Myers’ Math 140 & Math 135 Notes

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COMMON ERRORS
Study these before doing the first day’s Take-home quiz.

SAMPLE GATEWAY EXAMS (5)
These are practice exams for the Gateway Exams. To get a “C” or better, you must pass some Gateway exam with a score of at least 28 (7 problems out of 8).

PRACTICE EXAMS AND FORMULA REVIEWS 1-5
Before each exam, go over the corresponding practice exam thoroughly. You must be able to complete it within 50 minutes. Each exam problem has a worked example on a practice quiz sheet. Also take the Online exams.

LECTURES 1-30
These are almost verbatim copies of the lectures I will give. Lecture videos are on the web and on an optional $1.00 CD for Windows PCs.

HOMEWORK AND IN-CLASS WORKSHEETS 1-27
Fill in these worksheets and hand them in. Do not turn in scratch paper.

RECOMMENDED PROBLEMS AND WORKED EXAMPLES
These recommended problems are for practice only.

Print the syllabus from the class website: www.math.hawaii.edu/140
Math 140 Common Errors
Compiled by Myers and Andrew Conner

Exact answers. On exams and homework assignments such as Hw 12, 14, 18 you are asked to give exact answers rather than decimal answers. This means writing π and not its nonexact decimal approximation 3.14. This means writing \( \sqrt{2} \) instead of a finite decimal approximation given by a calculator. The exact answer for the area of a circle with radius 2 is \( \pi \cdot 2^2 = 4\pi \), the decimal approximation is \( 4(3.14) = 12.56 \). The exact answer for the solution of \( x^2 - 5 = 0 \) is \( x = \pm \sqrt{5} \). The decimal approximation to two places is \( \pm 2.24 \).

Take-home quiz. Note that \( \sqrt{2} = \sqrt{1^2 + 1^2} = \sqrt{1 + 1} = 2 \). In general \( \sqrt{a^2 + b^2} \) and \( \sqrt{a^2 - b^2} \) are not \( a + b \) and \( a - b \) and can not be simplified further. However \( \sqrt{a^2b^2} = ab \) is true for positive \( a, b \).

In a fraction, if you can factor out common factor from top and bottom, you can simplify. Otherwise you can’t. Hence \( \frac{2a-2b}{2b} = \frac{2(a-b)}{2b} = \frac{a-b}{b} \). However no cancellation is possible for \( \frac{2a-b}{2a} \), \( \frac{2a-b}{2} \).

Note that \( \frac{123^3}{83} = \frac{(4-3)^3}{4\cdot3^3} = \frac{3^3}{4\cdot3^3} = \frac{3}{4\cdot3} = \frac{3}{12} = \frac{1}{4} \). Secondly \( \frac{a^5-a^3}{a^3} = \frac{a^5-1}{a^3} = a^2 - 1 \).

Finally, \( \frac{2737-26}{26} = \frac{26(2737-1)}{26} = 2 \cdot 37 - 1 \).

Write with fewer symbols: \( -\frac{a-b}{2b-2a} = \frac{b-a}{2(b-a)} = \frac{1}{2} \).

\( f(x) = \sqrt{x+1} \), find \( f(f(x)) \) and \( f(x+y) \) \( f(f(x)) = \sqrt{\sqrt{x+1} + 1} = \sqrt{x+1} + 1 \). and \( f(x+y) = \sqrt{(x+y) + 1} = \sqrt{x+y+1} \).

Gateway problems. All Gateway exams have a problem which asks you to complete the square for a formula. The most common mistake is to try to make the formula into an equation and then complete the square for the equation. E.g., when asked to complete the square of \( x^2 - x + 1 \) many try to make this an equation \( x^2 - x + 1 = 0 \). Formulas and equations are two different things.

The most common error for factorization problems is to list the roots rather than give the factorization. For the problem “Factor \( x^2 - 4 \)”, the correct answer is \( “x^2 - 4 = (x-2)(x+2)” \) not \( “x = -2, x = 2” \).

Hw 0, Prob. 3, 4. Rewrite without \( |l|’s: |x+1| + 4|x| + 3l \) for \( x < -3 \).
\( |x+1| \) is either \( (x+1) \) or \( -(x+1) \).
To determine which, determine if \( x + 1 \) is \( \geq 0 \) or \( < 0 \).
If \( x + 1 \geq 0 \), then \( |x+1| = (x+1) \).
If \( x + 1 < 0 \), then \( |x+1| = -(x+1) \).

Hw 0, Prob. 6, 7. Write as an interval: \( \{x: |x-4| < 4\} \). Write as a union of two intervals: \( \{x: |x+5| \geq 2\} \).
\( A < \) inequality such as \( |x+6| < 2 \) is equivalent to \( -2 < x + 6 < 2 \) which in turn reduces to (subtract 6) \( -8 < x < -4 \) which gives one interval \( (-8, -4) \).
\( -8 < x < -4 \) can also be written \( -8 < x \) and \( x < -4 \).
\( A > \) inequality such as \( |x+6| > 2 \) is equivalent to \( x + 6 < -2 \) or \( 2 < x + 6 \) which is reduces to (subtract 6) \( x < -8 \) or \( -4 < x \) which gives two intervals \( (-\infty, -8) \cup (-4, \infty) \).
\( A > or \geq \) inequality involves an “or” instead of an “and” and two intervals instead of one.

Hw 1, Prob. 2. Solve \( \sqrt{3x+1} = 4 \)
When solving equations with radicals, get the radical by itself on one side and everything else on the other.
After solving the equations one must substitute the answers back into the original equation to determine if they are valid.
Hw 1, Prob 4. $\frac{x+4}{2x-5} \leq 0$.

The most common mistake is to multiply both sides of $\frac{x+4}{2x-5} \leq 0$ by $2x-5$, to get $x+4 \leq 0$. But things aren’t this simple. Since we don’t know whether $2x-5$ is positive or negative, we don’t know whether the new inequality is $x+4 \leq 0$ or the reverse $x+4 \geq 0$. Don’t multiply an inequality by a term such as $2x-5$ unless you know if it is positive or negative. Use the key number method given in Lecture 1 and see the worked examples for Homework 1.

Similarly, you can’t simplify $x - \frac{3}{x} \leq 2$ by multiplying both sides by $x$. First get the $2$ on the left, $x - \frac{3}{x} - 2 \leq 0$, then put the left side over a common denominator, $\frac{x^2 - 2x - 3}{x} \leq 0$, then factor. $\frac{(x-3)(x+1)}{x} \leq 0$, then find the key numbers where the fraction is 0 (where the numerator is 0) and where the fraction is undefined (where the denominator is 0).

Hw 2, Prob 2. $x^2 + y^2 - 10x + 2y + 17 = 0$.

To solve $x^2 + y^2 - 4x - 6y - 12 = 0$,

first group the $x$’s together and the $y$’s together and get the constant on the right.

$(x^2 - 4x) + (y^2 - 6y) = 12$

Now, complete the square by adding the square of one half the coefficient of $x$ or $y$.

Whatever is added to the left must also be added to the right side of the equation.

$(x^2 - 4x + (\frac{4}{2})^2) + (y^2 - 6y + (\frac{6}{2})^2) = 12 + (\frac{4}{2})^2 + (\frac{6}{2})^2$

$(x^2 - 4x + 4) + (y - 6y + 9) = 12 + 4 + 9$

$(x - 2)^2 + (y - 3)^2 = 25$

Now write the right side as a square.

$(x - 2)^2 + (y - 3)^2 = 5^2$.

The center is $(a,b) = (2,3)$ and the radius is $r = 5$.

Hw 2, Prob 9. Find the slope of the line through $(-3,0)$ and $(4,9)$.

When finding the slope of the line through $(x_1,y_1)$ and $(x_2,y_2)$, the most common error was invert the slope formula, i.e. use $m = \frac{x_1-x_2}{y_1-y_2}$ instead of the correct $m = \frac{y_1-y_2}{x_1-x_2}$ or equally correct $m = \frac{y_2-y_1}{x_2-x_1}$. Make sure that the order of the coordinates on top matches the order of the coordinates on the bottom, e.g. $m = \frac{y_1-y_2}{x_2-x_1}$ would be wrong.

Hw 3, Prob 3, 4.

Factor and find all roots $x^3 + 7x^2 + 11x + 5 = 0$. $-1$ is one root.

Likewise for $x^3 + 8x^2 - 3x - 24 = 0$. $-8$ is one root.

First error: not giving both the roots and the factorization.

Your answer should be of the form:

Factorization: $(x+3)(x+2)(x-1)$

Roots: $x = -3,-2,1$.

Second error: not knowing how to get started.

Notice the theorem of Lecture 3: $a$ is a root iff $(x-a)$ is a factor.

Secondly, if $(x-a)$ is a factor, then you get the other factor using long division.

Hence if 11 is a root, then $(x-11)$ is one factor. Divide the polynomial by $(x-11)$. The resulting quotient is the other factor. This quotient might have to be factored even further.

Hw 3, Prob 6. When calculating a domain, you must exclude all points which makes the denominator of a fraction 0.

Hw 3, Prob 11. Note that $f(g(x))$ is not multiplication. $f(x)g(x)$ is multiplication. $f(g(x))$ means replace each $x$ in $f(x)$ by $g(x)$. Hence if $f(x) = 2x + 3$, then $f(t) = 2t + 3$, $f(8) = 2(8) + 3$, $f(\frac{1}{2}) = 2\frac{1}{2} + 3$, $\frac{1}{f(x)} = \frac{1}{2x+3}$, $f(x^2) = 2x^2 + 3$,

$f(y+1) = 2(y+1) + 3$, $f(x+h) = 2(x+h) + 3$, $f(f(x)) = 2f(x) + 3 = 2(2x + 3) + 3$.

If $g(x) = x^2$, then $g(x+h) = (x+h)^2$, in particular, $g(x+h) \neq x^2 + h^2$. 


Hw 4, Quiz 4. When listing intervals of increase or decrease, separate the intervals with a comma rather than taking the union as is done when listing a function’s domain. Write \([0, 1) \cup (2, \infty)\) when giving a domain but write \([0, 1), (2, \infty)\) when listing two intervals of increase. Secondly, when listing an interval of increase or decrease, include the endpoint (i.e., use \([\) if the function is defined at the endpoint; exclude the endpoint (i.e., use (,)) if the function is undefined. Hence if \([0, 1), (2, \infty)\) are two intervals of increase for a function, then the function is defined at 0 but undefined at 1 and 2.

A common error is to wrongly assume that for any function \(f\), \(f(a + b) = f(a) + f(b)\). For multiplication by a number \(c\) it is true that \(c(a + b) = ca + cb\). But for functions, it is usually the case that \(f(a + b) \neq f(a) + f(b)\). Almost always, we have: \(\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}\), \(|a + b| \neq |a| + |b|\), \((a + b)^n \neq a^n + b^n\), \(x^{a+b} \neq x^a + x^b\), \(\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}\), \(\log(a + b) \neq \log a + \log b\), \(\sin(a + b) \neq \sin a + \sin b\), \(\cos(a + b) \neq \cos a + \cos b\), \(\tan(a + b) \neq \tan a + \tan b\).

Likewise, it is usually the case that \(\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}\) and \(|a - b| \neq |a| - |b|\), ... .

Hw 5, Prob 2(k). Do problem 2(k) on page 137 carefully; it is often missed. You will see variants in quizzes and exams.

Quiz 5, Prob 9, 14b. Problems 9 and 14(b) were often missed. Getting \(g(x + h)\) was the most common error in Problem 9.

If \(g(x) = x^2\), then \(g(x + h) = (x + h)^2\), in particular, \(g(x + h) \neq x^2 + h^2\) (see Hw 3, Prob 34).

If \(g(x) = 2 - \frac{3}{x}\), then the correct answer is \(g(x + h) = 2 - \frac{3}{x + h}\). Note that \(g(x + h) \neq 2 + \frac{3}{x} + h\), \(g(x + h) \neq 2 + \frac{3}{x} + \frac{2}{h}\), and \(g(x + h) \neq (2 + \frac{3}{x}) + (2 + \frac{3}{h})\).

Quiz 5, Prob 14b. The most common mistake was to solve the problem with the following incorrect sequence of steps
\[
\begin{align*}
\text{If } f(x) &= f(\sqrt{-x}) \\
\text{Then } f(-x) &= f(-x + 1) \\
\text{and } f(-x + 1) - 1 &= f(\sqrt{x}) - 1 \\
\text{as } f(-x + 1) &= f(-x - 1) \\
\text{The correct sequence is obtained by working forward from } f(-x + 1) \text{ to } f(x) \text{ and then to } f(x). \\
\text{This results in the sequence } f(x) \to f(\sqrt{x}) \to f(-x + 1). \\
\text{Note that this last step involves replacing } x \text{ by } x + 1 \text{ which shifts the graph right one unit.}
\end{align*}
\]

Practice Exam 1, Prob 14b. As with the preceding quiz problem, the majority of students can’t graph \(f(1-x) - 1\). They fail to rewrite the problem as \(f(-x + 1) - 1\) and fail to find the correct sequence which is
\[
\begin{align*}
\text{If } f(x) &= f(x + 1) \\
\text{Then } f(-x) &= f(-x + 1) \\
\text{and } f(-x + 1) &= f(-x + 1) - 1. \\
\text{Start with the value } f(-x + 1) \text{ work backward one step at a time to } f(x) \text{ and work forward to the desired value } f(-x + 1) - 1. (\text{see Quiz 5, Prob. 14b}).
\end{align*}
\]

Practice Exam 1, Prob. 12. Graphing piecewise defined functions is easy but many miss this problem.

Practice Exam 1, Prob. 16. Most get this problem but fail similar problems such as (1) if \(f(x) = 1 - x^2\), then find \(f(f(x))\) or (2) if \(g(x) = 1 - x^2\) then find \(\frac{g(x+h)-g(x)}{h}\) (see quiz 5 prob. 9, 14b).

Quiz 10. Many were able to find the vertical and horizontal asymptotes and x and y-intercepts but yet could not draw the graph. The best remedy is to plot more points until you see the shape of the graph.

Quiz 11, Prob. 9. Most mistakes here were due to confusion about the rules of exponent arithmetic — when you should add, subtract or multiply exponents. Here practice makes perfect, do the recommended exercises to develop competence with exponents.

Practice Exam 2, Prob. 1(b). To solve \(y^2 - 2y = x\) for \(y\), regard \(x\) as a constant and use the quadratic formula. Getting everything on the left gives \(y^2 - 2y - x = 0\). Note that this is \(ay^2 + by + c = 0\) where \(a = 1, b = -2, c = -x\). The quadratic formula then gives
\[
y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 + 4x}}{2} = 1 + \sqrt{1 + x}.
\]
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Practice Exam 2, Prob. 4. Similar problems on Exam 2 often require knowing formulas for volume and surface area. Know all the facts on the inside of the front cover of the text except Heron’s formula and the circular cone formulas.

Practice Exam 2, Prob. 5. You need to know that for similar triangles, ratios of corresponding sides are equal. In the answer’s picture the sides “6” and “s” of the small triangle correspond to the sides “10” and “x+s” of the larger triangle which is similar to the smaller one.

Hw 16. While you can calculate the trigonometric functions for the following angles using reference angles, they occur often enough that it is best be able to picture them graphically on the unit circle. Know the x and y coordinates of each point.

Here are the coordinates for the unit circle points for the first-quadrant angles: $P(0) = (1, 0), P(\pi/6) = (\sqrt{3}/2, 1/2), P(\pi/4) = (1/\sqrt{2}, 1/\sqrt{2}), P(\pi/3) = (1/2, \sqrt{3}/2)$. The x coordinate is the cosine and the y coordinate is the sine of the angle. The coordinates and hence sines and cosines of the other angles are the same as their first quadrant reference angles except for a change of sign. Points below the x-axis have a negative y coordinate. Points left of the y-axis have a negative x coordinate.

Hw 16. Don’t continue writing sin or cos after calculating sin or cos. $\sin(-\pi/4) = -\sin(\pi/4) = 1/\sqrt{2}$ is correct but $\sin(-\pi/4) = -\sin(\pi/4) = \sin(1/\sqrt{2})$ is wrong.

Hw 17. Common error when writing proofs: don’t write “iff” or $\iff$ between formulas; don’t write “=” between equations. Write “=” between formulas; write “iff” or $\iff$ between equations.

Practice Exam 3, Prob. 12, 13. The most common error (with a 1 point loss) was failure to insert the needed “iff”s between equalities or wrongly inserting “=’s between equalities. Some students made the opposite mistake of placing “iff”s between terms. E.g. “$2x + 4 = 2(x + 2)$” is correct but “$2x + 4 \iff 2(x + 2)$” is incorrect — you should place “=” not “iff” between two numbers. One the other hand “$2x = 4 \iff x = 2$” is correct but “$2x = 4 = x = 2$” is incorrect — you should place “iff” not “=” between two equations.

Exam 4 errors. When graphing tan and cot, remember that the period is $\pi$. The other trig functions have period $2\pi$. 

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While most students remember the graphs of \( \sin, \cos, \) and \( \tan \). Most fail to graph \( \csc, \sec \) or \( \cot \). These are obtained from \( \sin, \cos, \) and \( \tan \) respectively by taking reciprocals. This was done for \( \csc \) and \( \sec \) in the lecture notes and in the homework. The same technique will give the graph of \( \cot \).

When calculating phase shifts, remember that it is the number which is subtracted from \( x \). Hence the phase shift of \( \sin(x - \pi) \) is \( \pi \) and the graph of \( \sin \) is shifted \( \pi \) units to the right. The phase shift of \( \tan(\pi(x + 1)) \) is -1 since \( \tan(\pi(x + 1)) = \tan(\pi(x - (-1))) \) and this corresponds to shifting the graph of \( \tan(\pi x) \) one unit to the left.

**Practice Exam 4, Prob. 4.** Many try to apply the addition formula to \( \sin(x + y) \) and \( \cos(x + y) \). This makes the problem more complicated. One should note that the entire expression \( \sin(x + y) \cos(x - \cos(x + y)) \sin x \) is the right-hand side of the a subtraction formula for \( \sin: \sin((x + y) - x) = \sin(x + y) \cos x - \cos(x + y) \sin x \). Then \( \sin((x + y) - x) \) simplifies to \( \sin x \). In general, many students have a hard time remembering the sum and product rule identities. There is more memorization required here than in high school math classes but much less to remember than in an organic chemistry class.

**Practice Exam 4, Prob. 10.** Many who were able to solve this equation \( 2 \cos^2 x - 5 \cos x = -2 \) were not able to solve the related quiz-type equation \( 2 \sin^2 x + 5 \cos x = 4 \). The first step here would be to replace \( \sin^2 x \) by \( 1 - \cos^2 x \) and make this a problem involving only cosines. Many also were unable to solve \( \cos x + \sin x = 0 \). Since there is no squares another tactic is needed. Divide by \( \cos x \) to get \( 1 + \frac{\sin x}{\cos x} = 0 \) which is \( 1 + \tan x = 0 \) which is \( \tan x = -1 \). Then solve this.

**Practice Exam 4, Prob. 11a, 11b.** Most mistakes here involved ignoring what the restricted ranges are for the inverse trig functions. For any value \( x \), \( \cos^{-1}(x) \) must be an angle in the upper two quadrants, i.e., an angle in \([0, \pi]\). \( \sin^{-1}(x) \) must be an angle in the two right hand quadrants, i.e., an angle in \([-\pi/2, \pi/2]\).
1. Expand \((2s + 5t)^2 - 3st\) and then simplify with no parentheses in the final answer.

\[4s^2 + 17st + 25t^2\]

2. Factor \(81c^3 + 90c^2 + 25c\).

\[c(9c + 5)^2\]

3. Use inequalities to write \(|z + 1| > 0\) without absolute value signs.

\[z \neq -1\]

4. Let \(g(d) = \sqrt[4]{1/d}\). State the domain of \(g\) and compute \(g(1/16)\).

domain: \(d > 0\). \(g(1/16) = 2\)

5. Complete the square for \(3c^2 + 3c - 7\).

\[3(c + \frac{1}{2})^2 - \frac{31}{4}\]

6. Simplify \(y^{1/2}(\frac{1}{y} + 2\sqrt[3]{y} + y^{-1/3})\).

\[y^{-1/2} + 2y + y^{1/6}\]

7. Solve for \(n\): \(\frac{2^{5/3}}{4^{7/3}} = 2^n\).

\(n = -3\)

8. Find all real numbers \(b\) such that \(\sqrt[5]{b} = b\).

\(b = 0, 1\)
You must get at least 7 out of 8 problems on some Gateway Exam in order to get a C or better in Math 140. No calculators.

