.

For $\sin$, the restricted interval is $[-\pi/2, \pi/2]$.
For $\cos$, the restricted interval is $[0, \pi]$.
For $\tan$, the restricted interval is $(-\pi/2, \pi/2)$. 
Inverse trigonometric functions

\[ \sin^{-1}(x) = \text{the } \theta \in [-\pi/2, \pi/2] \text{ such that } \sin(\theta) = x, \]
\[ \cos^{-1}(x) = \text{the } \theta \in [0, \pi] \text{ such that } \cos(\theta) = x, \]
\[ \tan^{-1}(x) = \text{the } \theta \in (-\pi/2, \pi/2) \text{ such that } \tan(\theta) = x. \]

\( \sin^{-1}(x) \) and \( \cos^{-1}(x) \) have domain \([-1, 1]\).

\( \tan^{-1}(x) \) has domain \((-\infty, \infty)\).

**Notation**

\[
\begin{align*}
arcsin(x) &= \sin^{-1}(x) \\
arccos(x) &= \cos^{-1}(x) \\
arctan(x) &= \tan^{-1}(x)
\end{align*}
\]

**Warning.** \( \sin^{-1}(x) \neq 1/\sin(x) \).

\[
\begin{align*}
\sin^{-1}(x) &= \text{arcsin } x = \text{the inverse} \\
(\sin(x))^{-1} &= \frac{1}{\sin x} = \text{the reciprocal}
\end{align*}
\]