1. Simplify \( \frac{(5 \cdot 2)^8 - 2^7}{2^7} \).
   \[ 5^8 \cdot 2 - 1 \]

2. Expand \((x - 1)(x - 2)(x - 3)\).
   \[ x^3 - 6x^2 + 11x - 6 \]

3. Find all real numbers \( b \) such that \( 9b + 12 = -\frac{5}{b} \).
   no solutions

4. Factor \( 8E^3 - 27 \).
   \( (2E - 3)(4E^2 + 6E + 9) \)

5. Solve for \( a \): \( \frac{3^6}{9^{5/2}} = a^{33} \).
   \[ \sqrt[3]{3} \]

6. Let \( Y(j) = \sqrt[j]{(j + 3)^2} \). Compute \( Y(-12) \) and simplify it.
   3

7. Complete the square for \( 2L^2 + 4L - 7 \).
   \[ 2(L + 1)^2 - 9 \]

8. Use inequalities to write \( |z + 3| \geq 3 \) without absolute value signs.
   \[ z \leq -6 \text{ or } z \geq 0 \]
1. Factor \( x^2 + 9x + 8 \).

\[(x + 1)(x + 8)\]

2. Express the following inequalities by using one pair of absolute value signs: \( 3 < t \) or \( t < 0 \).

\[|t - \frac{3}{2}| > \frac{3}{2}\]

3. Simplify \( \frac{7a^4}{a^5} \) for \( a \neq 0 \).

\[\frac{7}{a}\]

4. Solve for \( c \): \( 125^{4/3}25^3 = c^5 \)

\[c = 25\]

5. Let \( L(z) = z^{1/3} \). Compute \( L((w + 1)^3) \) and simplify it.

\[w + 1\]

6. Complete the square for \( 2b^2 - 7b + 12 \).

\[2(b - \frac{7}{4})^2 + \frac{47}{8}\]

7. Find all real numbers \( a \) such that \( 5a^2 - 4a = -8a - 1 \).

no solution

8. Expand \( (a - 3)(b + c) - (ac + 2b) \) and then simplify with no parentheses in the final answer.

\[ab - 5b - 3c\]
1. Complete the square for \(6F^2 - 3F + 11\).

\[6(F - \frac{1}{4})^2 + \frac{85}{8}\]

2. Simplify \(\frac{(5\cdot 6)^{21} + 5^{20}}{5^{20}}\).

\[5 \cdot 6^{21} + 1\]

3. Let \(g(x) = \frac{1}{\sqrt[3]{x}}\). State the domain of \(g\) and compute \(g\left(-\frac{1}{27}\right)\).

domain: \(x \neq 0\). \(g\left(-\frac{1}{27}\right) = -3\)

4. Find all real numbers \(d\) such that \(6d^2 = d^2 + 7d - 2\).

\(d = \frac{2}{3}, 1\)

5. Expand \((x - 1)(x - 2)(x - 3)\).

\(x^3 - 6x^2 + 11x - 6\)

6. Use inequalities to write \(|t - 2| > \frac{1}{2}\) without absolute value signs.

\(t < \frac{3}{2}\) or \(\frac{5}{2} < t\)

7. Factor \(x^2 - 5x + 6\).

\((x - 2)(x - 3)\)

8. Solve for \(s\): \(24^{2/3} 3^{-2/3} = 8^{-s+1}\).

\(s = \frac{1}{3}\)
1. Use inequalities to write \(|t + \frac{3}{2}| < \frac{3}{2}\) without absolute value signs.

\[-5 < t < -2\]

2. Let \(g(x) = \frac{x}{x}\). What is the domain of \(g\) and what is \(g(2)\).

domain: \(x \neq 0\), \(g(2) = 1\)

3. Find all real numbers \(E\) such that \(17E^2 + 25E + 10 - E^3 = E^2 + E + 1 - E^3\).

\(E = -\frac{3}{4}\)

4. Factor \(25b^3 + 30b^2 + 9b\).

\(b(5b + 3)^2\)

5. Simplify \(\sqrt[3]{8b^6}\).

\(2b^2\)

6. Solve for \(d\): \(\frac{6^{3/2}}{(\sqrt{2})^3} = d^{1/2}\).

\(d = 27\)

7. Complete the square for \(20q^2 - 5q + 12\).

\(20(q - \frac{1}{8})^2 + \frac{187}{16}\)

8. Expand \((a + 3)(a - 2) - 5(a + 1)\) and then simplify with no parentheses in the final answer.

\(a^3 + 4a^2 - 8a - 23\)
Worked examples of gateway exam problems
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Solving quadratic equations
- Find all real numbers \( x \) such that \( 9x + 12 = -\frac{5}{x} \).
  First rewrite this with everything on the left, 0 on the right.
  \[
  9x + 12 = -\frac{5}{x}
  \]
  \[
  9x + 12 + \frac{5}{x} = 0
  \]
  \[
  9x^2 + 12x + 5 = 0
  \]
  First try factoring. In this case, it doesn’t work, so use the quadratic formula
  \[
  x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.\]

  Thus \( x = \frac{-12 \pm \sqrt{144 - 4(9)(5)}}{2(9)} = \frac{-12 \pm \sqrt{144-180}}{18} \) is undefined
  Since 144-180 is clearly negative, we can’t take the square root, hence there are no solutions. Don’t use the imaginary number \( i \) here, the problem asks for real numbers.
  Answer: no solution.

- Find all real numbers \( b \) such that \( \sqrt{b} = b \).
  Square to get rid of the square root, then put everything on the left.
  \[
  \sqrt{b} = b
  \]
  \[
  b = b^2
  \]
  \[
  b - b^2 = 0
  \]
  \[
  b^2 - b = 0 \quad \text{We multiply by -1 and put the highest power first to make things look more familiar.}
  \]
  Now factor.
  \[
  b(b - 1) = 0
  \]
  Set each factor to zero separately.
  Answer: \( b = 0, 1 \)

- Find all real numbers \( d \) such that \( 6d^2 = d^2 + 7d - 2 \).
  First rewrite this with everything on the left, 0 on the right.
  \[
  5d^2 - 7d + 2 = 0
  \]
  Now try to factor.
  \[
  (5d - 2)(d - 1) = 0
  \]
  Set each factor to zero separately.
  Answer: \( d = \frac{2}{5}, 1 \)

Completing the square
- Complete the square for \( 2x^2 - 7x + 12 \).
  Warning: completing the square for the formula \( 2x^2 - 7x + 12 \) and completing the square for the equation \( 2x^2 - 7x + 12 = 0 \) are two different problems. The common mistake is to assume the formula is an equation.
  For the equational version of completing the square, see the circle equation section of Lecture 2. Completing the square for formulas is different. With an equation, you can add a number to both sides. When you add something to a formula, you must subtract it off somewhere else in the formula.
  \[
  2x^2 - 7x + 12
  \]
  First factor the coefficient of \( x^2 \) out of the first two terms. Leave the constant alone. Factoring a 2 out of \( 7x \) is the same as dividing \( 7x \) by 2.
  \[
  (2x^2 - 7x) + 12
  \]
  \[
  2(x^2 - \frac{7}{2}x) + 12
  \]
  To complete the square of a factor \( (x^2 + ax) \), divide the coefficient of \( x \) by two, then square. This gives \( (\frac{a}{2})^2 \). Add this to the factor in parentheses to get
  \[
  (x^2 + ax + \left(\frac{a}{2}\right)^2) \quad \text{which is the perfect square} \ (x + \frac{a}{2})^2.
  \]
  In \( (x^2 - \frac{7}{2}x) \), \( a = -\frac{7}{2} \).
  Dividing by 2 and squaring gives: \( -\frac{7}{4} \rightarrow -\frac{7}{4} \rightarrow \frac{49}{16} \).
  Thus \( 2(x^2 - \frac{7}{2}x) + 12 \) becomes
  \[
  2(x^2 - \frac{7}{2}x + \frac{49}{16}) + 12 - 2(\frac{49}{16})
  \]
  Since we added \( \frac{49}{16} \), we compensate by subtracting it from the end of the formula. Where did the 2 come from? Since the \( \frac{49}{16} \) that we added was multiplied by 2 (the 2 at the beginning of the formula), the \( \frac{49}{16} \) we subtract must also be multiplied by 2. Failure to do this multiplication is a common error.
  Hint. Cancellation almost always occurs at this step.
  Don’t multiply \( 2 \times 49 \); cancel 2 off the 16. You should get \( \frac{49}{8} \) rather than \( \frac{98}{16} \).
  \[
  2(x^2 - \frac{7}{2}x + \frac{49}{16}) + 12 - \frac{49}{8}
  \]
  \[
  2x - \frac{7}{4}
  \]
  \[
  (\frac{12(8)}{8} - \frac{49}{8})
  \]
  \[
  2x - \frac{7}{4}
  \]
  \[
  \frac{96-49}{8}
  \]
  \[
  2(x - \frac{7}{4})^2 + \frac{47}{8}
  \]
  Answer: \( 2(x - \frac{7}{4})^2 + \frac{47}{8} \)

Be careful when adding fractions; partial credit isn’t allowed on gateway exams.
Inequalities and absolute values
You can replace absolute value inequalities with a pair of algebraic inequalities.
\[ |a| > 0 \iff a \neq 0 \]
\[ |a| > 2 \iff a < -2 \text{ or } 2 < a \]
\[ |a| < 2 \iff -2 < a < 2 \]

\[ |a| \]

Use inequalities to write \( |z + 1| > 0 \) without absolute value signs.
\[ |z + 1| > 0 \iff z + 1 \neq 0 \iff z \neq -1 \]
Answer: \( z \neq -1 \)

Use inequalities to write \( |z + 3| \geq 3 \) without absolute value signs.
\[ |z + 3| \geq 3 \iff z + 3 \leq -3 \text{ or } 3 \leq z + 3 \]
\[ \iff z \leq -6 \text{ or } 0 \leq z \]
Answer: \( z \leq -6 \) or \( z \geq 0 \)

Use inequalities to write \( |t + \frac{7}{2}| < \frac{3}{2} \) without absolute value signs.
\[ |t + \frac{7}{2}| < \frac{3}{2} \iff \frac{-3}{2} < t + \frac{7}{2} < \frac{3}{2} \]
\[ \iff \frac{-3}{2} - \frac{7}{2} < t < \frac{3}{2} - \frac{7}{2} \]
\[ \iff -5 < t < -2 \]
Answer: \(-5 < t < -2\)

Express the following inequalities by using one pair of absolute value signs:
\[ t \leq 1 \text{ or } 4 \leq t \]

Draw a picture of the set; find the midpoint between the endpoints; find the distance between the midpoint and the endpoints.
The distance between \( a \) and \( b \) is \( |a - b| \).
The midpoint between \( a \) and \( b \) is \( (a + b)/2 \).

Picture:

\[
\begin{array}{ccc}
1 & 5/2 & 4 \\
\cdot & \bullet & \square \\
3/2 & 3/2 & \\
\end{array}
\]

The midpoint is \( (1+4)/2 = 5/2 \).
The distance between the midpoint and the endpoints is \( 5/2 - 1 = 3/2 \).
\( t \) is in the set iff the distance between \( t \) and \( 5/2 \) is greater than \( 3/2 \).
\[ \iff |t - 5/2| \geq 3/2 \]

Answer: \( |t - 5/2| \geq 3/2 \)

Domain problems

Let \( g(d) = \sqrt{1/d} \). State the domain of \( g \) and compute \( g(1/16) \).
You can’t divide by 0, you can’t take the square root of a negative.
Hence we must have \( d \neq 0 \) and we must have \( 1/d \geq 0 \).
That latter means \( d \geq 0 \). This plus \( d \neq 0 \) gives
Answer: domain: \( d > 0 \).

\[ g(1/16) = \sqrt{1/(1/16)} = \sqrt{16} = 4 = 2 \]
Answer: \( g(1/16) = 2 \).

Let \( g(x) = x/x \). What is the domain of \( g \) and what is \( g(2) \).
You can’t divide by 0. Hence we must have \( x \neq 0 \).
Answer: domain: \( x \neq 0 \).
Note: Domains must be calculated without simplifications. If we simplified \( x/x \) to 1, we would wrongly conclude that the domain was \( (-\infty, \infty) \).
\[ g(2) = 2/2 = 1 \]
Answer: \( g(2) = 1 \).
Exponential equation: variable in exponent

Rewrite the equation so that both sides are powers of the same base $b^p = b^q$. Then equate the exponents $p = q$ and solve for the variable.

- **Solve for $n$:** $\frac{2^{5/3}}{4^{1/3}} = 2^n$.
  
  Since $4 = 2^2$, everything can be written as a power of 2.
  
  $\frac{2^{5/3}}{2^{2/3}} = 2^{n}$
  
  $2^{5/3 - 2/3} = 2^n$  
  
  You don’t have to do this much detail.

- **Solve for $x$:** $24^{2/3} 3^{-2/3} = 8^{-x+1}$.
  
  First simplify.
  
  $24^{2/3} 3^{-2/3} = 8^{-x+1}$
  
  $(3 \cdot 8)^{2/3} 3^{-2/3} = 8^{-x+1}$
  
  $3^{2/3} 8^{2/3} 3^{-2/3} = 8^{-x+1}$ since $3^{2/3} 3^{-2/3} = 3^{2/3 - 2/3} = 3^0 = 1$
  
  $8^{2/3} = 8^{-x+1}$
  
  Now equate exponents.
  
  $2/3 = -x + 1$
  
  $x + 2/3 = 1$
  
  $x = 1 - 2/3 = 1/3$
  
  Answer: $x = \frac{1}{3}$

Exponential equation: variable in base

- **Solve for $x$:** $\frac{6^{3/2}}{(\sqrt{2})^6} = x^{1/2}$.
  
  To get $x$, square both sides.
  
  $(\frac{6^{3/2}}{(\sqrt{2})^6})^2 = (x^{1/2})^2$
  
  $\frac{(6^{3/2})^2}{(\sqrt{2})^{12}} = x$
  
  $x = \frac{6^3}{(2^{1/2})^6}$
  
  $x = \frac{6^3}{2^3} = \frac{(2 \cdot 3)^3}{2^3} = \frac{2^3 3^3}{2^3} = 3^3 = 27$
  
  Answer: $x = 27$

- **Solve for $x$:** $\frac{3^6}{9^{5/2}} = x^{33}$.
  
  To get $x$, take the 33rd root of both sides.
  
  $x^{33} = \frac{3^6}{9^{5/2}}$
  
  $(x^{33})^{1/33} = \left(\frac{3^6}{9^{5/2}}\right)^{1/33}$
  
  $(x^{33})^{1/33} = \left(\frac{3^6}{3^{5/2}}\right)^{1/33}$
  
  $x = \left(\frac{3^6}{3^{5/2}}\right)^{1/33} = \left(3^6 \cdot 3^{5/2}\right)^{1/33}$
  
  $x = (3^1)^{1/33} = 3^{1/33} = 3^{1/3}$
  
  Answer: $\sqrt[33]{3}$

Worked examples of gateway exam problems

You must get at least 7 out of 8 problems on some Gateway Exam in order to get a C or better in Math 140. No calculators.
1(5). Rewrite without | |s: |x - 3| + 2|x - 1| for x ∈ [3,6].
2(5). Write as a union of two intervals: \{x: |x+2|≥1\}.
3(4). Find all real solutions x. Write \( \emptyset \) if there are none.
\[ \sqrt{3x+4} = x \]
4(16). Solve for x. Write your answer in interval notation.
(a) \( \frac{3}{x-3} < 1 \) One interval
(b) \( 1 - \frac{x}{2\sqrt{x}} \) A union of two intervals
5(6). Find the circle \( x^2 - 2x + y^2 = 7 \): 
(a) Find the center and radius.
(b) Find the x and y-intercepts.
6(4). Find the equation for the line through (-1,-1) and (2,3).
7(5). Find the domain of \( f(x)=\frac{1}{1-x^2+1} \). Union of two intervals.
8(6). Factor and find all roots given that -1 is one root.
\[ x^3 + x^2 - 3x - 3 = 0 \]
9(4). Simplify to a single polynomial:
\[ g(x) = x^3 \]
\[ [g(x+h) - g(x)]/h = \]
10(3). For the function \( h \) pictured,
\[ \text{for which values of } x \text{ does } h(x)=0? \]
11(14). For the domains and ranges, use interval notation, [1,5], or set notation, \{1,3,5\}. The answer may be \( \emptyset \).
<table>
<thead>
<tr>
<th>function</th>
<th>( 1/x )</th>
<th>( \sqrt{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain</td>
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<td>range</td>
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<td>interval(s) of increase</td>
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<tr>
<td>interval(s) of decrease</td>
<td></td>
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</tr>
</tbody>
</table>
12(6). Graph \( f(x) \) where
\[ f(x) = \begin{cases} 
1/x & \text{if } x < -1 \\
-x & \text{if } -1 \leq x \leq 1 \\
1/x & \text{if } x > 1 
\end{cases} \]
13(4). Graph \( 1 - \sqrt{x+1} \).
14(6). The graph of \( f(x) \) is a line segment from (0,2) to (2,0).
(a) Graph \( y = f(x+1)+1 \). (b) Graph \( y = f(1-x) - 1 \).
15(4). \( f(x) = (1/x)+1 \), \( g(x)=(1/x)-1 \), find and simplify \( (f \circ g)(x) \).
16(2). \( G(-4) = -3 \), \( G(-3) = 1 \), \( G(-1) = 2 \), \( G(1) = 5 \).
\( F(3) = 4 \), \( F(1) = -3 \), \( F(2) = -1 \), \( F(5) = 1 \). Find \( (G \circ G \circ F)(1) \).
17(6). Write each function as a composition \( f(g(x)) \) of two simpler functions \( f(x) \) and \( g(x) \).
(a) \( G(x) = x^4 + x^2 - 1 \). (b) \( F(x) = \frac{1}{\sqrt{x}} + \sqrt{x} \).

Answers
1. \( 3x-7 \)
2. \( (-\infty,-3]\cup[-1,\infty) \)
3. \( x = 4 \), (not \( x = -1 \), it doesn't work in the original equation.)
4a. \( (-3,3) \) \hspace{1cm} 4b. \( (-2,-1]\cup(0,2) \)
5. (a) center=(1,0), radius=2\( \sqrt{2} \),
(b) x-intercept: \( \pm 2\sqrt{2} \), y-intercept: \( \pm \sqrt{7} \)
6. \( y = \frac{4}{3}x + \frac{1}{3} \)
7. \( [-1,0]\cup(0,\infty) \)
8. \( (x+\sqrt{3})(x-\sqrt{3})(x+1) \) roots \( -\sqrt{3},-1,\sqrt{3} \).
9. \( 3x^2+3xh+h^2 \)
10. \( x = 1, 9 \)
11. See homework problem Hw 4:7
12.
13.
14(a). \hspace{1cm} 14(b).
15. \( 1/(1/x)-1)+1 = 1/(1-x) \).
16. \( G(G(F(1))) = G(G(-3)) = G(1) = 5 \)
17. (a) \( G(x) = f(g(x)) \) where \( g(x) = x^2 \) and \( f(x) = x^2+x-1 \).
(b) \( F(x) = f(g(x)) \) where \( g(x) = \sqrt{x} \) and \( f(x) = 1/x+x \).
Know the area and volume formulas for triangles, rectangles, circles, boxes, cans (including curved surface area).

1. (a) \( f^{-1}(x) = \frac{x^3}{x^3+1} \), find \( f(x) \).
   (b) \( f(x) = x^2 - 2x \) for \( x \geq 1 \), find \( f^{-1}(x) \).

2. The given lines are the main diagonal and the graph of \( f(x) \). Graph \( f^{-1}(x) \) if it exists. If not, write “not 1-1”.

3. Graph and on the graph mark the vertex and give both coordinates. Also mark the intercepts.

   \( y = -3x^2 - 6x \)

In 4 and 5,

Picture: Draw the picture. On the picture indicate your variables.

Given: Write the equations which relate the variables.

Answer: Solve for the wanted quantities.

4. A square is inscribed inside a circle. Express the area \( A \) of the circle as a function of the width \( x \) of the square.

5. A 6’ man stands \( x \) feet away from a 10’ high lamp. Express the length \( s \) of his shadow as a function of \( x \).

6. Graph. Mark the intercepts and the asymptotes.
   (a) \( y = \frac{2x-x^2}{(x-1)^2} \).
   (b) \( y = \frac{(x-1)^2}{2x-x^2} \).

7. Estimate as a power of 10. \( 2^{30} \approx \)

8. Solve for \( t \): \( 3^{2t-3} = \sqrt{3} \)

9. Simplify to a rational number.
   (a) \( \log_2(1/\sqrt{2}) \)
   (b) \( \ln(e^{\sqrt{e}}) \)

10. Graph. Mark the intercepts and asymptotes and list the domain and range. \( y = 1 - \ln(2-x) \)

11. Find the exact answer. No decimals.
   (a) Solve for \( w \): \( (3^{2w})^5 = 10 \). Use an appropriate logarithm.
   (b) Solve using natural logarithms: \( 3^x = 2^{x+1} \).

Answers

1a. \( f(x) = \frac{3}{\sqrt{1-x}} \)
   1b. \( f^{-1}(x) = 1 + \sqrt{1 + x} \)

2.

3. 

4. Picture:

   Given: \( A = \pi r^2 \)
   \( x^2 + x^2 = (2r)^2 \)
   Answer: \( A = \frac{\pi r^2}{2} \)

5. Picture:

   Given: \( \frac{10}{x+y} = \frac{6}{y} \)
   Answer: \( s = \frac{3x}{2} \)

6a.

6b.

7. \( 10^9 \)

8. \( t = 7/4 \)

9a. \(-\frac{1}{2}\)  (b) \( 3/2 \)

10. Hor. asymp.: none. Vert. asymp.: \( x = 2 \).

11a. \( w = \log_3(10)/10 \)  11b. \( x = \ln(2)/\ln(3/2) \)

Know volume, area, and circumference formulas (they are inside the front cover of the text).
1. Combine into a single logarithm: \(2 \ln\left(\frac{1}{x}\right) + 3 \ln(x+1)\)

2. Write as a sum / difference / multiple of the simplest possible logarithms: \(\log_b(x+1)\)

3. Find the one valid solution \(x\).
   \[\ln x + \ln(x-1) = \ln 2\]

4. Solve for \(x\).
   \[\ln(2x) = 1 + \ln(x-1)\]

5. Express in terms of natural logarithms: \(\log_{10}14\)

6. Initially, 200 bacteria are present in a colony. Eight hours later there are 500.
   (a) Determine the growth constant \(k\).
   (b) What is the population two hours after the start?
   (c) How long did it take for the population to double?

7. The half-life of a radioactive substance is 9 years. Initially a sample has 20 grams. How many grams remain after 8 years?

8. Convert 300° to radian measure using \(\pi\).

9. Find the radian measure of an angle which intercepts a 10 inch arc on a circle of radius 36 inches.

10. A point rotates around a circle of radius 3 feet at 20 revolutions/minute.
   (a) Find its angular speed. Exact radian answer using \(\pi\).
   (b) Find its linear speed. Exact answer using \(\pi\).

11. \(\cos \theta = -4/5\) and \(\pi < \theta < 3\pi/2\). Find \(\sin \theta\).
   (b) The side opposite angle \(\theta\) of a right triangle is 6 and the hypotenuse is \(x^2\), find \(\tan(2\pi - \theta)\) and \(\tan\left(\frac{\pi}{2} - \theta\right)\)
   (c) The two legs of a right triangle are 2 and 3. What is the cosine of the smallest angle? Hint: the smallest angle is opposite the smallest side.

12. Prove: \(\tan^2 A + 1 = \sec^2 A\)

13. Prove: \(\cos^2 \theta - \sin^2 \theta = \frac{1-\tan^2 \theta}{1+\tan^2 \theta}\)

14. (a) Find \(\cot(11\pi/3)\). Give the exact answer.
   (b) Rewrite \(\cos(-16\pi/5)\) with a reference angle in \([0, \pi/2]\).
   Don't give the decimal or exact answer.

15. Simplify to 3 symbols: \(\frac{\cot^2 t(\sec^2 t-1)}{\sec^2 t - \tan^2 t + 1}\)

16. Simplify to 5 or 7 symbols: \(\tan(t)[\cos(t) + \cos(-t)]\)

Answers

1. \(\frac{3}{x^2}\)

2. \(\frac{1}{3} \log_b x - \log_b(x+1)\)

3. \(x = 2\)

4. \(x = e/(e-2)\)

5. \(\ln(14)/\ln(10)\)

6. (a) \(k = (1/8) \ln(5/2)\)
   (b) \(N(2) = 200e^{(1/4)\ln(5/2)}\)
   (c) \((8\ln(2))/\ln(5/2)\)

7. \(N(8) = 20e^{(8/9)\ln(1/2)}\)

8. \(5\pi/3\)

9. \(5/18\) radians

10. (a) \(40\pi\) radians/minute
    (b) \(120\pi\) ft/min

11. (a) \(-3/5\)
   (b) \(-6/\sqrt{x^4-36}, \sqrt{x^4-36}/6\)
   (c) \(3/\sqrt{13}\)

12. \(\frac{\sin^2 A}{\cos^2 A} + 1 = \frac{1}{\cos^2 A}\) iff \(\sin^2 A + \cos^2 A = 1\) iff true.

13. \(\frac{1-\tan^2 \theta}{1+\tan^2 \theta} = \frac{1-\sin^2 \theta/\cos^2 \theta}{1+\sin^2 \theta/\cos^2 \theta} = \frac{\cos^2 \theta-\sin^2 \theta}{\cos^2 \theta+\sin^2 \theta} = \frac{\cos^2 \theta-\sin^2 \theta}{1} = \cos 2\theta\)

14. (a) \(-1/\sqrt{3}\)
    (b) \(-\cos(\frac{\pi}{3})\)

15. \(1/2\)

16. \(2 \sin t\)
1. amplitude = 2, period = 6, \( y = -2 \cos \frac{\pi}{3} x \)

2. \( y = -2 \sin(\pi x - \pi) \).

3. \( y = \tan(-x/2 + \pi) \) over one period. List the \( x \)-intercepts and the vertical asymptotes which occur in this period.

4. \( y = \sec(x + \pi) \). Draw the asymptotes as dotted lines. Be prepared to graph cot and csc.

5. Simplify \( \sin(x + y) \cos x - \cos(x + y) \sin x \).

6. Simplify \( \frac{\tan(\pi/5) - \tan(\pi/30)}{1 + \tan(\pi/5) \tan(\pi/30)} \).

7. \( x = \sin(\theta)/2, \frac{\pi}{2} < \theta < \pi \).
   (a) Find \( \sin 2\theta \)
   (b) Find \( \sin(\theta/2) \)
   (c) Find \( \tan(\theta/2) \)

8. Write as a sum or difference of trig functions. \( \cos(x-1) \sin(x+1) \)

9. Find all solutions to \( \cos \theta + \frac{\sqrt{3}}{2} = 0 \).
   Two sets of solutions.

10. Find all solutions of \( 2 \cos^2 x - 5 \cos x = -2 \).
    Two sets of solutions.

11. Find the exact value (no decimals) of
    (a) \( 3 \cos^{-1}(-\sqrt{3}/2) \)
    (b) \( 3 \tan^{-1}(-\sqrt{3}) \)
    (c) \( 4 \sin^{-1}(\sin(5\pi/7)) \)
    (d) \( 4 \tan(\arcsin(3/5)) \)

12. \( \frac{1}{\sqrt{3}} \) \( \frac{1}{\sqrt{1 - \sqrt{1 - 4x^2}}} \)

13. \( -4x \sqrt{1 - 4x^2} \)
    (b) \( \sqrt{\frac{1 + \sqrt{1 - 4x^2}}{2}} \)
    (c) \( \frac{2x}{1 - \sqrt{1 - 4x^2}} \)

14. \( \frac{1}{2} [\sin(2) + \sin(2x)] \)

15. \( \theta = -5\pi/6 + 2\pi n, 5\pi/6 + 2\pi n \)

16. \( x = \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n \)

17. (a) \( 5\pi/6 \) (b) \( -\pi/3 \) (c) \( 2\pi/7 \) (d) \( 3\pi/4 \)
1(6). Find the area (exact value) of a triangle with sides \(a=2\), \(b=3\), and an included angle \(\angle C=25^\circ\).

2(10). An antenna sits on top of a 1000 ft building. From a point level with the base of the building, the building has elevation 20° and the antenna has elevation 25°. How high is the top of the antenna above the ground?

\[
\text{Height} = \text{Building Height} + \text{Elevation Difference} = 1000 + (25 - 20) = 1005 \text{ ft}
\]

3(8). Convert the given polar coordinates to rectangular coordinates:
(a) \(2, \pi/4\)
(b) \((-1, 2\pi)\)
(c) Convert the given rectangular coordinates to polar coordinates: \((\pi, \pi)\)

4(9). \(a=3\), \(\angle C=60^\circ\), \(\angle A=50^\circ\), solve for \(b\).

5(12). \(a=3\), \(b=2\), \(\angle B=30^\circ\), solve for \(\angle C\).

6(13). Find the perimeter of a pentagon inscribed in a unit circle (exact answer using radians rather than degrees).

7(3). Convert the given polar coordinates to rectangular coordinates:
(a) \((2, \pi/4)\)
(b) \((-1, 2\pi)\)
(c) \(\pi, \pi\)

8(6). (a) Convert the polar equation to a rectangular equation: \(r^2 = \cos 2\theta\)
(b) Convert the rectangular equation to a polar equation: \(x^2 - y^2 = 1\)

9(14). Find all axes and focus or focal points and draw the graph of \(4y^2 - x^2 + 2x - 2 = 0\)

10(14). Find all axes and focus or focal points and draw the graph of \(4y^2 + x^2 - 2x = 0\)
**Review of Theorems and Formulas for Exam 1**

All exams will consist solely of homework type problems. All numeric answers must be exact, no decimals, no mixed fractions. E.g., $\frac{3}{2}$, $\sqrt{2}$ not, $1 \frac{1}{2}$ or 1.5 or 1.414...

### Absolute Value

- $|a| = x$ if $x \geq 0$.
- $|a| = -x$ if $x < 0$.

### Distance

Distance. $d = |a-b|$ = the distance between $a$ and $b$.

### Definition

- **Domain** of a function (real values, real variables) is the set of numbers for which it is defined.
- **Graph** of an equation is the set of all points which satisfy the equation.

### Distance Formula

The distance $d$ between points $(x_1, y_1)$ and $(x_2, y_2)$ is $d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

### Circle Equation

The circle of radius $r$ with center $(a,b)$ has equation: $(x-a)^2 + (y-b)^2 = r^2$.

### Completing the Square

To make $x^2 + ax$ a perfect square, add $(\frac{a}{2})^2$. $x^2 + ax + (\frac{a}{2})^2 = (x + \frac{a}{2})^2$.

### Facts

- For positive $a$, $a = (\sqrt{a})^2$. For all $a$, $\sqrt{a^2} = |a|$.
- **Intercept** is the x-coordinate of a point where the graph crosses the y-axis. A y-intercept is the y-coordinate of a point where the graph crosses the y-axis.
- **Slope** of a line is the ratio of the vertical (height) change in $y$ over a horizontal change in $x$.

### Theorem

- The slope of the line through $(x_1,y_1)$ and $(x_2,y_2)$: $m = \frac{y_2-y_1}{x_2-x_1}$
- The equation of the line through $(x_1,y_1)$ with slope $m$: $y - y_1 = m(x - x_1)$
- The equation of the line with slope $m$ and y-intercept $b$: $y = mx + b$
- The slope of a horizontal line is 0.
- The slope of a vertical line is undefined.
- The equation of the horizontal line through $(a,b)$ is $y = b$.
- The equation of the vertical line through $(a,b)$ is $x = a$.

### Line Equation Format

Line equations must be in the one of these four forms:

- $y = mx + b$
- $y = mx$, $y = b$, $x = a$.

### Theorem

If $a > 0$, $x^2 - a = (x - \sqrt{a})(x + \sqrt{a})$

### Division Law

If $p(x)/d(x)$ has quotient $q(x)$ and remainder $r(x)$ then $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$ and thus $p(x) = d(x)q(x) + r(x)$

### X-Intercept

$a$ is a root or zero of $p(x)$ iff $p(a) = 0$.

### Theorem

$a$ is a root of $p(x)$ iff $(x-a)$ is a factor of $p(x)$.

### Definition

For sets $A$ and $B$, a function from $A$ to $B$ is a rule which assigns a value $f(x)$ in $B$ to each $x$ in $A$. The domain of $f$ is $A$; the range of $f$ is the set of all possible values $f(x)$.

### Definition

The graph of a function $f$ is the set of all points $(x,y)$ with $y = f(x)$. The height $y$ is the function value $f(x)$.

### Fact

A curve is the graph of a function iff no vertical line intersects it more than once.

### Definition

A function $f$ increases on an interval if the value $f(x)$ increases as $x$ moves from left to right in the interval. $f$ decreases if $f(x)$ decreases as $x$ moves from the left to the right. $c$ is a turning point of $f$ if $f$ increases on one side of $c$ and decreases on the other.

### Theorem

$f(a) = c$ is a maximum value iff $c$ is $\geq$ all other values.

### Theorem

Translating a graph $f(x)$ or reflecting it across an axis, changes the function as follows:

<table>
<thead>
<tr>
<th>up 1 unit</th>
<th>down 1</th>
<th>*left 1</th>
<th>*right 1</th>
<th>reflect in x-axis</th>
<th>reflect in y-axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)+1$</td>
<td>$f(x)-1$</td>
<td>$f(x+1)$</td>
<td>$f(x-1)$</td>
<td>$f(-x)$</td>
<td>$f(-x)$</td>
</tr>
</tbody>
</table>

* Horizontal changes are the opposite of what one would expect.

### Definition

For functions $f$ and $g$, define $f + g, f - g, f \cdot g, f/g$ by

- $(f+g)(x) = f(x)+g(x)$, Note: $(f+g)(x)$ is not $(f+g)(x)$
- $(f-g)(x) = f(x)-g(x)$, The first is function application.
- $(fg)(x) = f(x)g(x)$, The second is multiplication.
- $(f/g)(x) = f(x)/g(x)$.

### Definition

For functions $f$ and $g$, define $f \circ g$, the composition of $f$ and $g$, by

- $(f \circ g)(x) = f(g(x))$

$f$ is the outer function; $g$ is the inner function.
Review of Theorems and Formulas for Exam 2

All exams will consist solely of homework type problems.
All numeric answers must be exact, no decimals, no mixed fractions. E.g., $\frac{3}{2}$, $\sqrt{2}$ not, $1\frac{1}{2}$ or 1.5 or 1.414...

**Definition.** $f^{-1}$, the inverse of $f$, is the function, if any, such that, $f(f^{-1}(x)) = x$ when $f^{-1}(x)$ is defined and $f^{-1}(f(x)) = x$ when $f(x)$ is defined.

**Theorem.** The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the major diagonal $y = x$.

**Definition.** $f$ is 1-1 (“one-to-one”) iff $x \neq y$ implies $f(x) \neq f(y)$.

**Theorem.** $f$ has an inverse iff $f$ is 1-1 iff no horizontal line intersects its graph more than once.

**Definition.** A quadratic function is a degree-2 polynomial $y = ax^2 + bx + c$ with $a \neq 0$.

The graph is a parabola.
- If $a > 0$, the parabola opens up.
- If $a < 0$, the parabola opens down.
- If $a = 0$, the graph is a horizontal line.

**Theorem.** Every quadratic function may be written in the form:

$$y = a(x - h)^2 + k,$$

where $(h, k)$ is the vertex (nose) of the parabola.

For an expanded polynomial $a x^n + \ldots + c$ with $a x^n$ the term of highest degree: $a x^n$ is the leading term, $a$ is the leading coefficient, and $n$ is the degree. The $y$-intercept is the constant term $c$.

- For large $x$ (near $\pm \infty$), graph looks like the leading term $ax^n$.
- As $x$ goes to $\infty$, $y$ goes to $+\infty$ if $a > 0$, to $-\infty$ if $a < 0$.
- Graphs of odd degree go to $+\infty$ in one direction, $-\infty$ in the other, like $y = x^3$, $y = -x^3$.
- Graphs of even degree either go to $+\infty$ in both directions or to $-\infty$ in both directions, like $y = x^2$, $y = -x^2$.
- At roots of degree 1, the graph crosses x-axis like $y = x$ or $y = -x$.
- At roots of odd degree $> 1$, the graph crosses the x-axis like $y = x^3$ or $y = -x^3$.
- At roots of even degree, the graph touches but doesn’t cross the x-axis, like $y = x^2$ or $y = -x^2$.

**Definition.** A ratio of two polynomial functions is a rational function. It is reduced if the top and bottom have no common factors.

In the graphs below, $x = 0$ (the y-axis) is the vertical asymptote, $y = 0$ (the x-axis) is the horizontal asymptote.

For odd degree vertical asymptotes, one side goes to $+\infty$, the other to $-\infty$.
For even degree vertical asymptotes, both sides go to $+\infty$ or both go to $-\infty$.

For a reduced rational function:
- x-intercepts (roots) occur where the top is 0.
- If the root has degree $n$, the x-intercept goes like that of $y = x^n$ or $y = -x^n$.
- If the bottom is 0 at $a$, then $x = a$ is a vertical asymptote.
- If the factor has degree $n$, the vertical asymptote looks like that of $y = 1/x^n$ or $y = -1/x^n$.

As $x \to \pm \infty$, the graph resembles the graph of the leading term which is either a constant $b$ or of the form $a x^n$ or $ax^n$.

1. If a constant $b$, then $y = b$ is a horizontal asymptote.
2. If it is $a x^n$, then $y = 0$ is a horizontal asymptote.
3. If it is $ax^n$, there is no horizontal asymptote.

**Definition.** An exponential function is of the form $y = b^x$ with the base $b > 0$.

$b^1 = 1$, $b^0 = b^0$, $b^2 = b^2$, ...

$b^n = \sqrt[n]{b}$, the $n^{th}$ root of $b$.

$b^{p/q} = (b^p)^{1/q} = (b^q)^{1/p}$

**Examples**

$(b^p)^m = b^{pm}$

$(5^2)^3 = 5^6$

$b^a b^m = b^{a+m}$

$5^2 5^3 = 5^5$

$b^{p/m}$

$b^{p-m}$

$\frac{5^3}{5^2} = 5^{-1}$

$\frac{5^2}{5^3} = 5^{-2}$

$(ab)^n = a^n b^n$

$2^2 3^3 = 6^5$

$(a^p)^q = a^{pq}$

$(\frac{1}{2})^5 = \frac{1}{32}$

Property: If $b \neq 1$, $b^x = b^y \iff x = y$. Hence $b^x$ is 1-1.

$a \approx b$ means $a$ is approximately equal to $b$.

Fact: $2^{10} \approx 10^3$, i.e., $2^{10}$ is approximately equal to $10^3$.

The graph of $y = 1^x$ is the horizontal line $y = 1$.
Otherwise, the graph of $y = b^x$:
- has y-intercept 1 but no x-intercept,
- it goes to $\infty$ in one direction,
- it has the horizontal asymptote $y = 0$ in the other.

For $b > 1$, the graph of $b^x$ is like the graph of $2^x$ as below.

For $0 < b < 1$, the graph is like $(\frac{1}{2})^x$.

**Definition.** $e^t$ is the unique exponential function $b^t$ whose the tangent at $(0,1)$ has slope 1. Fact: $e \approx 2.7$.

**Definition.** $\log_b(x)$, the log of $x$ to the base $b$, is the inverse of the exponential function $b^x$.

$\ln(x)$, the natural logarithm, $= \log_e(x)$ is the inverse of $e^x$.

Inverses act in opposite directions and inverses cancel.:

$y = \log_b(x)$ if $b^y = x$, $y = \ln(x)$ if $e^y = x$.

$\log_b(b^y) = y$, $b^{\log_b(x)} = x$, $\ln(e^y) = y$, $e^{\ln(x)} = x$.

If the exponent base is $e$, $\ln$ and $e^t$ don’t completely cancel; the bottom left equation becomes:

$\ln(b^t) = t \ln(b)$.

The exponent comes down to the outside.

Fact: $e^t = 1 \Rightarrow 0 = \ln(1)$. $e^t = e \Rightarrow 1 = \ln(e)$

Know area, circumference, volume formulas for triangles, rectangles, boxes and cans. See inside front cover of the text.
CONVERSION FORMULAS. 180° = \pi \text{ radians}

The speed of an object rotating through angle \( t \) radians with a constant angular speed \( \omega \) is:

\[ \text{linear speed} = \omega \cdot r \]

where \( r \) is the radius.

Most properties of logarithms are the same as for exponentiation but

with the symbols

<table>
<thead>
<tr>
<th>( \log_b )</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_b x )</td>
<td>( b^x )</td>
<td>( \log_b y )</td>
<td>( b^y )</td>
</tr>
</tbody>
</table>

When \( b = e \), \( \log_e x \) is written as \( \ln x \).

\[ \ln \left( \frac{x}{y} \right) = \ln x - \ln y \]

\[ \ln \left( x^y \right) = y \ln x \]

\[ \ln(1) = 0 \]

\[ \ln(x^y) = y \ln x \]

In the base-e form lemma, every exponential function can be written in the form

\[ N(t) = N_0 e^{kt} \]

where \( N_0 \) is the initial amount.

\[ N(t) \text{ is the growth constant.} \]

If \( k > 0 \), \( N(t) \) measures exponential growth.

If \( k < 0 \), \( N(t) \) measures exponential decay.

DEFINITION. Suppose the vertex of an angle is at the center of a circle of radius \( r \). Let \( s \) be the length of the arc the angle intercepts on the circle. Then

\[ \theta = \frac{s}{r} \]

is the radian measure of the angle. For unit circles, the radius \( r = 1 \) and radian measure equals arc length: \( \theta = s \).

Clockwise angles \( \theta \) are negative.

CONVERSION FORMULAS. 180° = \pi \text{ radians. Thus}

\[ 1^\circ = \frac{\pi}{180} \text{ radians and 1 radian} = \frac{180}{\pi} \text{ degrees.} \]

DEFINITION. If an object travels a distance \( d \) in time \( t \), its linear speed is \( \frac{d}{t} \).

If an object rotates around a circle of radius \( r \), then its linear speed is \( \omega r \).

The unit circle has radius one and center \((0,0)\). There are four quadrants I, II, III, IV as pictured.

An angle is in standard position if its vertex is at the origin \((0,0)\) and its initial side is on the positive x-axis. The other side of the angle is the terminal side.

The trigonometric functions of \( \theta \) are:

\[ \sin \theta = \frac{y}{r} \quad \tan \theta = \frac{y}{x} \quad \sec \theta = \frac{r}{x} \quad \csc \theta = \frac{r}{y} \quad \cot \theta = \frac{x}{y} \]

FACTS. Know the sin and cos (and hence tan, cot, sec and csc) of:

\[ 0, \pi/6, \pi/4, \pi/3, \pi/2. \]

The Pythagorean identities:

\[ \sin^2 \theta + \cos^2 \theta = 1 \]

DEFINITION. Given an angle of radian measure \( \theta \) in standard position, let \((x,y)\) be the coordinates of the intersection of the terminal side with the unit circle.

\[ \cos \theta = x \quad \sin \theta = y \quad \tan \theta = \frac{y}{x} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta} \quad \cot \theta = \frac{1}{\tan \theta} \]

Pythagorean identities:

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Pythagorean identities:

\[ \sin^2 \theta + \cos^2 \theta = 1 \]
Graph of \( \sin(x) \). Shift the above graph left \( \pi/2 \) units.

**Definition.** For any periodic function \( f(x) \), the amplitude of \( f \) is half the distance between the min value and the max value of \( f \). Like periods, amplitudes are always positive.

**Theorem.** For \( y = \sin(Bx) \) or \( y = \cos(Bx) \) with \( B > 0 \),
- period = \( 2\pi/B \),
- amplitude = |A|. Reflect the graph around the x-axis if \( A < 0 \).

**Trigonometric Addition Formulas.**
\[
\begin{align*}
\sin(s \pm t) &= \sin s \cos t \pm \cos s \sin t \\
\cos(s \pm t) &= \cos s \cos t \mp \sin s \sin t \\
\cos(s - t) &= \cos s \cos t + \sin s \sin t \\
\tan(s \pm t) &= \frac{\tan s \pm \tan t}{1 \mp \tan s \tan t} \\
\tan(s - t) &= \frac{\tan s - \tan t}{1 + \tan s \tan t}
\end{align*}
\]

**Double-angle Formulas.**
\[
\begin{align*}
\sin 2\theta &= 2 \sin \theta \cos \theta \\
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
\tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta}
\end{align*}
\]

**Half-angle Formulas.**
\[
\begin{align*}
\sin \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \text{if } \theta/2 \text{ is in quadrant I or II.} \\
\cos \frac{\theta}{2} &= \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad \text{if } \theta/2 \text{ is in quadrant I or IV.} \\
\tan \frac{\theta}{2} &= \frac{\sin \theta}{1 + \cos \theta}
\end{align*}
\]

**Product-to-Sum Formula.**
\[
\begin{align*}
\sin x \sin y &= \frac{1}{2} [\cos(x - y) - \cos(x + y)] \\
\cos x \cos y &= \frac{1}{2} [\cos(x - y) + \cos(x + y)] \\
\sin x \cos y &= \frac{1}{2} [\sin(x - y) + \sin(x + y)]
\end{align*}
\]

**Definition.** \( \sin^{-1} \), \( \cos^{-1} \), \( \tan^{-1} \) are the inverses of the above restrictions of \( \sin \), \( \cos \), and \( \tan \). Thus
\[
\begin{align*}
\sin^{-1}(x) &= \theta \big| \theta \in [-\pi/2, \pi/2] \big| \text{ such that } \sin(\theta) = x, \\
\cos^{-1}(x) &= \theta \big| \theta \in [0, \pi] \big| \text{ such that } \cos(\theta) = x, \\
\tan^{-1}(x) &= \theta \big| \theta \in (-\pi/2, \pi/2) \big| \text{ such that } \tan(\theta) = x.
\end{align*}
\]

\( \sin^{-1}(x) \) and \( \cos^{-1}(x) \) have domain \([-1, 1]\), for \( x \in [-1, 1] \), they are undefined. \( \tan^{-1}(x) \) has domain \(( -\infty, \infty ) \).

**Notation.** \( \sin^{-1}(x) \), \( \cos^{-1}(x) \) and \( \tan^{-1}(x) \) are also written: \( \arcsin(x) \), \( \arccos(x) \), and \( \arctan(x) \).

Since inverses undo each other, we have
\[
\begin{align*}
\sin^{-1}(\sin x) &= x \text{ if } x \in [-\pi/2, \pi/2] & \sin(\sin^{-1} x) &= x \text{ if } x \in [-1, 1] \\
\cos^{-1}(\cos x) &= x \text{ if } x \in [0, \pi] & \cos(\cos^{-1} x) &= x \text{ if } x \in [-1, 1] \\
\tan^{-1}(\tan x) &= x \text{ if } x \in (-\pi/2, \pi/2) & \tan(\tan^{-1} x) &= x \text{ for any } x
\end{align*}
\]

**Theorem.** For \( x \in [-1, 1] \),
\[
\begin{align*}
\cos(\sin^{-1} x) &= \sqrt{1 - x^2} \\
\sin(\cos^{-1} x) &= \sqrt{1 - x^2}
\end{align*}
\]

In a triangle with hypotenuse 1 and side \( x \),
\[
\begin{align*}
\sin^{-1} x &= \text{ angle opposite } x \\
\cos^{-1} x &= \text{ angle adjacent to } x
\end{align*}
\]

Note that \( \sin^{-1} x + \cos^{-1} x = \pi/2 \quad (= 90^\circ) \)
Theorem. The area of a triangle with sides \( a \) and \( b \) and included angle \( \theta \) is \( \frac{1}{2}ab \sin(\theta) \).

**Straight Angle Sum Fact.** The sum of a triangle’s 3 angles is a straight angle.

\[ \angle A + \angle B + \angle C = \pi \quad (=180^\circ) \]

**Convention.** Assume side \( a \) is opposite angle \( A \), side \( b \) is opposite angle \( B \) and side \( c \) is opposite angle \( B \).

**Sine Laws.**

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

**Cosine Laws.**

\[
\begin{align*}
\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
\cos B &= \frac{c^2 + a^2 - b^2}{2ac} \\
\cos C &= \frac{a^2 + b^2 - c^2}{2ab}
\end{align*}
\]

**Ellipses**

Theorem. Let \( a \) be half the major axis’ length, \( b \) be half the minor axis’s length, and \( c \) be half the distance between the foci. Then \( a^2 = b^2 + c^2 \). \( c = \sqrt{a^2 - b^2} \).

Theorem. For \( a \geq b > 0 \), the graphs of \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) and \( \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \) are ellipses. The center is \((0,0)\). The major axis has length \( 2a \); the minor axis has length \( 2b \); the distance between foci is \( 2c \) where \( c = \sqrt{a^2 - b^2} \).

- For \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), the ellipse is horizontal, the foci are \((-c,0)\) and \((c,0)\), the major axis is the line segment \((-a,0)(a,0)\), the minor axis is \((0,-b)(0,b)\).

- For \( \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \), the ellipse is vertical, the foci are \((0,-c)\) and \((0,c)\), the major axis is the line segment \((0,-a)(0,a)\), the minor axis is \((-b,0)(b,0)\).

**Hyperbolas**

**Vertical Parabola Theorem.** For \( p \neq 0 \), the graph of \( x^2 = ky \) is the vertical parabola with focus \((0,p)\) and directrix \( y = -p \) where \( p = \frac{k}{4} \). The axis is the \( y \)-axis; the vertex is \((0,0)\).

**Horizontal Parabola Theorem.** For \( p \neq 0 \), the graph of \( y^2 = kx \) is a horizontal parabola with focus \((p,0)\) and directrix \( x = -p \) where \( p = \frac{k}{4} \). The axis is the \( x \)-axis, the vertex =\((0,0)\).

**Elliptical Parabola Theorem.** For \( p \neq 0 \), the graph of \( 4px = y^2 \) is an elliptical parabola with focal-width point \((p,0)\) and directrix \( x = -p \) where \( p = \frac{k}{4} \). The axis is the \( y \)-axis, the vertex =\((0,0)\).
Simplify fully to a formula with fewer symbols. $\frac{13\sqrt{x}}{y}$ has 6 symbols. Circle the 4 formulas (excluding the example) which can not be simplified.

Do not write in the gray area. Go to the top menu in www.math.hawaii.edu/140, read “Common Errors”.

Example: $\frac{6x^2}{3x^3} = \frac{2}{x}$

Example: $\sqrt{x^2 - 1} = \ldots$

1. $\sqrt{a+b} = \ldots$
   Don't write in gray areas.

2. $\sqrt{x^2 + 25} = \ldots$

3. $\sqrt{25b} = \ldots$

4. $\sqrt{25b^4} = \ldots$

5. $\sqrt{25(a^2 + b^2)^2} = \ldots$

6. $\sqrt{\frac{9x}{16}} = \ldots$
   -- 5 symbols --

7. $\frac{3x+5}{3y+7} = \ldots$

8. $\frac{3x}{3y+5} = \ldots$

9. $\frac{3x}{3x+6} = \ldots$

10. $\frac{3x^2y}{3y^2x} = \ldots$
   -- 3 symbols --

11. $\frac{6^7}{3^7} = \ldots$
   -- single exponent, e.g., $5^9$ --

12. $\frac{3^8-3^7}{3^7} = \ldots$
   -- single digit --

13. $-\frac{x^2}{x-3} = \ldots$
   -- 7 symbols, $\frac{2-x}{3-x}$ is wrong. --

14. $-\frac{x}{3-x} = \ldots$
   -- 5 symbols --

Evaluate the following functions, simplify to the given # of symbols.

Example. $g(x) = x^2 - 1$, $g(x+h) = (x+h)^2 - 1$

$g(g(x)) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$

Example. $h(x) = 1 - 2x$, $h(y^2) = 1 - 2y^2$

15. $f(x) = x^2$, $f(x^2) = \ldots$
   -- 2 symbols --

16. $f(x) = x^2$, $f(x+h) = \ldots$
   -- 6 symbols --

17. $g(x) = x + 3$, $g(g(x)) = \ldots$
   -- 3 symbols --

18. $g(x) = x + 3$, $g(x+h) - g(x) = \ldots$
   -- 1 symbol --

19. $f(x) = \frac{1}{x}$, $f(x+h) = \ldots$
   -- 5 symbols, not $\frac{1}{x} + h$ --

Write as a single reduced fraction. No fractions in numerator or denominator.

20. $\frac{a}{2b} = \ldots$
   -- 5 symbols --

21. $\frac{4b+2}{2a+4} = \ldots$
   -- 11 symbols --
Simplify fully to a formula with fewer symbols. $\frac{13\sqrt{x}}{y}$ has 6 symbols. Circle the three formulas which cannot be simplified. Go to www.math.hawaii.edu/140 and read “Common Errors.”

Example: $\frac{6x^2}{3x^3} = \frac{2}{x}$ 3 symbols Example: $\sqrt{x^2 - 1} = \text{shaded areas.}$

The following are variants of often missed problems from Take-home Quiz 1.

2. $\sqrt{9-x^2} =$

Don’t write in shaded areas.

7. $\frac{2x+3}{2x+2} =$

8. $\frac{4x}{2x+6} =$

9. $\frac{4x}{2x+5} =$

12. $\frac{2^{10}-2^8}{2^7} =$ -- single digit --

13. $\frac{-3-x}{2-x} =$ -- 7 symbols, $\frac{x-3}{x-2}$ is wrong. --

14. $\frac{-x}{x-2} =$ -- 5 symbols --

17. $g(x) = h - 2x, \quad g(g(x)) =$ -- 4 symbols --

19. $f(x) = \frac{1}{x}, \quad f(h-x) =$ -- 5 symbols --

Write as a single reduced fraction. No fractions in numerator or denominator.

20. $\frac{c}{2b} =$ -- 4 symbols --

21. $\frac{3a+4}{4+\frac{7}{b}} =$ -- 11 symbols --

Evaluate the following functions

Example. $g(x) = x^2 - 1, \quad g(x+h) = (x+h)^2 - 1$

$g(g(x)) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$

Example. $h(x) = 1 - 2x, \quad h(y^2) = 1 - 2y^2$

18. $\frac{6}{x^3} = \frac{2}{x}$ 3 symbols Example: $\sqrt{x^2 - 1} = \text{shaded areas.}$
Practise on the Hw 0 recommended-problem worksheet or the recommended textbook problems listed above.

1=1+1 has 5 symbols, \((-\infty,0]\) has 6 symbols, \((\frac{1}{2})x - 3\) has 8 symbols, \(\sqrt{x^2 - 2x}\) has 6 symbols, \(\frac{13}{12}\) has 5 symbols, \(0 \leq x\) has 3 symbols.

Do not turn in scratch paper; put your work in the space provided.

You get extra credit if you are the first to find an error in the lecture note packet sold at the Bookstore.

Problems marked with a superscript \(\hat{e}\) have notes in the note-packet section titled “Errors”.

1(1). Is the inequality true or false? \(\pi^2 < 12\)

2(1). Draw the interval on the number line: \((-4, 0]\)

3\(\hat{e}\)(1). Rewrite without \(||\)'s: \(|x-3| + |x-4|\) for \(x > 4\)
   Answer has 4 symbols

4\(\hat{e}\)(1). Rewrite without \(||\)'s: \(|x+1| + 4|x+3|\) for \(x < -3\)
   6 symbols

5(1). Write using \(||\)'s:
   The distance between \(x\) and 1 is less than \(\frac{1}{2}\).
   9 symbols

6\(\hat{e}\)(1). Write as an interval: \(\{x: |x-4| < 4\}\)
   5 symbols

7\(\hat{e}\)(1). Write as a union of two intervals: \(\{x: |x+5| \geq 2\}\)
   14 symbols
Math 140  Quiz  0  Name ________Practice____________ Score___/14pts,10mins

See if you can do these "practice quiz" problems on your own in the time limit given. They will be covered in the lab sessions. Problems on the exams will be very similar to these problems and problems on the practice exams. You should be able to complete each quiz in the time listed after the score. You need this level of proficiency to complete an exam on time. “(5)” after the problem number means the corresponding final exam problem will probably have 5 points.

1(5). Rewrite without |’s: 2|4 − x| − |x − 7| for x ∈ [4, 6].

2|4 − x| − |x − 7| = 2(x − 4) − (7 − x) = 2x − 8 − 7 + x = 3x − 15

Answer:

2(4). Write in interval notation: \{x : |x + 2| ≤ 3\}.

|x + 2| ≤ 3 ⇔
−3 ≤ x + 2 ≤ 3 ⇔
−5 ≤ x ≤ 1 ⇔
Answer: x ∈ [−5, 1]

3(5). Write in interval notation: \{x : |x − 1| > 2\}.

|x − 1| > 2 ⇔
x − 1 < −2 or 2 < x − 1 ⇔
x < −1 or 3 < x ⇔
Answer: x ∈ (−∞, −1) ∪ (3, ∞)
Rewrite each expression in a form that does not contain absolute values.

A. \(|x - 3| + |x - 4|\) where \(x < 3\).

B. \(|x - 3| + |x - 4|\) where \(3 < x < 4\).

C. \(|x + 1| + 4|x + 3|\) where \(-\frac{5}{2} < x < -\frac{3}{2}\).

Rewrite each statement using absolute values.

D. The distance between \(x\) and 4 is 8.

E. The distance between \(x\) and 1 is \(\frac{1}{2}\).

F. The distance between \(x\) and 1 is at least \(\frac{1}{2}\).

G. The distance between \(y\) and -4 is less than 1.

H. The number \(y\) is less than three units from the origin.

Write the following in interval notation.

I. \(|x| < 4\).

J. \(|x| > 1\).

K. \(|x - 5| < 3\).

L. \(|x - 3| \leq 4\).

M. \(|x + \frac{1}{2}| < \frac{3}{2}\).

N. \(|x - 5| \geq 2\).

Answers

A. \(-2x + 7\).

B. 1.

C. \(3x + 11\).

D. \(|x - 4| = 8\).

E. \(|x - 1| = \frac{1}{2}\).

F. \(|x - 1| \geq \frac{1}{2}\).

G. \(|y + 4| < 1\).

H. \(|y| < 3\).

I. \((-\infty, -1) \cup (1, \infty)\).

J. \((-\infty, -1) \cup (1, \infty)\).

K. (2, 8).

L. \([-1, 7]\).

M. \((-\frac{11}{6}, \frac{7}{6})\).

N. \((-\infty, 3] \cup [7, \infty)\).
Math 140     Hw 0     Worked examples of selected recommended problems.

Rewrite each expression in a form that does not contain absolute values.

C. \(|x + 4| + 4|x + 3| \) where \(-\frac{5}{2} < x < -\frac{3}{2}\).

First we need the following facts from Lecture 1:

\(|w| = w \) if \(w \geq 0\). Example, \(|3| = 3\).

\(|w| = -w \) if \(w < 0\). Example, \(|-3| = -(-3) = 3\).

In \(-w\), the minus sign cancels the minus sign in \(w\).

To get rid of the absolute value sign, you must determine whether \(w\) is positive or negative.

Is \(|x + 4| \) positive or negative?

\(x < -\frac{5}{2} \Rightarrow x + 4 < -\frac{5}{2} + 4 = -\frac{3}{2} < 0\).

\(\therefore x + 4 \) is not negative.

\(\therefore |x + 4| = -(x + 4)\).

Is \(|x + 3| \) positive or negative?

\(-\frac{5}{2} < x \Rightarrow -\frac{5}{2} + 3 < x + 3 \Rightarrow 0 < \frac{1}{2} < x + 3\).

\(\therefore x + 3 \) is positive.

\(\therefore |x + 3| = x + 3\).

\(\therefore |x + 4| + 4|x + 3| = -(x + 1) + 4(x + 3)\)

\(-x - 1 + 4x + 12 = 3x + 11\)

\(\text{Answer: } 3x + 11\)

Each of problems 51 - 61 can be solved by

- using key numbers,
- restating the absolute value as a distance, or
- restating the problem in terms of inequalities.

Use any method you wish.

Write the following in interval notation.

L. \(|x - 3| < 4\).

We’ll solve this one using key numbers. First solve the equality. The solutions to the equality are “key numbers” which divide the line into “key intervals”.

Plug in numbers from each interval to determine which intervals satisfy the inequality.

First solve the equality \(|x - 3| = 4\).

In general, \(|w| = a \) iff \(w = a \) or \(w = -a\).

For example, \(|w| = 3 \) iff \(w = 3 \) or \(w = -3\).

\(\therefore |x - 3| = 4 \) iff \(x - 3 = 4 \) or \(x - 3 = -4\)

if \(x = 7 \) or \(x = -1\).

The key numbers: \(-1\), \(7\).

The key intervals: \((-\infty, -1\), \([-1, 7\), \([7, \infty)\).

To find out which intervals satisfy the inequality, pick a number from each interval and see if it makes the inequality true.

\(|x - 3| \leq 4 \) is false for \(x = -2 \in (-\infty, -1\) \)

\(|x - 3| \leq 4 \) is true for \(x = 0 \in [-1, 7]\)

\(|x - 3| \leq 4 \) is false for \(x = 10 \in [7, \infty\) \)

\(\therefore \text{Answer: } |x - 3| \leq 4 \) iff \(x \in [-1, 7]\).

Note that we use “[ ]” rather than “( )” since the inequality \(\leq\) means the endpoints are included. If the inequality have been <, the endpoints would be excluded and the answer would have been \((-1, 7\) instead of \([-1,7]\).

M. \(|x + \frac{1}{3}| < \frac{3}{2}\).

We’ll solve this problem by restating it in terms of a distance.

\(|a - b|\) is the distance between \(a\) and \(b\).

\(\therefore |x + \frac{1}{3}| = |x - (-\frac{1}{3})| = \) the distance between \(x\) and \(-\frac{1}{3}\).

\(\therefore |x + \frac{1}{3}| < \frac{3}{2}\)

iff the distance between \(x\) and \(-\frac{1}{3}\) is less than \(\frac{3}{2}\)

iff \(x\) is between \(-\frac{1}{3} - \frac{3}{2}\) and \(-\frac{1}{3} + \frac{3}{2}\).

iff \(-\frac{7}{6} < x < \frac{5}{6}\)

\(\therefore \text{Answer: } |x + \frac{1}{3}| < \frac{3}{2}\) iff \(x \in \left(-\frac{7}{6}, \frac{5}{6}\right)\).

If the problem had been \(|x + \frac{1}{3}| > \frac{3}{2}\), the answer would have been \(x \in (-\infty, -\frac{11}{6}) \cup \left(\frac{5}{6}, \infty\right)\).

N. \(|x - 5| \geq 2\).

We’ll solve this one using inequalities. You can solve inequalities using the same rules you use for equalities with the following exceptions.

When you multiply or divide by a negative number, you must change the direction of the inequality.

You may not multiply or divide by a number if you do not know its sign (positive or negative).

In general,

\(1\) \(|w| \leq a \) iff \(-a \leq w \) and \(w \leq a \) iff \(-a \leq w \leq a\).

\(2\) \(|w| \geq a \) iff \(w \leq -a\) or \(a \leq w\).

In case (2) we have an “or” rather than the “and” of case (1).

\(\therefore \)

\(|x - 5| \geq 2\)

iff \((x - 5) \leq -2\) or \(2 \leq (x - 5)\) (add \(5\) to both side)

iff \(x \leq 3\) or \(7 \leq x\)

\(\therefore \text{Answer: } x \in (-\infty, 3] \cup [7, \infty)\)

If the problem had been \(|x - 5| < 2\), the solution would have been \(|x - 5| < 2 \iff -2 < x - 5 < 2 \iff 3 < x < 7\)

And the answer would have been: \(x \in (3, 7]\).
Math 140  Lecture 1

Intervals, absolute value, domains, polynomials

**Interval notation.**

- \( x \in [1, 3] \) iff \( 1 \leq x \leq 3 \)
- \( x \in (1, 3) \) iff \( 1 < x < 3 \)
- \( x \in (1, \infty) \) iff \( x < 1 \)
- \( x \in (-\infty, -1) \cup [1, \infty) \) iff \( x \leq -1 \) or \( 1 \leq x \)

**Absolute value.**

\[ |3| = 3, \quad |3| = 3, \quad |0| = 0. \]

Negating a negative makes it positive, \((-3)=3). In general

\[ |x| = x \text{ if } x \geq 0. \]

\[ |x| = -x \text{ if } x < 0. \quad \text{Alternatively, } |x| = \text{ its distance to } 0. \]

- **Rewrite without \( |\cdot|'s: \**

\[ \pi - \sqrt{2} = \pi - \sqrt{2} \quad \text{since } \pi > \sqrt{2} \Rightarrow \pi - \sqrt{2} > 0 \]

\[ \sqrt{2} - \pi = -(\sqrt{2} - \pi) = \pi - \sqrt{2} \]

- **Rewrite without \( |\cdot|'s: \**

\[ x - 2 | x - 6 | \text{ for } x \in (3, 5). \]

\[ 3 < x < 5 \Rightarrow 3 - 2 < x - 2 < 5 - 2 \Rightarrow 1 < x - 2 \Rightarrow x = 2 - 1 \]

\[ 3 < x < 5 \Rightarrow 3 - 6 < x - 6 < 5 - 6 \Rightarrow x - 6 < 1 \Rightarrow \]

\[ |x - 6| = -(x - 6) = -x + 6 \]

\[ \therefore \quad |x - 2| + 2|x - 6| = (x - 2) + 2(-x + 6) = -x + 10 \]

**Distance.** \( |a - b| = \text{ the distance between } a \text{ and } b. \)

**Note.** \( |x| < 3 \iff -3 < x < 3 \) iff \( x \in (-3, 3) \).

\[ |x| > 3 \iff x < -3 \text{ or } 3 < x \text{ iff } x \in (-\infty, -3) \cup (3, \infty). \]

- **Write as an interval:** \( \{ x : x + 7 < 3 \} \)

\[ x + 7 < 3 \iff x < -4 < 3 \text{ iff } -10 < x < -4. \]

Answer: \( \{ x : x + 7 < 3 \} = (-10, -4). \)

- **Write as a union of two intervals:** \( \{ x : x - 5 \geq 2 \} \)

\[ x - 5 \geq 2 \iff x - 5 \leq -2 \text{ or } 2 \leq x - 5 \]

\[ x \leq 3 \text{ or } 7 < x. \]

Answer: \( \{ x : x - 5 \geq 2 \} = (-\infty, 3] \cup [7, \infty). \)

**Definition.** The domain of a function (real values, real variables) is the set of numbers for which it is defined.

- **Where is \( \frac{x - 1}{(x - 2)(x - 3)} \) not defined?** Its domain is \( x \neq 2, 3. \)

- **Where is \( \sqrt{x - 2} \) not defined?** Its domain is \( x \geq 2. \)

- **Solve \( \sqrt{3x + 4} - x = 0. \) Get radical on one side. Square. Check validity: put solutions into the original equation.**

\[ \sqrt{3x + 4} = x, \quad 3x + 4 = x^2, \quad x^2 - 3x - 4 = 0, \quad (x + 1)(x - 4) = 0, \quad x = -1, 4. \]

\[ \sqrt{3(-1) + 4} - (-1) = \sqrt{1} + 1 = 0, \quad x = -1 \text{ is invalid.} \]

\[ \sqrt{3(4) + 4} - (4) = \sqrt{16} - 4 = 0, \quad x = 4 \text{ is valid.} \]

Know what the following terms mean: polynomial, degree, coefficients, quadratic function, expanded form, factored form mean. Know how to factor a polynomial and use the quadratic formula.

\[
(x + 1)^2 = x^2 + 2x + 1 \\
(x + 1)^2 \quad \text{is the factored form,} \\
x^2 + 2x + 1 \quad \text{is the expanded form.}
\]

**Solving inequalities**

As with equalities, you may add or subtract anything from both sides. You may multiply or divide both sides by a positive number. If you multiply or divide by a negative number, you must change the direction of the inequality (multiplying \( 5 > 3 \) by -1 gives \( -5 < -3 \). If you don’t know if a number is positive or negative, don’t multiply or divide by it.

- \( 2x \leq 6 \Rightarrow x \leq 3 \quad \text{(divide by } 2) \)
- \( -2x \leq 6 \Rightarrow x \geq -3 \quad \text{(divide by } -2; \text{ change sign direction)} \)
- \( x < \frac{1}{2} \text{ doesn’t imply } x^2 < 1 \text{ (don’t know if } x > 0 \text{ or } x < 0) \)

To solve problems like \( x < \frac{1}{2} \), use the **key-number method.**

- **Rewrite with 0 on the right. Factor the } f(x) \text{ on the left. } \]

\( f(x) < 0, f(x) > 0, f(x) \leq 0, \text{ or } f(x) \geq 0. \)

- **Find key numbers } x \text{ where } f(x) = 0 \text{ or } f(x) \text{ is undefined.} \]

- **On the key intervals before, between, and after key numbers, } f(x) \text{ is either } > 0 \text{ or } < 0. \text{ To find out which, evaluate } f(x) \text{ at some point inside the interval. You don’t have to find the value, just the sign.} \]

- **Use } ( \text{‘s with } <, \text{ with } >, \text{ around } \pm \infty, \text{ and where } f(x) \text{ is undefined.} \]

- **Use } [ \text{‘s if } f(x) \text{ is defined and the inequality is } \leq \text{ or } \geq . \]

- **Solve for } x: \quad x < \frac{1}{x}. \text{ Write as a union of two intervals.} \]

\[
x < \frac{1}{x} \\
\text{iff } x - \frac{1}{x} < 0 \\
\text{iff } \frac{x^2 - 1}{x} < 0 \\
\text{iff } \frac{(x-1)(x+1)}{x} < 0 \quad \therefore f(x) = \frac{(x-1)(x+1)}{x} \]

**Key numbers:** \( x = -1, 0, 1. \)

**Key intervals:** \( (-\infty, -1), (-1, 0), (0, 1), (1, \infty). \)

\[ f(-2) = - + , f(-\frac{1}{2}) = - + , f(\frac{1}{2}) = - , f(2) = +. \]

We picked \(-2 \in (-\infty, -1), -\frac{1}{2} \in (-1, 0), \frac{1}{2} \in (0, 1), 2 \in (1, \infty). \)

\[ \therefore x < \frac{1}{x} \quad \text{iff } f(x) < 0 \quad \text{iff} \]

Answer: \( x \in (-\infty, -1) \cup (0, 1) \)

- **Solve for } x: \quad x^2 + x - 6 \leq 0. \text{ Put answer in interval notation.} \]

First, factor the function.

\( f(x) = x^2 + x - 6 = (x + 3)(x - 2). \)

\( f(x) = 0 \text{ iff } x = -3, 2. \text{ These are the key numbers.} \)

The 3 key intervals: \( (-\infty, -3), [-3, 2], [2, \infty). \)

Picking \(-4, 0, 3): \quad f(-4) > 0, \quad f(0) < 0, \quad f(3) > 0. \)

\[ \therefore x^2 + x - 6 \leq 0 \text{ iff} \]

Answer: \( x \in [-3, 2] \)

- **Write as a union of two intervals:** \( \frac{x - 1}{x^2 - x - 2} \geq 0 \)

\[ \frac{x - 1}{x^2 - x - 2} \geq 0 \text{ iff } \frac{x - 1}{(x - 1)(x + 2)} \geq 0. \]

**Key numbers:** \(-1, 1, 2. \text{ Undefined at } -1, 2 \)

**Key intervals:** \( (-\infty, -1), (-1, 1), [1, 2], (2, \infty). \)

\[ f(-2) < 0, f(0) > 0, f(3/2) < 0, f(3) > 0 \]

\[ \therefore \quad \text{Answer: } x \in (-1, 1] \cup (2, \infty) \]
Practice 3(4). Find all real solutions \( x \). Write \( \emptyset \) if there are none.

\[
\sqrt{3 + 2x} + x = 0
\]

\[
\Rightarrow x = -\sqrt{3 + 2x}
\]

Get radical by itself on the right side, everything else on the left.

\[
\Rightarrow x^2 = 3 + 2x
\]

Since this is \( \Rightarrow \) rather than \( \Leftrightarrow \), we must check for validity at the end.

E.g. \( x = \sqrt{3} \Rightarrow x^2 = 3 \) but \( x^2 = 3 \) does not imply \( x = \sqrt{3} \) since we could have \( x = -\sqrt{3} \).

\[
\Rightarrow x^2 - 2x - 3 = 0
\]

\[
\Rightarrow (x + 1)(x - 3) = 0
\]

\[
\Rightarrow x = -1, x = 3
\]

Now check for validity.

Stick the proposed solutions in the original equation. Check that the square roots are defined and that equality holds.

For \( x = 3 \),

\[
\sqrt{3 + 2x} + x = 0
\]

\[
\Rightarrow \sqrt{3 + 2(3)} + (3) = 0
\]

\[
\Rightarrow \sqrt{9} + 3 = 0
\]

\[
\Rightarrow 3 + 3 = 0 \Leftrightarrow \text{false}
\]

\[\therefore x = 3 \text{ is invalid}\]

For \( x = -1 \),

\[
\sqrt{3 + 2x} + x = 0
\]

\[
\Rightarrow \sqrt{3 + 2(-1)} + (-1) = 0
\]

\[
\Rightarrow \sqrt{1} - 1 = 0 \Leftrightarrow \text{true}
\]

Answer: \( x = -1 \)

Practice 4(7). Solve for \( x \). \( x - 2 \leq \frac{3}{x} \) Write the answer as a union of two intervals.

\[
x - 2 \leq \frac{3}{x}
\]

\[
\Rightarrow x - 2 - \frac{3}{x} \leq 0
\]

Get everything on the left side, 0 on the right.

\[
\Rightarrow \frac{x^2 - 2x - 3}{x} \leq 0
\]

\[
\Rightarrow \frac{(x + 1)(x - 3)}{x} \leq 0
\]

Key numbers:

\( x = -1, 0, 3 \)

Key intervals:

\( (-\infty, -1], [-1, 0), (0, 3], [3, \infty) \)

Key values:

\[
f(x) = \frac{(x + 1)(x - 3)}{x} \leq 0
\]

\[
f(-2) = \frac{(-)(-)}{(-)} = -\checkmark
\]

\[
f(-\frac{1}{2}) = \frac{(+)(-)}{(-)} = +
\]

\[
f(1) = \frac{(+)(-)}{(+) = -\checkmark}
\]

\[
f(4) = \frac{(+)(+)}{(+)} = +
\]

\[\therefore f(x) \leq 0 \Leftrightarrow\]

Answer: \( x \in (-\infty, -1] \cup (0, 3] \)
Classwork 2 is hard, work on it before class.

Problem 3. Find all real solutions \( x \). \( \sqrt{5 + 4x} + x = 0 \) Write \( \emptyset \) if there are none.

Get radical by itself on the right side, everything else on the left. Square and solve. \( \sqrt{\_\_\_\_} / 2 \)

Then check for validity; show which is valid/invalid. Stick each solution into the original equation.

If equality holds, the solution is valid; if not, it is invalid. The one correct answer is a single positive or single negative digit. Show why one works and the other does not. Show your work in the space below, not on a separate sheet of paper.

Problem 4. Solve for \( x \). \( x \geq \frac{2}{x+1} \)

Get everything on the left side, 0 on the right. Combine fractions, factor. E.g. like \( \frac{x+4}{(x-5)(x+2)} < 0 \) or \( \frac{(x+3)(x-4)}{x} \geq 0 \).

List the key numbers (where left side is 0 or undefined) and plot them on the line. Should be two solid circles, one hollow circle.

In at least one of the four intervals, pick a point and calculate the sign of its key value - show your work.

On the number line, mark each key interval with “+” or “−”.

The answer is a union of two intervals. 13 symbols after “\( x \in \)”. Should be 2 brackets, 2 parentheses. \( \_\_\_\_ / 2 \)
Find all real solutions $x$. Write $\emptyset$ if there are none.
One problem has no solutions, one has one, one has two.

1. $x(x-1) = -4$

2. $\sqrt{3x+1} = 4$
   One solution, a positive integer

3. $\frac{5}{x+2} - \frac{2x-1}{5} = 0$
   Two solutions: one integral, the other a rational

In 6, $A$, rewrite with 0 on the right. List the key numbers.
Solve for $x$. Write the answer in interval notation.

6(2). $\frac{2x}{x-2} < 3$
   Union of two intervals with 12 symbols, integral endpoints

§1.5 page 55 problems 75-78, 81-83, 94.
For superscript $E$ problems, see notes in the section titled “Errors”.

§1.7 page 84 problems 47-60. Odds have answers in the back.
Solve for $x$. Write answer in interval notation. Two points each. A number is integral if it is an integer (positive or negative).

4$E$(2). $\frac{x+4}{2x-5} \leq 0$
   Single interval, 8 symbols

5(2). $\frac{x^2-1}{x^2+8x+15} \geq 0$
   Union of three intervals, 21 symbols, integral endpoints

7(2). $x - \frac{10}{x-1} \geq 4$
   Union of two intervals with 12 symbols, integral endpoints
Find all real solutions $x$. Write $\emptyset$ if there are no solutions.

A. $(3x - 2)^2 = 3x - 2$

B. $3x^2 + 4x - 3 = 0$

C. $x^2 = 24$

D. $x(x - 1) = 1$

E. $\frac{1}{2}x^2 - x - \frac{1}{3} = 0$

F. (a) $\sqrt{x - 8} = 4$
   (b) $\sqrt{x - 8} = 2x + 1$

G. $\sqrt{1 - 3x} = 2$

H. $\frac{3}{x+5} + \frac{4}{x} = 2$

I. $1 - x - \frac{2}{6x+1} = 0$

J. $\frac{3x^2 - 6x - 3}{(x+1)(x-2)(x-3)} + \frac{5-2x}{x^2-5x+6} = 0$

K. $\frac{2x}{x^2-1} - \frac{1}{x+3} = 0$

Solve for $x$. Write your answer in interval notation.

L. $(x-4)(2x^2 - 6x - 1) < 0$

M. $x^3 + 2x^2 - x - 2 > 0$

N. $\frac{x-1}{x+1} \leq 0$

O. $\frac{2-x}{3-2x} \geq 0$

P. $\frac{x^2-8x-9}{x} < 0$

Q. $\frac{2x^3+5x^2-7x}{3x^2+7x+4} > 0$

R. $\frac{x}{x+1} > 1$

S. $\frac{3-2x}{3+2x} > \frac{1}{x}$

T. $1 + \frac{1}{x} \geq \frac{1}{x+1}$

Answers

A. $x = \frac{2}{3}, 1$

B. $x = (-2 \pm \sqrt{13})/3$

C. $x = \pm 2\sqrt{6}$

D. $x = (1 \pm \sqrt{5})/2$

E. $x = (3 \pm \sqrt{15})/3$

F. (a) $x = 24$ (b) $\emptyset$

G. $x = -1$

H. $x = \frac{5}{2}, -4$

I. $x = \frac{1}{2}, \frac{1}{3}$

J. $x = 1$

K. $x = -3 \pm 2\sqrt{2}$

L. $(-\infty, \frac{3-\sqrt{11}}{2}) \cup (\frac{3+\sqrt{11}}{2}, 4)$

M. $(-2, -1) \cup (1, \infty)$

N. $(-1, 1]$  

O. $(-\infty, \frac{5}{2}) \cup [2, \infty)$

P. $(-\infty, -1) \cup (0, 9)$

Q. $(-\frac{7}{2}, -\frac{4}{3}) \cup (-1, 0) \cup (1, \infty)$

R. $(-\infty, -1)$

S. $(-\frac{3}{2}, 0)$

T. $(-\infty, -1) \cup (0, \infty)$
Find all real solutions $x$. Write \( \emptyset \) if no solution.

A \((3x-2)^2 = 3x-2\)

Don’t divide both sides by \(3x-2\), you could be dividing by 0. First get everything on the left, 0 on the right. Then factor or use the quadratic formula.
\[
(3x-2)^2 - (3x-2) = 0 \\
(3x-2)(3x-3) = 0 \\
3(3x-2)(x-1) = 0.
\]

Answer: \( x = \frac{2}{3}, 1 \)

D. \( x(x-1) = 1 \)

\[
x(x-1) - 1 = 0 \\
x^2 - x - 1 = 0
\]

Doesn’t factor, so use quadratic formula.
\[
Answer: x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}
\]

F. \( \sqrt{x-8} = 4 \)

Square both sides (when you do this, you may introduce invalid answers. Delete any invalid answers at the end.)
\[
x - 8 = 16 \\
x = 24
\]

Sticking \( x \) back into \( \sqrt{x-8} = 4 \) gives \( 24 - 8 = 4 \) which is true.
Answer: 24.

(b) \( \sqrt{x-8} = 2x + 1 \)

\[
x - 8 = (2x+1)^2 \\
x - 8 = 4x^2 + 4x + 1 \\
x - 8 - 4x^2 - 4x - 1 = 0 \\
-4x^2 - 3x - 9 = 0 \\
4x^2 + 3x + 9 = 0
\]

Doesn’t factor, so use quadratic formula.
\[
x = \frac{-3 \pm \sqrt{-153}}{8}. \\
Answer: no solutions: \( \emptyset \).
\]

I. \( 1 - x - \frac{2}{6x+1} = 0 \)

\[
(6x+1)(1-x) - 2 = 0 \\
(6x+1)(-6x^2+5x+1)-2 = 0 \\
-6x^2+5x+1 = 0 \\
-6x^2+5x+1 = 0 \\
-3x(1)(2x-1) = 0 \\
A rational is zero iff the top is zero. \\
Answer: x = \frac{1}{2}, \frac{1}{3}
\]

J. \( \frac{3x^2-6x-3}{(x+1)(x-2)(x-3)} + \frac{5-2x}{x^2-5x+6} = 0 \)

First factor as much as possible and simplify.
\[
\frac{3x^2-6x-3}{(x+1)(x-2)(x-3)} + \frac{5-2x}{(x-3)(x-2)} = 0 \\
\frac{3x^2-6x-3}{(x+1)(x-2)(x-3)} + \frac{5-2x}{(x+1)(x-2)(x-3)} = 0 \\
\frac{3x^2-6x-3}{(x+1)(x-2)(x-3)} + \frac{-2x^2+3x+5}{(x+1)(x-2)(x-3)} = 0 \\
\frac{x^2-3x+2}{(x+1)(x-2)(x-3)} = 0
\]
Proof. To find the slope, set y to 0 and solve for x. To find the y-intercepts, set x to 0 and solve for y.

\[ \text{Find the x and y-intercepts.} \]

\[ y - x^2 + 4 = 0 \]

x-intercepts (set y = 0):

\[ -x^2 + 4 = 0, \quad x = \pm 2 \]

The slope of the line through \((a, b)\) and \((c, d)\) is given by:

\[ m = \frac{b - d}{a - c} \]
Practice 5(10). For the circle \( x^2 - 6x + y^2 + 4y + 4 = 0 \):

(a) Find the center and radius.

(b) Find the x and y-intercepts.

Given equation: \( x^2 - 6x + y^2 + 4y + 4 = 0 \)

Write the equation in the form: \( (x - a)^2 + (y - b)^2 = r^2 \)

Get the constant on the right, group the variables together.

\[ (x^2 - 6x) + (y^2 + 4y) = -4 \]

Complete the square:

Divide the coefficient of \( x \) (-6) and the coefficient \( y \) (4) by 2 \((\pm\frac{3}{2})\) and square \((x^2)\).

\[ \left(x - \frac{3}{2}\right)^2 + \left(y + 2\right)^2 = 3^2 \]

Answer: (a) center = \((3, -2)\), radius = 3.

(b)

x-intercept (set \( y = 0 \)): Use the original equation.

\[ x^2 - 6x + 0^2 + 4(0) + 4 = 0 \]

\[ x^2 - 6x + 4 = 0 \]

Try factoring first. Then try the quadratic formula.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{6^2 - 4(1)(4)}}{2} = \frac{6 \pm \sqrt{20}}{2} \]

\[ \frac{6 \pm \sqrt{20}}{2} = 3 \pm \frac{\sqrt{5}}{5} \]

Answer: x-intercepts: \( x = 3 \pm \frac{\sqrt{5}}{5} \)

y-intercept (set \( x = 0 \)): -2

\[ (0)^2 - 6(0) + y^2 + 4y + 4 = 0 \]

\[ y^2 + 4y + 4 = 0 \]

\[ (y + 2)(y + 2) = 0 \]

Answer: y-intercept: \( y = -2 \)

Practice 6(4). Find the equation for the line through \((-1, 4)\) and \((4, -2)\). Write the equation in the form \( y = mx + b \).

Given: \((x_1, y_1) = (-1, 4) \)

\((x_2, y_2) = (4, -2) \)

Slope: \( m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-2)}{(-1) - 4} = \frac{6}{-5} = -\frac{6}{5} \)

Equation:

\[ y - y_1 = m(x - x_1) \]

\[ y - 4 = -\frac{6}{5}(x - (-1)) \]

\[ y = -\frac{6}{5}x + \frac{6}{5} + \frac{20}{5} \]

\[ y = -\frac{6}{5}x + \frac{14}{5} \]

Answer: \( y = \left(-\frac{6}{5}\right)x + \frac{14}{5} \)
5. Write in completed square form: 
\[(x - a)^2 + (y - b)^2 = r^2\]
\[x^2 + (y + 3)^2 = 16 \quad \text{___}/.5\]

5b. For the circle \(x^2 - 8x + y^2 + 6y + 9 = 0\):

(a) Write the equation in completed square form:
\[(x - a)^2 + (y - b)^2 = r^2 \quad \text{___}/.5\]
Move the constant to the right side.
The \(x\) terms are already grouped together. So are the \(y\) terms.
In each group add the constant needed to complete the square.
Add the constants to both sides of the equation.

The form must be \((x - a)^2 + (y - b)^2 = r^2\) not \((x - a)^2 + (y + b)^2 = c\).
19 symbols.

(b) Find the center.
Remember “()”, e.g., “center = (1,2)” not “center = 1,2”. 6 symbols.
\[\text{___}/.5\]

(c) Find the radius. 1 symbol.
\[\text{___}/.5\]

(d) Find the \(x\)-intercept (use original equation). Set \(y = 0\) and solve for \(x\) using the quadratic formula.
\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{5 symbols after “x=”}. \]
\[\text{___}/.5\]

6. Find the equation for the line through (3, -1) and (1, 2).
Write the equation in the form \(y = mx + b\). 4 symbols.

(a) Find the slope: 
\[m = \frac{y_1 - y_2}{x_1 - x_2} \quad ? \quad 4 \text{ symbols.} \]
\[\text{___}/.5\]

(b) Find the equation. Use one of the given points, say \((x_1, y_1) = (3, -1)\) and the slope \(m\) above.
In the point-slope formula \(y - y_1 = m(x - x_1)\),
fill in the values for \(y_1, x_1, m\).
Rewrite in the form \(y = mx + b\). 11 symbols.
\[\text{___}/.5\]
§1.8 page 99: problems 89-94. Odds have answers in the back.

1(2). \(x^2 + (y + 1)^2 = 20\)

Integral center, radius has a radical

2E(2). \(x^2 + y^2 - 10x + 2y + 17 = 0\)

Integral center and radius

3(2). \(4x^2 - 4x + 4y^2 - 63 = 0\)

Integral radius, center has one fraction

5. \(3x - 4y = 12\)

6. \(y = x^2 - 4x - 12\)

7. \(y = x^2 - 4x + 12\)

8. \((x - 2)^2 - (y - 1)^2 = 9\)

9E. Find the slope of the line through (-3, 0) and (4, 9). The slope is a fraction.

Write the equations for the lines below in one of the following forms: \(y = mx + b\), \(y = mx\), \(y = b\) or \(x = a\).

In the remaining problems, the coefficients are all integers.

10. The line with slope 22 through (0, 0).

11. The line through (7, 9) and (-11, 9).

12. The line through (12, 13) and (13, 12).

13. The line with slope 0 and \(y\)-intercept 14.
Find the center and radius.

A. \((x - 3)^2 + (y - 1)^2 = 25\)

B. \(x^2 + y^2 = \sqrt{2}\)

C. \(x^2 + y^2 + 8x - 6y = -24\)

D. \(9x^2 + 54x + 9y^2 - 6y + 64 = 0\)

Find the x-intercept(s) and y-intercept(s). Write “none” if there are none. You don’t need to draw the graphs.

E. \(3x + 4y = 12\)

F. \(y = 2x - 4\)

G. \(x + y = 1\)

H. \(y = x^2 + 3x + 2\)

I. \(y = x^2 + x - 1\)

Find the slope of the line through the given points.

J. (a) \((-3, 2), (1, -6)\)
   (b) \((2, -5), (4, 1)\)
   (c) \((-2, 7), (1, 0)\)
   (d) \((4, 5), (5, 8)\)

K. (a) \((1, 1), (-1, -1)\)
   (b) \((0, 5), (-8, 5)\)
   (c) \((-1, 1), (1, -1)\)
   (d) \((a, b), (b, a)\)

Write the equations for the lines below in one of the following forms: \(y = mx + b\), \(y = mx\), \(y = b\) or \(x = a\).

L. (a) Slope \(m = -5\), \((-2, 1)\) is on the line.
   (b) Slope \(m = 4\), \((4, -4)\) is on the line.
   (c) Slope \(m = \frac{1}{3}\), \((-6, -2/3)\) is on the line.
   (d) Slope \(m = -1\), \((0, 1)\) is on the line.

M. (a) \((4, 8), (-3, -6)\) are on the line.
   (b) \((-2, 0), (3, -10)\) are on the line.
   (c) \((-3, -2), (4, -1)\) are on the line.

N. (a) Slope \(m = -4\), y-intercept = 7
   (b) Slope \(m = 2\), y-intercept = 3/2
   (c) Slope \(m = -4/3\), y-intercept = 14

Answers
A. center \((3, 1)\), radius 5
B. center \((0, 0)\), radius \(\sqrt{2}\)
C. center \((-4, 3)\), radius 1
D. center \((-3, \frac{1}{2})\), radius \(\sqrt{2}\)

E. x-intercept = 4; y-intercept = 3
F. x-intercept = 2; y-intercept = -4
G. x-intercept = 1; y-intercept = 1
H. x-intercept = -1, -2; y-intercept = 2
I. x-intercept = \((-1 \pm \sqrt{5})/2\); y-intercept = -1

J. (a) -2, (b) 3, (c) -7/3 , (d) 3
K. (a) 1, (b) 0, (c) -1, (d) -1
L. (a) \(y = -5x - 9\)  (b) \(y = 4x - 20\)
   (c) \(y = \frac{1}{3}x + \frac{4}{3}\)  (d) \(y = -x + 1\)
M. (a) \(y = 2x\)  (b) \(y = -2x - 4\)
   (c) \(y = \frac{2}{7}x - \frac{11}{7}\)
N. (a) \(y = -4x + 7\)  (b) \(y = 2x + \frac{3}{2}\)
   (c) \(y = -\frac{4}{3}x + 14\)
Find the center and radius.

C. $x^2 + y^2 + 8x - 6y = -24$
   $(x^2 + 8x) + (y^2 - 6y) = -24$
   $(\frac{8}{2})^2 = 4^2 = 16$, $(\frac{6}{2})^2 = 3^2 = 9$
   $(x^2 + 8x + 16) + (y^2 - 6y + 9) = -24 + 16 + 9$
   $(x + 4)^2 + (y - 3)^2 = 1$
   $(x - (-4))^2 + (y - 3)^2 = 1^2$
   Answer: center $(-4, 3)$, radius 1

D. $9x^2 + 54x + 9y^2 - 6y + 64 = 0$
   $(9x^2 + 54x) + (9y^2 - 6y) = 64$
   Divide by 9 to make to coefficients of $x^2$ and $y^2$ are 1.
   $(x^2 + 6x) + (y^2 - \frac{2}{3}y) = 64/9$
   $(\frac{6}{2})^2 = 3^2 = 9$, $(\frac{2}{3})^2 = (\frac{1}{3})^2 = \frac{1}{9}$
   $(x^2 + 6x + 9) + (y^2 - \frac{2}{3}y + \frac{1}{9}) = \frac{64}{9} + 9 + \frac{1}{9}$
   $(x + 3)^2 + (y - \frac{1}{3})^2 = \frac{64}{9} + 9 = -7 + 9 = 2$
   $(x - (-3))^2 + (y - \frac{1}{3})^2 = (\sqrt{2})^2$
   Answer: center $(-3, \frac{1}{3})$, radius $\sqrt{2}$

Find the x-intercept(s) and y-intercept(s). Write “none” if there are none. You don’t need to draw the graphs.

H. $y = x^2 + 3x + 2$
   - Intercepts (set $y = 0$):
     - $0 = x^2 + 3x + 2$
     - $(x + 2)(x + 1) = 0$
     - Answer: x intercepts are: $x = -2, -1$
   - Intercepts (set $x = 0$):
     - $y = 2$
     - Answer: y-intercept is: $y = 2$

I. $y = x^2 + x - 1$
   - Intercepts (set $y = 0$):
     - $0 = x^2 + x - 1$
     - $x^2 + x - 1 = 0$ Does't factor, use the quadratic formula.
     - $x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$
     - Answer: x intercepts are: $x = \frac{-1 \pm \sqrt{5}}{2}$
   - Intercepts (set $x = 0$):
     - $y = -1$
     - Answer: y-intercept is: $y = -1$

Find the slope of the line through the given points.

J. (a) $(-3, 2), (1, -6)$
   Answer: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-6)}{1 - (-3)} = \frac{8}{4} = -2$

Write the equations for the lines below in the one of the following forms: $y = mx + b, y = mx, y = b$ or $x = a$.

L. (d) Slope $m = -1$, $(0, 1)$ is on the line.
   - $y = mx + b$
   - $y = (-1)x + 1$
   - Answer: $y = -x + 1$

M. (c) $(-3, -2), (4, -1)$ are on the line.
   - $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(4) - (-2)}{(4) - (-3)} = -1 = \frac{1}{7}$
   - $y - y_1 = m(x - x_1)$
   - $y - (-2) = \frac{1}{7}(x - (-3))$
   - $y + 2 = \frac{1}{7}x + \frac{3}{7}$
   - $y = \frac{1}{7}x + \frac{3}{7}$
   - Answer: $y = \frac{1}{7}x + \frac{11}{7}$

N. (c) Slope $m = -4/3$, y-intercept = 14.
   - $y = mx + b$
   - Answer: $y = -\frac{4}{3}x + 14$
Factors and roots

Theorem. If \( a > 0 \), \( x^2 - a = (x - \sqrt{a})(x + \sqrt{a}) \). But \( x^2 + a \) has no roots and can’t be factored any more.

Division Law. If \( p(x)/d(x) \) has quotient \( q(x) \) and remainder \( r(x) \) then
\[
\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.
\]
Multiply by \( d(x) \) to get \( p(x) = d(x)q(x) + r(x) \).

\( d(x) \) divides into \( p(x) \) evenly iff the remainder is 0 iff \( p(x) = d(x)q(x) \) iff \( d(x) \) is a factor of \( p(x) \).

If \( d(x) \) is a factor of \( p(x) \), the other factor of \( p(x) \) is \( q(x) \), the quotient of \( p(x)/d(x) \).

Given \( p(x)/d(x) \), divide to get the quotient \( q(x) \) and remainder \( r(x) \). Write the answer in division law form:
\[
p(x) = d(x)q(x) + r(x).
\]

\[
\begin{align*}
x^3 + 1 & = x^2 + x + 1 + \frac{2}{x-1}, \quad x^3 + 1 = (x - 1)(x^2 + x + 1) + 2 \\
\end{align*}
\]

Check that the answer is correct for \( x = 0 \). For \( x = 0 \), we get \( 0 + 1 = (-1)(0 + 0 + 1) + 2 \), \( 1 = 1 \). √

X-intercept. \( a \) is a root or zero of \( p(x) \) iff \( p(a) = 0 \).

Theorem. \( a \) is a root of \( p(x) \) iff \( (x - a) \) is a factor of \( p(x) \). Note, rewrite \( (x + 3) \) as \( (x - (-3)) \).

To find all roots of \( p(x) \), completely factor \( p(x) \).

Factor the polynomial and find all roots.

- \( x + 2 \) Root: -2
- \( x^2 + 2 \) Fully factored as is, no roots.
- \( x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}) = (x - (\sqrt{2}))(x + \sqrt{2}) \) Rts: \( -\sqrt{2}, \sqrt{2} \)
- \( x^2 - 4x + 4 = (x - 2)^2 \) One repeated factor. Root: 2
- \( x^3 + 5x^2 + 8x + 4 \) given that -1 is a root.
\[
\frac{x^3 + 5x^2 + 8x + 4}{x+1} = x^2 + 4x + 4 = (x + 2)(x + 2) = (x - (-2))^2
\]

\( x^3 + 5x^2 + 8x + 4 = (x - (-1))(x - (-2))^2 \) Rts: -2, -1. List in order.
- \( x^3 - x^2 - 2x + 2 \) given that 1 is a root.
\[
\frac{x^3 - x^2 - 2x + 2}{x-1} = x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}) = (x - (\sqrt{2}))(x + \sqrt{2}),
\]

\( x^3 - x^2 - 2x + 2 = (x - 1)(x - (\sqrt{2}))(x + \sqrt{2}) \)
Roots: \( -\sqrt{2}, 1, \sqrt{2} \).
- \( 2x^2 + 2x - 2 \). Factor out the coefficient of \( x^2 \); find the roots with the quadratic formula; factor.
\[
2(x^2 + x - 1) \text{ Roots: } x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2a} = \frac{-1 \pm \sqrt{5}}{2} \\
\text{Factorization: } 2(x - \frac{-1 - \sqrt{5}}{2})(x - \frac{-1 + \sqrt{5}}{2})
\]

Functions

Definition. For sets \( A \) and \( B \), a function from \( A \) to \( B \) assigns a value \( f(x) \) in \( B \) to each \( x \) in \( A \). The domain of \( f \) is \( A \); the range of \( f \) is the set of all possible values \( f(x) \).

\( f(x) = x^2 \) is a function from real numbers to real numbers.
\[ \text{domain} = (-\infty, \infty) \text{ since } x^2 \text{ is defined for all numbers.} \]
\[ \text{range} = [0, \infty) \text{ since } x^2 \text{ can never be negative.} \]

Notation. Sometimes, instead of writing \( f(x) = x^2 \), we define a function by writing \( y = x^2 \).

Thus \( y \) is the value of the function. Since it depends on \( x \), \( y \) is the dependent variable. Since \( x \) ranges freely over the domain, it is the independent variable.

A function may assign only one value to each \( x \).
Thus \( y = \pm \sqrt{x} \) is not a function.

- Of \( f \) and \( g \), which are functions? (\( f \) isn’t, \( g \) is)

- Write each domain in interval notation.
  - \( y = 1 - x \) \((-\infty, \infty)\)
  - \( y = \frac{1}{1-x} \) \((-\infty, 1) \cup (1, \infty)\)
  - \( y = \sqrt{1-x} \) \((-\infty, 1]\)

\( f(x) = x^2 \). First add \( x \)'s around each \( x \): \((x)^2\). Simplify to an expanded polynomial.

\[
\frac{f(x) - f(a)}{x-a} = \frac{(x)^2 - (a)^2}{x-a} = x + a
\]

\[
\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - (x)^2}{h} = 2x + h
\]

To get \( f(x+h) \), replace \( x \) in \( f(x) = x^2 \) by \((x+h)\) to get \( f(x+h) = (x+h)^2 \). Note, \( f(x+h) \neq x^2 + h^2 \).

- \( g(x) = \frac{1}{x} - x \). Rewrite as \( \frac{1}{x} - (x) \). Simplify
\[
g(g(x)) = \frac{1}{g(x)} - g(x) = \frac{1}{\left(\frac{1}{x} - x\right)} = \frac{x}{x(1-x)} = \frac{x^2 - x}{x(1-x^2)} = \frac{x}{(1-x^2)}
\]

\( h(x) = \frac{1}{x} \). Simplify
\[
h(x+h) = \frac{1-(x+h)}{(h(x))} = \frac{1-x-h}{h} = \frac{1-x}{h} \]

\( h(h(x)) = \frac{1-(h(x))}{h} = \frac{1-\frac{1}{x}}{h} = \ldots = 2x-1 \).
7(6). Find the domain of \( f(x) = \frac{1}{x - \sqrt{x+30}} \). Interval notation.

- What is under the radical, \( x + 30 \), must be \( \geq 0 \).
  \[ x + 30 \geq 0 \]
  \[ \text{iff} \; x \geq -30 \]

- The denominator \( x - \sqrt{x+30} \) must not be 0.
  \[ x - \sqrt{x+30} = 0 \]
  \[ \text{iff} \; x = \sqrt{x+30} \]
  \[ \text{iff} \; x^2 = x + 30 \]
  \[ \text{iff} \; x^2 - x - 30 = 0 \]
  \[ \text{iff} \; (x - 6)(x + 5) = 0 \]
  \[ \text{iff} \; x = 6, -5 \]
  \[ 6 \text{ is valid,} \]
  \[ -5 \text{ is invalid} \]
  \[ \therefore x - \sqrt{x+30} = 0 \text{ iff } x = 6 \]
  \[ \therefore x - \sqrt{x+30} \neq 0 \text{ iff } x \neq 6 \]

- The domain consists of points satisfying both conditions.

  \text{Draw the number line. Cross off all points which don’t satisfy both conditions. What is left is your answer.}

  - \( x \in \text{the domain} \)
  - \( \text{iff } x \neq 6 \text{ and } x \geq -30 \text{ iff} \)
  - \( \text{Answer: } x \in [-30, 6) \cup (6, \infty) \)

  \text{Problems like this are often missed. Practice on homework 3’s recommended problems.}

8(6). Factor and find all roots given that -1 is one root.

\[ x^3 + x^2 - 3x - 3 \]

- Use the give root to get one factor.
  \( -1 \text{ a root } \Rightarrow x - (-1) = x + 1 \text{ is a factor.} \)

- Divide by this factor to get the quotient which is the other factor.
  \[ \frac{x^3 + x^2 - 3x - 3}{x+1} = x^2 - 3 \]

- Factor this quotient if possible.
  \[ x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3}) \]

- Complete factorization: \( x^3 + x^2 - 3x - 3 = (x + 1)(x + \sqrt{3})(x - \sqrt{3}) = (x - (-1))(x - (-\sqrt{3}))(x - \sqrt{3}) \)

- Roots in increasing order: \( x = -\sqrt{3}, -1, \sqrt{3} \)
17. Find the domain of \( f(x) = \frac{\sqrt{2-x}}{x^2-9} \).

Radical arguments must be \( \geq 0 \). Denominators must not be 0.

• When is the radical argument \( \geq 0 \)? __/5

• When is the denominator = 0? __/5

• Hence the denominator \( \neq 0 \) when \( x \neq \_ \) and \( x \neq \_ \).

• Find the domain. __/5

The answer is a union of two intervals. 14 symbols. 3 ( )'s, and 1 [ ]. If you answer includes numbers > 2, it is wrong.

18. Factor and find all roots given that \( x = -3 \) is one root. \( x^3 + 3x^2 - 3x - 9 \)

• What is the factor determined by the root \( x = -3 \)? __/5

• Divide the given polynomial \( x^3 + 3x^2 - 3x - 9 \) by this factor to get the quotient which is the other factor. 4 symbols __/5

• Factor this new factor. __/5

Write factors in the form \((x - \_\_\_\_\_\_\_)\) instead of \((x + \_\_\_\_\_\_)\).

• Find the factorization and the roots. __/5

Factorization:
\[ x^3 + 3x^2 - 3x - 9 = (x - \_\_\_\_) (x - \_\_\_\_\_) (x - \_\_\_\_) \]

List the 3 roots in increasing order: 3 roots, 2 with radicals.
\[ x = \_\_\_\_, \_\_\_\_, \_\_\_\_\_ \]
For superscript \( E \) problems, see notes in the section titled “Errors”. §3.2 page 270 problems 1-6. Odds have answers in back.

1(1). \( \frac{x^3-4x^2+x-2}{x-5} \)
2 digit positive remainder

2(1). \( \frac{x^6+64}{x-2} \)
3 digit positive remainder

3.2 271:55,56. §3.3 279:11-20.
Factor the two polynomials and find all the roots (zeros).
3(2). \( x^3 + 7x^2 + 11x + 5 = 0 \).  \(-1\) is one root.
2 roots (both integral), 2 factors (one repeated)

4(3). \( x^3 + 8x^2 - 3x - 24 = 0 \).  \(-8\) is one root.
3 roots (one integral), 3 factors

§1.7 85: 87-90. §2.1 156:41-45, 47-52, 54-57.
5(1). Find the domain.  \( y = 3x + 12 \)
1 big interval

6(1). Find the domain.  \( y = \frac{1}{3x+12} \)
union of 2 intervals

7(1). Find the domain.  \( y = \sqrt{3x + 12} \)
1 interval

§2.2 168:55-56.
8(1). List which (there are three) of \( f, g, F, G \) are functions.

9(½). \( f(x) = 4 - 3x.  f(x^2) = \)
2 term polynomial, 5 symbols

10(½). \( f(x) = 4 - 3x.  f(1/x) = \)
2 terms, one fractional, 5 symbols

11(½). \( f(x) = 4 - 3x.  f(f(x)) = \)
2 term polynomial, 4 symbols

12(½). \( f(x) = 4 - 3x.  x^2 f(x) = \)
2 term polynomial, 7 symbols

13(½). \( f(x) = 4 - 3x.  1/f(x) = \)
Single fraction, 6 symbols

14(1). \( f(x) = -2x + 9.  \frac{f(x+h)-f(x)}{h} = \)
Assume \( h \neq 0 \). 2-symbol integer.

15(1). \( g(x) = -2x^2 + 9.  \frac{g(x)-g(a)}{x-a} = \)
Assume \( x \neq a \). 6-symbol polynomial.
Math 140      Hw 3   Recommended problems, don't turn this in.

Divide to get the quotient $q(x)$, remainder $r(x)$. Write in the form: $p(x) = d(x)q(x) + r(x)$.
A. $\frac{x^2 - 8x + 4}{x - 3}$
B. $\frac{x^2 - 6x - 2}{x + 5}$
C. $\frac{6x^3 - 2x + 3}{2x + 1}$
D. $\frac{x^5 + 2}{x + 3}$
E. $\frac{x^6 - 64}{x - 2}$

Factor the polynomials and find all the roots (zeros).
F. $x^3 - 4x^2 - 9x + 36 = 0$ given $-3$ is a root.
G. $x^3 + x^2 - 7x + 5 = 0$ given $1$ is a root.
H. $3x^3 - 5x^2 - 16x + 12 = 0$ given $-2$ is a root.
I. $2x^3 + x^2 - 5x - 3 = 0$ given $-3/2$ is a root.

Write the domain in interval notation.
J. (a) $y = -5x + 1$.
   (b) $y = 1/( -5x + 1)$.
   (c) $y = \sqrt{-5x + 1}$.
   (d) $y = \sqrt[3]{-5x + 1}$.
K. (a) $f(t) = t^2 - 8t + 15$.
    (b) $g(t) = 1/(t^2 - 8t + 15)$.
    (c) $h(t) = \sqrt{t^2 - 8t + 15}$.
    (d) $k(t) = \sqrt[3]{t^2 - 8t + 15}$.

Let $f(x) = 3x^2$. Find the following.
(a) $f(2x)$.
(b) $2f(x)$.
(c) $f(x^2)$.
(d) $[f(x)]^2$.
(e) $f(x)/2$.
(f) $f(x)/2$.

Let $H(x) = 1 - 2x^2$. Find the following.
(a) $H(0)$.
(b) $H(2)$.
(c) $H(\sqrt{2})$.
(d) $H(5/6)$.
(e) $H(x + 1)$.
(f) $H(x + h)$.
(g) $H(x + h) - H(x)$.
(h) $\frac{H(x + h) - H(x)}{h}$.

Find the quotient $\frac{f(x + h) - f(x)}{h}$ for $f(x) = 8x - 3$.

Find the quotient $\frac{g(x) - g(a)}{x - a}$ for $g(x) = 4x^2$.

Find the quotient $\frac{g(x) - g(a)}{x - a}$ for $g(x) = x^2 - 2x + 4$.

Answers
A. $x^2 - 8x + 4 = (x - 3)(x - 5) - 11$
B. $x^2 - 6x - 2 = (x + 5)(x - 11) + 53$
C. $6x^3 - 2x + 3 = (2x + 1)(3x^2 - \frac{3}{2}x - \frac{1}{4}) + \frac{13}{4}$
D. $x^3 + 2 = (x + 3)(x^2 - 3x + 9x^2 - 27x + 81) - 241$
E. $x^6 - 64 = (x - 2)(x^5 + 2x^4 + 4x^3 + 8x^2 + 16x + 32)$
F. $±3, 4$. G. $1, -1 ± \sqrt{6}$. H. $-2$, $\frac{2}{3}$, $3$.
I. $-\frac{3}{2}$, $(1 ± \sqrt{5})/2$.
J. (a) $(−∞, ∞)$. (b) $(−∞, \frac{1}{3}]$.
   (c) $(−∞, \frac{1}{3}]$. (d) $(−∞, ∞)$.
K. (a) $(−∞, ∞)$. (b) $(−∞, 3) \cup (3, 5) \cup (5, ∞)$.
   (c) $(−∞, 3] \cup [5, ∞)$. (d) $(−∞, ∞)$.
L. (a) $12x^2$. (b) $6x^2$. (c) $3x^4$. (d) $9x^4$. (e) $\frac{3}{4}x^2$. (f) $\frac{3}{2}x^2$
M. (a) 1. (b) -7. (c) -3. (d) $-\frac{7}{18}$.
   (e) $-2x^2 - 4x - 1$. (f) $1 - 2x^2 - 4xh - 2h^2$.
   (g) $-4xh - 2h^2$. (h) $-4x - 2h$.
N. 8. O. $4x + 4a$. P. $x + a - 2$.  
Divide to get the quotient \( q(x) \), remainder \( r(x) \). Write in the form: \( p(x) = d(x)q(x) + r(x) \).

B. \( \frac{x^2 - 6x - 2}{x + 5} = \frac{x - 11}{x + 5} + \frac{53}{x + 5} \)  

\[ x^2 - 6x - 2 \]
\[ x + 5 \]
\[ x^2 + 5x \]
\[ = -11x - 2 \]
\[ + -11x - 55 \]
\[ = 53 \]

\[ \therefore \frac{x^2 - 6x - 2}{x + 5} = (x - 11) + \frac{53}{x + 5} \]

Multiplying by \( x + 5 \) gives

**Answer:** \( x^2 - 6x - 2 = (x + 5)(x - 11) + 53 \)

Factor the polynomials and find all the roots (zeros).

G. \( x^3 + x^2 - 7x + 5 = 0 \) given 1 is a root.

Since 1 is a root, divide \( x^3 + x^2 - 7x + 5 \) by \( x - 1 \).

The quotient is \( \frac{x^3 + x^2 - 7x + 5}{x - 1} = x^2 + 2x - 5 \).

Hence \( x^3 + x^2 - 7x + 5 = (x - 1)(x^2 + 2x - 5) \)

Now we must factor \( x^2 + 2x - 5 \).

The first factors to try are \( x \pm 1 \) and \( x \pm 5 \). Neither work, thus we must use the quadratic formula.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

\[ = \frac{-2 \pm \sqrt{4 - 4(1)(-5)}}{2(1)} \]

\[ = -1 \pm \sqrt{6} \]

Hence the factorization of \( x^2 + 2x - 5 \) is

\[ x^2 + 2x - 5 = [x - (-1 - \sqrt{6})][x - (-1 + \sqrt{6})] \]

Hence \( x^3 + x^2 - 7x + 5 \) factors into

**Answer:** \( (x - 1)[x - (-1 - \sqrt{6})][x - (-1 + \sqrt{6})] \)

The three roots are

**Answer:** \( 1, -1 \pm \sqrt{6} \).

H. \( 3x^3 - 5x^2 - 16x + 12 = 0 \) given -2 is a root.

Since -2 is a root, divide by \( x - (-2) = x + 2 \).

The quotient is \( \frac{3x^3 - 5x^2 - 16x + 12}{x + 2} = 3x^2 - 11x + 6 \)

Now we must factor \( 3x^2 - 11x + 6 \)

\[ 3x^2 - 11x + 6 = (3x - 2)(x - 3) \]

Hence \( 3x^3 - 5x^2 - 16x + 12 \) factors into

**Answer:** \( (x + 2)(3x - 2)(x - 3) = (x - 2)(3x - 2)(x - 3) \)

The three roots are

**Answer:** \( -2, \frac{2}{3}, 3 \).

Write domain in interval notation.

K. (a) \( f(t) = t^2 - 8t + 15 \).

There are no divisions by 0, no square roots of negative numbers. Hence \( f(t) \) is defined everywhere.

**Answer:** Domain \( (-\infty, \infty) \).

(b) \( g(t) = 1/(t^2 - 8t + 15) \).

Factoring gives

\[ g(t) = \frac{1}{(t-3)(t-5)} \]

Thus \( g(t) \) is undefined at \( t=3, 5 \).

**Answer:** Domain \( (-\infty, 3) \cup (3, 5) \cup (5, \infty) \).

(c) \( h(t) = \sqrt{t^2 - 8t + 15} \).

This is defined provided \( t^2 - 8t + 15 \geq 0 \).

By part (b), the key numbers are 3, 5.

The key intervals are \( (-\infty, 3), [3, 5], [5, \infty) \).

We use \( [ ] \) since the inequality is \( \geq \) rather than >.

Now evaluate \( t^2 - 8t + 15 \) at a point in each interval.

\[ 0 \in (-\infty, 3), 0^2 - 8 \cdot 0 + 15 = 15 = + \]

\[ 4 \in [3, 5], 4^2 - 8 \cdot 4 + 15 = -1 = - \]

\[ 10 \in [5, \infty), 10^2 - 8 \cdot 10 + 15 = 35 = + \]

Hence \( h(t) \) is undefined between 3 and 5.

**Answer:** (c) Domain \( (-\infty, 3] \cup [5, \infty) \).

(d) \( k(t) = \sqrt[3]{t^2 - 8t + 15} \).

You can't take the square root of a negative number but you can take the cube root of any number. Hence \( k(t) \) is defined everywhere.

**Answer:** Domain \( (-\infty, \infty) \).

L. Let \( f(x) = 3x^2 \). Find the following.

(c) \( f(x^2) \).

\[ f(x^2) = 3(x^2)^2 = 3x^4 \]

Answer: \( 3x^4 \).

(d) \( f([x]) \).

\[ f([x]) = [3x^2]^2 = 9x^4 \]

Answer: \( 9x^4 \).

(e) \( f(x/2) \).

\[ f(x/2) = 3(x/2)^2 = 3\frac{x^2}{4} \]

Answer: \( \frac{3}{4}x^2 \).

M. Let \( H(x) = 1 - 2x \). Find the following.

(f) \( H(x + h) \).

\[ H(x + h) = 1 - 2(x + h)^2 = 1 - 2(x^2 + 2xh + h^2) \]

Answer: \( 1 - 2x^2 - 4xh - 2h^2 \).

(h) \( \frac{H(x+h) - H(x)}{h} \).

\[ \frac{H(x+h) - H(x)}{h} = \frac{[1-2x^2-4xh-2h^2] - [1-2x^2]}{h} \]

\[ = \frac{1-2x^2-4xh-2h^2 - 1+2x^2}{h} \]

\[ = \frac{-4xh-2h^2}{h} = -4x - 2h \]

Answer: \( -4x - 2h \).

O. Find the quotient \( \frac{g(x)-g(a)}{x-a} \).

\[ g(x) = 4x^2 \]

\[ \frac{g(x)-g(a)}{x-a} = \frac{[4x^2] - [4a^2]}{x-a} = \frac{4(x^2-a^2)}{x-a} = \frac{4(x+a)(x-a)}{x-a} \]

Answer: \( 4x + 4a \)
Math 140 Lecture 4

Gateway exam. Practice exams included with lecture notes.

Definition. The graph of a function \( f \) is the set of all points \((x, y)\) such that \( y = f(x) \). The height \( y \) is the function’s value \( f(x) \).

Vertical line test. A curve is the graph of a function iff no vertical line intersects it more than once.

Which curve is the graph of a function?

\[
\begin{align*}
\text{Domain: } & (-1, 3] \\
\text{Range: } & \{2\}
\end{align*}
\]

Find the domain and range of each function.

Find the values.

\[
\begin{align*}
& f(1) = 2 \\
& f(5) = \text{undefined}
\end{align*}
\]

For what \( x \) is \( f(x) = 2? \)

\[
\begin{align*}
& x = 1, 3
\end{align*}
\]

Compute \( [g(x) - g(1)]/[x - 1] \) when \( x = 3.25 \).

\[
\begin{align*}
& x = 2
\end{align*}
\]

Fill in the table.

\[
\begin{array}{ccc}
\text{domain} & [-2, 5] & [0, 4] \\
\text{range} & [-1, 4] & [0, 3] \\
\text{turning points} & x = 0, 2 & x = 2 \\
\text{maximum value} & f(0) = 4 & g(2) = 3 \\
\text{minimum value} & f(2) = -1 & g(0) = g(4) = 0 \\
\text{interval(s) of increase} & [-2, 0], [2, 5] & [0, 2] \\
\text{interval(s) of decrease} & [0, 2] & [2, 4]
\end{array}
\]

Graph.

\[
f(x) = \sqrt{x^2 - 1}
\]

\[
\begin{align*}
& \frac{1}{x} \text{ if } x < 0 \\
& \sqrt{x} \text{ if } x \geq 0
\end{align*}
\]

Graph.

\[
f(x) = \begin{cases} 
1 & \text{if } x < 0 \\
1 & \text{if } 0 \leq x \leq 1 \\
1 & \text{if } 1 < x
\end{cases}
\]

Gateway problem. Complete the square of the formula.

\[
2x^2 - 3x + 5 = \ldots = 2(x - \frac{3}{4})^2 + \frac{31}{8}
\]
Math 140 Quiz 4 Name ________ Practice ___________________________ Total _____/26pts 19 mins

Practice problem numbers correspond to numbers of similar problems on the practice exams. Try to do practice quizzes in the time indicated at the top right corner. Classwork 4 is hard, work on it in advance. Your TA will probably skip problem 9(4) to give you an extra five minutes of group work time.

9(4). Simplify to a single reduced polynomial:

(a) \( g(x) = \frac{1}{1+x} \), rewrite \( g(x) = \frac{1}{1+x} \).

\[
\frac{1}{1+(x+h)} - \frac{1}{1+x} = \frac{1}{h} \left( \frac{1}{1+(x+h)} - \frac{1}{1+x} \right) = \frac{1}{h} \left( \frac{(1+x)-(1+x+h)}{1+(x+h)(1+x)} \right) = \frac{1}{h} \left( \frac{-h}{1+(x+h)(1+x)} \right) = -\frac{1}{1+(x+h)(1+x)}
\]

(b) \( f(x) = x + \frac{1}{x} \), rewrite \( f(x) = (x) + \frac{1}{(x)} \).

\[
\frac{1}{(x+h) \cdot (x+\frac{1}{x})} = \frac{1}{hx(x+h)}
\]

11(16). Fill in the table. For domains and ranges, use interval notation, e.g., \([1,5)\) or set notation, e.g., \(\{1,3,5\}\). Sometimes the correct answer is “none”. \(h(x)\) is graphed above. \(f(x)\) is the function listed, it is not graphed.

<table>
<thead>
<tr>
<th>function</th>
<th>( f(x) = 1 - x^2 )</th>
<th>( h(x) ), see graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>roots, ( x )</td>
<td>( x = -1, 1 )</td>
<td>( x = -2, 2, 6 )</td>
</tr>
<tr>
<td>domain</td>
<td>((-\infty, \infty))</td>
<td>([-2, 6])</td>
</tr>
<tr>
<td>range</td>
<td>((-\infty, 1])</td>
<td>([-2, 2])</td>
</tr>
<tr>
<td>turning point(s), ( x )</td>
<td>( x = 0 )</td>
<td>( x = 0, 4 )</td>
</tr>
<tr>
<td>max value(s) ( f(a) = b )</td>
<td>( f(0) = 1 )</td>
<td>( h(4) = 2 )</td>
</tr>
<tr>
<td>min value(s) ( f(a) = b )</td>
<td>none</td>
<td>( h(0) = -2 )</td>
</tr>
<tr>
<td>interval(s) of increase</td>
<td>((-\infty, 0])</td>
<td>([0, 4])</td>
</tr>
<tr>
<td>interval(s) of decrease</td>
<td>([0, \infty))</td>
<td>([-2, 0], [4, 6])</td>
</tr>
</tbody>
</table>

- In intervals of increase and decrease, include the endpoints if the function is defined, exclude them if not.
- For domains we take the union of the intervals \([-2, 0] \cup [4, 6]\). When listing the intervals of increase and decrease, list the intervals separately \([-2, 0], [4, 6]\).
9. Simplify the fraction over $h$ to a reduced ratio of two polynomials, e.g. $\frac{2x-1}{3x+h}$, not $\frac{x-1/x}{3+1/(x+h)}$.

(a) $g(x) = \frac{1}{3-x}$, rewrite as $\frac{1}{3-(x)}$ $g(x) + h = \frac{1}{3-(x)} + h$. $g(y + 5) = \frac{1}{3-(y+5)} = \frac{1}{3-y-5}$

Find the following:

$$g(x + h) =$$

Note: $\frac{1}{3-x} + h$ and $\frac{1}{3-x+h}$ are wrong. Distribute the "." to get 7 symbols. $\frac{g(x+h) - g(x)}{h} = $ The answer is $\frac{1}{(3-x)(3-x-h)}$ To get credit, you must show your work.

11. Fill in the table. For domains and ranges, use interval notation $[1,5)$ or set notation $\{1,3,5\}$. Sometimes the correct answer is none. Three answers are given. Note the format. When asked for a max value, write the answer in the form $f(a) = b$, not just the value $b$.

$$f(x) = -x^2 \text{ see graph above}$$

<table>
<thead>
<tr>
<th>function</th>
<th>$f(x) = -x^2$ see graph above</th>
<th>$h(x)$, see graph above</th>
</tr>
</thead>
<tbody>
<tr>
<td>roots</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>domain</td>
<td></td>
<td>3 roots</td>
</tr>
<tr>
<td>range</td>
<td>$(-\infty, 0]$</td>
<td></td>
</tr>
<tr>
<td>turning point(s)</td>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>max value(s) $f(a)=b$</td>
<td>$f(0) = 0$</td>
<td></td>
</tr>
<tr>
<td>min value(s) $f(a)=b$</td>
<td>none</td>
<td>$h(-2) = h(2) = -2$</td>
</tr>
<tr>
<td>interval(s) of increase</td>
<td>$(-\infty, 0]$</td>
<td>[0, 2], [6, 8]</td>
</tr>
<tr>
<td>interval(s) of decrease</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In intervals of increase and decrease, include the endpoints if the function is defined, exclude them if not.

When listing the intervals of increase and decrease, list the intervals separately $[-2, 0]$, $[4, 6]$. In contrast, for domains we take the union $[-2, 0] \cup [4, 6]$. For roots and turning points, we list only the $x$ coordinate, e.g. $x = 3$ not both coordinates $(3, 8)$. 

Math 140 Classwork 4

No credit unless you work with a group of three or more people. For examples, see Practice Quiz 4. Hard.
1(1). In the graph of \( y = \sqrt{1 - x^2} \), find the y-coordinate of the point with x-coordinate \(-\frac{1}{2}\). 4 symbols, exact answer, no decimals.

§2.2 167:24,23. 169:57-60.
In 2-4, list the domain and range of each graph (see text). Each domain is a single interval. For the ranges: 3, 4 use set notation e.g., \{1,3,5\}, 2 uses a combination e.g. \((-1,1] \cup \{2\} \).

2(1).

domain =
range =

3(1).

domain =
range =

4(1).

domain =
range =

5(2). For the function \( h \) pictured,

(a) \( h(a) = \)

(b) \( h(b) = \)

(c) \( h(c) = \)

(d) \( h(d) = \)

(c) For which values of \( x \) does \( h(x) = 0? \)

(e) As \( x \) increases from \( c \) to \( d \), does \( h(x) \) increase or decrease?

(f) As \( x \) increases from \( a \) to \( b \), does \( h(x) \) increase or decrease?

6(1). For the \( f \) pictured, compute

(a) \( \left[ f(x) - f(2) \right] / [x - 2] \) when \( x = 3 \).

(b) \( \left[ f(x) - f(-2) \right] / [x + 2] \) when \( x = -3 \).

Continued on next page
7(7). Fill in the table for $1/x$ and $\sqrt{x}$.
Seven of the answers are “none”. ½ point for each entry.

<table>
<thead>
<tr>
<th>function</th>
<th>$1/x$</th>
<th>$\sqrt{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>domain</td>
<td></td>
<td></td>
</tr>
<tr>
<td>range</td>
<td></td>
<td></td>
</tr>
<tr>
<td>turning point</td>
<td></td>
<td></td>
</tr>
<tr>
<td>max value</td>
<td></td>
<td></td>
</tr>
<tr>
<td>min value</td>
<td></td>
<td></td>
</tr>
<tr>
<td>interval(s) of increase</td>
<td></td>
<td></td>
</tr>
<tr>
<td>interval(s) of decrease</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

§2.2 168:38-44.  
8(2). Graph $U(x)$ where

$$U(x) = \begin{cases} 
  x^3 & \text{if } -1 \leq x < 0 \\
  \sqrt{x} & \text{if } 0 \leq x < 1 \\
  1/x & \text{if } 1 \leq x < 3 
\end{cases}$$

9(5/2). For the function pictured, find the range (single interval)

max value

min value

interval of increase

interval of decrease

10(5/2). For the function pictured, find the range (single interval)

max value

min value

two intervals of increase

two intervals of decrease

Continued from previous page
In A-D, list the domain and range of each graph.

A. For the function pictured,

\[\text{domain= }\]
\[\text{range= }\]

B. For the function pictured,

\[\text{domain= }\]
\[\text{range= }\]

C. For the function pictured,

\[\text{domain= }\]
\[\text{range= }\]

D. For the function \( f \) pictured,

\[\begin{array}{c}
\text{(a) Is } f(0) \text{ positive or negative?} \\
\text{(b) Find } f(-2) \quad f(1) \\
\quad f(2) \quad f(3) \\
\text{(c) Which is larger, } f(2) \text{ or } f(4)? \\
\text{(d) Find } f(4)-f(1). \\
\text{(e) Find } |f(4)-f(1)|. \\
\text{(f) Find (in interval notation) the domain and range of } f. \\
\text{domain} \\
\text{range}
\end{array}\]

E. Fill in the table for \(|x|\), \(x^2\), \(x^3\).

\[\begin{array}{c|c|c|c}
\text{function} & |x| & x^2 & x^3 \\
\hline
\text{domain} & & & \\
\text{range} & & & \\
\text{turning point} & & & \\
\text{max value} & & & \\
\text{min value} & & & \\
\text{interval(s) of increase} & & & \\
\text{interval(s) of decrease} & & & \\
\end{array}\]

Continued on next page.
In F, G, find the range, max value, min value, intervals of increase, and intervals of decrease. \(1/2\) point each part.

F. For the function pictured,

- range
- max value
- min value
- interval(s) of increase
- interval(s) of decrease

G. For the function pictured,

- range
- max value
- min value
- interval(s) of increase
- interval(s) of decrease

In H, I, A, graph the given function.

H. \(A(x) = \begin{cases} x^3 & \text{if } -2 \leq x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}\)

I. \(C(x) = \begin{cases} x^3 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x > 1 \end{cases}\)

J. \(f(x) = \begin{cases} 1/x & \text{if } x < -1 \\ x & -1 \leq x \leq 1 \\ 1/x & 1 < x \end{cases}\)
In $A-D$, list the domain and range of each graph.

$A.$ (see the graph in Hw 4 recommended problem 9).

Domain = the set of $x$-coordinates of points on the graph. The smallest $x$-coordinate is -3 from the point (-3, -2). The largest $x$-coordinate is 4 from the point (4, 1).

Answer: Domain = $[-3, 4]$

Range = the set of $y$-coordinates of points on the graph.
The lowest $y$-coordinate is -2 from the point (-3, -2). The highest $y$-coordinate is 2 from the point (1, 2).

Answer: Range = $[-2, 2]$

$B.$ (see the graph in Hw 4 recommended problem 11).

Domain = the set of $x$-coordinates of points on the graph. The smallest $x$-coordinate is -4 from the point (-4, 2). The largest $x$-coordinate is 4 from the point (4, -2). There is hollow circle above $x = -1$, thus the function is undefined here and $x = -1$ is not in the domain.

Answer: Domain = $[-4, -1) \cup (-1, 4]$

Range = the set of $y$-coordinates of points on the graph.
The smallest $y$-coordinate would be 3 from the point (-1, 3). However, there is a hollow circle here, so this point is excluded.

Answer: Range = $[-2, 2]$

$D.$ (see the graph of $f$ in Hw 4 recommended problem 11).

Find $f(-2)$

To find $f(-2)$, find the point on the graph with $x$-coordinate -2. The point is (-2, 4). Then $f(-2) = y$ of this point which is $y = 4$.

Answer $f(-2) = 4$

$F.$ For the $f$ function pictured,

\[
\begin{align*}
&\text{range} = [-1, 1] \\
&\text{max value } f(1) = 1 \\
&\text{min value } f(3) = -1 \\
&\text{interval(s) of increase} \quad [0,1], [3,4] \\
&\text{interval(s) of decrease} \quad [1,3].
\end{align*}
\]

Note, we don't write $[0,1] \cup [3,4]$ as we would if we were finding a domain. The function increases on each of the two intervals separately but not on their union.

$F.$ For the $f$ function pictured,

\[
\begin{align*}
&\text{range} = [-1, 1] \\
&\text{max value } f(1) = 1 \\
&\text{min value } f(3) = -1 \\
&\text{interval(s) of increase} \quad [0,1], [3,4] \\
&\text{interval(s) of decrease} \quad [1,3].
\end{align*}
\]

Note, we don't write $[0,1] \cup [3,4]$ as we would if we were finding a domain. The function increases on each of the two intervals separately but not on their union.

Interval(s) of decrease $[1,3]$.
We include the endpoints in an interval of increase or decrease when the function is defined at those points.
If there had been a hollow circle at (3, -1), then 3 would not be included in the interval of decrease (or increase) and the interval of decrease would then have been $[1,3)$.

$J.$ $f(x) = \begin{cases} 
1/x & x < -1 \\
1/x & 1 < x \\
x & -1 \leq x \leq 1
\end{cases}$

On the plane to the left of $x = -1$, and to the right of $x = 1$, the graph is the same as the graph of $1/x$. On the vertical strip of points between the vertical lines $x = -1$ and $x = 1$, the graph is the same as the graph of $x$ which is the major diagonal.
Basic graphs
Know these graphs. Note: tangent to $x^3$ at the origin is horizontal.

Given $f(x)$, find the functions for the other graphs.

<table>
<thead>
<tr>
<th>1</th>
<th>1, $x$, $x^2$, $x^3$, $1/x$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)+1$</td>
<td>$(x+1)$ Shift right 1</td>
</tr>
<tr>
<td>$f(x)-1$</td>
<td>$(-x+1)$ Shift left 1</td>
</tr>
<tr>
<td>$f(x)+1$</td>
<td>$f(x)-1$</td>
</tr>
<tr>
<td>$f(-x)$</td>
<td>$f(-x)$</td>
</tr>
</tbody>
</table>

**Translations and reflections**

**Theorem.** Changing the value $f(x)$ changes the vertical position; changing the argument $x$ changes the horizontal position in *opposite the expected direction.

<table>
<thead>
<tr>
<th>1</th>
<th>1, $x$, $x^2$, $x^3$, $1/x$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>up 1 unit</td>
<td>down 1</td>
</tr>
<tr>
<td>$f(x)+1$</td>
<td>$f(x)-1$</td>
</tr>
<tr>
<td><em>left 1</em></td>
<td><em>right 1</em></td>
</tr>
<tr>
<td>reflect in x-axis</td>
<td>reflect in y-axis</td>
</tr>
</tbody>
</table>

**Horizontal moves with argument changes**

$x = $ the $x$-axis position. Changing $x$, changes the horizontal position of the coordinate system. Replacing $x$ by $x+2$ shifts the coordinate system 2 units to the left.

For a formula with several shifts and reflections of a function $f$, rewrite it in the graph-translation form

$$af(b(x-c)) + d$$

Then the shifts and reflections occur in the left-to-right order:

A negative $a$ gives a vertical reflection.
A negative $b$ gives a horizontal reflection.

The horizontal shift is determined by the $c$: right if $c$ is positive, left if $c$ is negative.

The vertical shift is determined by $d$: up if $d$ is positive, down if $d$ is negative.

**Vertical moves with value changes**

Given $f(x) = |x|$, graph $f(x)+2$, $f(x)-2$, $-f(x)$.

$f(x) = |x|$, $f(x)+2 = |x|+2$, $f(x)-2 = |x|-2$, $-f(x) = -|x|$.

The value $f(x) = $ the height = the vertical position of a point on the graph. Changing $f(x)$ changes the vertical position of the graph.

Adding 2 raises the graph 2 units.
Negating $f(x)$ reflects the graph vertically across the $x$-axis.

- Given $f(x) = \sqrt{x}$, graph $f(x+2)$, $f(x-2)$, $f(-x)$.
  
  $f(x) = \sqrt{x}$, $f(x+2) = \sqrt{x+2}$, $f(x-2) = \sqrt{x-2}$, $f(-x)$.

- Given $f(x) = \frac{1}{x}$, graph $1/x$.
  
  $1/x$, $\sqrt{x}$, $|x|$, $\sqrt{1-x^2}$.

- $g(x) = x^2 - x$. Graph Parabola with roots 0, 1
  
  $g(1+x)$ Graph Shift left 1
  
  $g(1-x)$ Graph Reflect in y-axis, shift right 1.
13(6). Graph \(1 - \sqrt{1 - (x + 2)^2}\).

The known graph is \(\sqrt{1 - x^2}\) (upper unit semicircle).

Rewrite the function in the form \(a\sqrt{1 - [b(x - c)]^2} + d\).

\[
1 - \sqrt{1 - (x + 2)^2} = -\sqrt{1 - (1)[x - (-2)]^2} + 1
\]

Hence \(a = -1, b = 1, c = -2, d = 1\)

Start with the known function \(\sqrt{1 - x^2}\).

Then apply, in order, vert. reflect. \((a = -1)\), no hor. reflect. \((b = 1)\), left shift 2 \((c = -2)\), shift up 1, \((d = 1)\).

\[
\begin{align*}
\sqrt{1 - x^2} &\rightarrow \text{vert. reflect.}, \quad -\sqrt{1 - x^2} \rightarrow \text{left shift 2}, \quad -\sqrt{1 - [x - (-2)]^2} \rightarrow \text{up 1}, \quad -\sqrt{1 - (x + 2)^2} + 1
\end{align*}
\]

Number the graphs.

Practice 14(6). The graph of \(f(x)\) is given. Graph (a) \(y = -f(-x)\) and (b) \(y = f(2 - x)\).

(a) \(f(x) \rightarrow f(-x) \rightarrow -f(-x)\) Reflect vertically, then horizontally.

(b) First rewrite: factor out any coefficient of \(x\). \(f(2 - x) = f(-x + 2) = f(-(x - 2))\)

This is \(af(b(x - c)) + d\) for \(a=1, b=-1, c=2, d=0\)

Hence no vert. refl. \((a=1)\), hor. reflect. \((b=-1)\), right shift 2 \((c=2)\), no upward shift \((d=0)\).
13. Graph $1 - \sqrt{1-x}$.

Factor out the "-" under the radical (you can’t pull it outside of the radical) Rewrite it in the form $a\sqrt{b(x-c)} + d$.

$$\_\sqrt{\_ (x - \_)} + \_$$

Start with the graph (1) of $\sqrt{x}$ and apply the shifts and reflections needed to get the graph of $1 - \sqrt{1-x}$.

Graph 1 is given. Number the other graphs, 2, 3, 4, 5. 5 being the final answer.

The graph has an $x$-intercept. Be careful to get the $x$-intercept right (it is not a fraction).
13. Graph $1 - |2 - x|$. 

Factor out the “−” under the radical. Rewrite it in the form $a|b(x - c)| + d$.

$$1 - |2 - x| = -| - x + 2| + 1 = -| (x - 2)| + 1$$

Graph number:

<table>
<thead>
<tr>
<th>a = -1</th>
<th>b = -1</th>
<th>c = 2</th>
<th>d = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>lxl → -lxl</td>
<td>-lxl</td>
<td>-l(x - 2)</td>
<td>-l2 - xl + 1</td>
</tr>
<tr>
<td>vert. reflection</td>
<td>hor. reflection</td>
<td>right 2</td>
<td>up 1</td>
</tr>
</tbody>
</table>

1 2 3 4 5
§2.4 190:1-20. Odds have answers in back.

1(5). List the combination of translations and/or reflections needed to get the given function from \( f(x) \).

Careful with \( k \).

A. Left 2, down 3.
B. Left 3, up 2.
C. Right 3, up 2.
D. Left 3, down 2.
E. Right 3, down 2.
F. Reflect in \( y \)-axis, up 2.
G. Reflect in \( x \)-axis, right 2.
H. Reflect in \( x \)-axis, left 2.
I. Reflect in \( y \)-axis, right 2.
J. Right 2, up 3.
K. Left 2, up 3.
L. Right 2, down 3.

\[
\begin{align*}
\text{(a)} & \quad f(x+2)+3 \\
\text{(c)} & \quad f(x-2)+3 \\
\text{(d)} & \quad f(x-2)-3 \\
\text{(j)} & \quad -f(x-2) \\
\text{(k)} & \quad f(2-x) \\
\text{(l)*} & \quad f(-x)+2 \\
\end{align*}
\]

F: Reflect in \( y \)-axis, up 2.  *Example


Graph the functions.

2(2). \( y = (x-4)^2 + 1 \)

3(2). \( y = -x^2 - 3 \)

4(2). \( y = -(x-3)^2 + 3 \)

The graph of \( f(x) \) is a line segment from \((-4,1)\) to \((-1,2)\).

5(1). \( y = f(x) - 2 \)

6(1). \( y = f(x-4) - 2 \).
A. Describe the combination of translations and/or reflections needed to get the given function from $f(x)$.

(a) $y = f(x - 1)$  
(b) $y = f(x) - 1$  
(c) $y = f(x) + 1$  
(d) $y = f(x + 1)$  
(e) $y = f(-x) + 1$  
(f) $y = f(-x) - 1$  
(g) $y = -f(x) + 1$  
(h) $y = -f(x + 1)$  
(i) $y = -f(x) - 1$  
(j) $y = f(1 - x) + 1$  
(k) $y = -f(-x) + 1$

Graph the functions.

B. $y = x^3 - 3$

C. $y = (x + 4)^2$

D. $y = (x - 4)^2$

E. $y = -x^2$

F. $y = -(x - 3)^2$

The graph of $f(x)$ is a line segment from (-4,1) to (-1,2).

Graph the following.

G. (a) $y = -f(x)$  
(b) $y = f(-x)$  
(c) $y = -f(-x)$  
(d) $y = -f(x - 1)$  
(e) $y = -f(1 - x)$  
(f) $y = 1 - f(1 - x)$

H. Given $g(x)$, graph (a), (b), (c).

(a) $y = g(-x)$  
(b) $y = -g(x)$  
(c) $y = -g(-x)$
**A. Describe the combination of translations and/or reflections needed to get the given function from** \( f(x) \).

- (a) \( y = f(x - 1) \) \( \text{right 1} \)
- (b) \( y = f(x) - 1 \) \( \text{down 1} \)
- (c) \( y = f(x) + 1 \) \( \text{up 1} \)
- (d) \( y = f(x + 1) \) \( \text{left 1} \)
- (e) \( y = f(-x) + 1 \) \( y\)-axis reflection, up 1
- (f) \( y = f(-x) - 1 \) \( y\)-axis reflection, down 1
- (g) \( y = -f(x) + 1 \) \( x\)-axis reflection, up 1
- (h) \( y = -f(x + 1) \) \( \text{left 1, x-axis reflection} \)
- (i) \( y = -f(x) - 1 \) \( x\)-axis reflection, down 1
- (j) \( y = f(1 - x) + 1 \) \( y\)-axis reflection, right 1, up 1

Note: the order is important: \( y\)-axis reflection, left 1, up 1 is wrong.

- (k) \( y = -f(-x) + 1 \) \( x\)-axis reflection, \( y\)-axis rel., up 1

**Graph the functions.**

B. \( y = x^3 - 3 \)

Start with the graph of \( x^3 \), then move it down 3 units.


C. \( y = (x + 4)^2 \)

Start with the graph of \( x^2 \), then shift if left 4 units.


**F.** \( y = -(x - 3)^2 \)

Start with \( x^2 \), reflect it vertically around the x-axis to get \( -x^2 \). Shift it right 3 units to get \( -(x - 3)^2 \).


**Graph the following.**

G. (a) \( y = -f(x) \) \hspace{2cm} (b) \( y = f(-x) \)


(c) \( y = -f(-x) \) \hspace{2cm} (d) \( y = -f(x - 1) \)


(e) \( y = -f(1 - x) \) \hspace{2cm} (f) \( y = 1 - f(1 - x) \)


**Math 140  Hw 5  Worked examples of selected recommended problems.**
Math 140 Lecture 6

Study Practice Exam 1 and the recommended exercises.

Functions can be added and multiplied just like numbers.

**Definition.** For functions \( f, g \), define \( f + g, f - g, f g, f / g \) by
\[
(f + g)(x) = f(x) + g(x), \quad f(x) + g(x) = f(g)(x) - f(g)(x),
\]
\[
(f - g)(x) = f(x) - g(x), \quad f(x) - g(x) = f(g)(x) - g(f)(x),
\]
Note: \( f + g \) is not \( f + g(x) \)
The first is function addition.
\[
(f g)(x) = f(x) g(x), \quad f(x) g(x) = f(g)(x) g(f)(x),
\]
The second is multiplication.
\[
(f / g)(x) = f(x) / g(x), \quad f(x) / g(x) = f(g)(x) / g(f)(x),
\]

**Example.**

Find \( f(g(x)) \) and \( g(f(x)) \) for the functions
\[
\begin{align*}
  f(x) & = x - 2, \\
  g(x) & = x^2, \\
\end{align*}
\]

- **Example (a)**
  \[
  f(x) = x - 2 \implies f(g(x)) = f(x^2) = x^2 - 2,
  \]
- **Example (b)**
  \[
  g(x) = x^2 \implies g(f(x)) = g(x - 2) = (x - 2)^2 = x^2 - 2x + 4.
  \]

Note that \( f o g \neq g o f \). For composition, order matters.

**Example (b)’**

Find \( (f o g)(2) \) and \( (g o f)(2) \) directly without \( (f o g)(x) \),
\[
\begin{align*}
  f(x) & = x - 2, \\
  g(x) & = x^2, \\
  f(2) & = 2 - 2 = 0, \\
  g(0) & = 0^2 = 0, \quad (g o f)(2) = g(f(2)) = g(0) = 0.
  \end{align*}
\]

- **Example (c)**
  \[
  h(x) = c, \quad h(8) = c, \quad h(8 - 1) = c, \quad h(g(x)) = c.
  \]

- **Example (d)**
  \[
  f(x) = 3x + 4, \quad g(x) = 5, \quad f(g(x)) = f(5) = 3 \cdot 5 + 4 = 19, \\
  f(x) = 3x + 4, \quad g(x) = 5, \quad g(f(x)) = g(19) = 5.
  \]

For \( f \) and \( g \) above, note that
\[
\begin{align*}
  f(-3) & = 1, \quad f(-1) = -2, \quad f(2) = 3, \quad f(4) = 2, \\
  g(-2) & = -1, \quad g(-1) = -2, \quad g(1) = 1, \quad g(2) = 2.
  \end{align*}
\]

**Definition.** \( id(x) = x \) is called the identity function.

Hence \( id(5) = 5, \quad id(0) = 0, \quad id(x - 1) = x - 1, \ldots \).

**Theorem.** For any function \( f(x) \), \( f \circ id = f \) and \( id \circ f = f \).

**Proof.**

\[
\begin{align*}
  (f \circ id)(x) & = f(id(x)) = f(x), \\
  (id \circ f)(x) & = id(f(x)) = f(x).
  \end{align*}
\]

0 is the identity for addition, since \( f + 0 = f \).
1 is the identity for multiplication, \( f \cdot 1 = f \).
\( id(x) \) is the identity for composition, since \( f \circ id = id \circ f = f \).
15(4). \( f(x) = x - \frac{1}{x} \), find \( (f \circ f)(x) \). Simplify to a single polynomial or fraction.

\[
(f \circ f)(x) = f(x) - \frac{1}{f(x)} = (x - \frac{1}{x}) - \frac{1}{x - \frac{1}{x}} = x - \frac{1}{x} - \frac{x}{x^2 - 1} = \frac{x^2 - 1 - x^2 - x^2}{x(x^2 - 1)} = \frac{x^4 - 3x^2 + 1}{x(x^2 - 1)}
\]

16(2).
\( f(1) = 2, f(2) = 4, f(3) = 1, f(4) = 3 \).
\( g(1) = 4, g(2) = 3, g(3) = 1, g(4) = 2 \).
\( (f \circ g \circ f)(4) = ? \)

Work from inside out. \( (f \circ g \circ f)(4) = (f(g(f(4)))) = f(g(3)) = f(1) = 2 \).

17. (a)(3) Write \( G(x) = (x^4 + 3)x^2 \) as a composition \( f(g(x)) \) of two polynomials \( f(x) \) and \( g(x) \) of lower degree. \( g(x) \) is the inner function.

Write this as \( ((x^2)^2 + 3)x^2 \)

The inner function \( g(x) = x^2 \)

The outer function applies to the inner function; it does what remains to be done.

To get the outer function, replace each occurrence of the inner function by \( x \).

\( f(x) = (x^2 + 3)x \)

(b)(3) Write \( F(x) = \left(\frac{1 - \sqrt{x}}{1 + \sqrt{x}}\right)^{10} \) as a composition \( h(f(g(x))) \) of three simpler functions \( h(x) \), \( f(x) \) and \( g(x) \). \( g(x) \) is the inner function.

Inner function: \( g(x) = \sqrt{x} \)

Replace each occurrence of \( \sqrt{x} \) by \( x \).

Intermediate function \( f(x) \): \( f(x) = \frac{1 - x}{1 + x} \)

Outer function \( h(x) \) \( h(x) = x^{10} \)
17. (a) Write \((x^{12} - 3)(x^4 + 6)\) as a composition \(f(g(x))\) of two polynomials \(f(x)\) and \(g(x)\) of lower degree. \(g(x)\) is the inner function.

\[
\begin{align*}
g(x) &= \\
f(x) &= 
\end{align*}
\]

17. (b) Write \(\frac{\sqrt{2+x}}{3}\) as a composition \(f(h(g(x)))\) of three nontrivial simpler functions \(f, h,\) and \(g(x)\).

The functions must not be the trivial function \(x\). Hence \(g(x) = x\) is always a wrong answer. \(g(x)\) is the innermost function, find it first. Circle it.

\(h\) applies next. To find \(h\), look at the circled inner function, and ask what happens next to this inner function? \(f\) is the outermost. It does what remains to be done after \(g\) and \(h\) have been applied.

\[
\begin{align*}
g(x) &= \\
h(x) &= \\
f(x) &= 
\end{align*}
\]

17. (c) Write \(3\sqrt{1+x^2} - \frac{2}{1+x^2}\) as a composition \(f(h(g(x)))\) of three nontrivial simpler functions \(f, h,\) and \(g(x)\).

Of the functions \(g, h, f\), one has 2 symbols, one has 3 symbols, one has 7 symbols.

\[
\begin{align*}
g(x) &= \\
h(x) &= \\
f(x) &= 
\end{align*}
\]

1. To get back \(x\) from \(x+2\), you have to subtract 2. Hence the inverse of \(x+2\) is \(x-2\).

To get back \(x\) from \(3x\), you have to divide by 3. Hence the inverse of \(3x\) is \(x/3\).

To get back \(x\) from \(\sqrt[5]{x}\), you have to? Hence the inverse of \(\sqrt[5]{x}\) is Answer has 2 symbols.
Ignore * problems; they are just examples.

 Exam 1.

§2.7 219:1-6.
\[ f(x) = 2x - 1, \ g(x) = x^2 - 3x - 6, \ h(x) = x^3, \ k(x) = 2, \ m(x) = x^2 - 9. \]
* (g h)(x) = g(x) h(x) = (x^2 - 3x - 6)(x^3) = x^5 - 3x^4 - 6x^3 example

1. \( f h \)(x) = \]
   2 terms, 6 symbols

2. \( h f \)(x) = \]
   fraction, 7 symbols

3. \( f h \)(1) = \]
   digit

\[ \]

§2.7 220:17-22.
\[ f(x) = 1 - 2x^2, \ g(x) = x + 1. \]

* (g o g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2. -- example

4. \( f o g \)(x) = \]
   3 terms, 9 symbols

5. \( f o g \)(-1) = \]
   digit

6. \( g o f \)(x) = \]
   2 terms, 5 or 6 symbols

7. \( g o f \)(-1) = \]
   digit

8. \( f o f \)(x) = \]
   3 terms, 10 symbols

\[ \]

§2.7 220:45-54.
* Write \( F(x) = \frac{1}{x^4 + x^2 + 1} \) as a composition of a 3-symbol function \( f(x) \), a 6-symbol function \( g(x) \) and a 2-symbol function \( h(x) \).
\[ F(x) = f(g(h(x))) \] where \( f(x) = 1/x, \ g(x) = x^2 + x + 1, \ h(x) = x^2. \]

16. Write \( G(x) = 1/(1+x^4) \) as a composition of a 3-symbol function \( f(x) \) and a 4-symbol function \( g(x) \).
\[ G(x) = f(g(x)) \]
\[ f(x) = \]
\[ g(x) = \]
Math 140  Hw 6   Recommended problems, don’t turn this in.

\(f(x)=2x-1, \ g(x)=x^2-3x-6, \ h(x)=x^3, \ k(x)=2, \ m(x)=x^2-9.\)

A. (a) \((f+g)(x)\)

(b) \((f-g)(x)\)

(c) \((f-g)(0)\)

B. (a) \((m-f)(x)\)

(b) \((f-m)(x)\)

C. (a) \((f \circ k)(x)\)

(b) \((k \circ f)(x)\)

(c) \((f \circ k)(1) - (kf)(2)\)

D.\(f(x)=3x+1, \ g(x)=-2x-5\)

(a) \((f \circ g)(x)\)

(b) \((f \circ g)(10)\)

(c) \((g \circ f)(x)\)

(d) \((g \circ f)(10)\)

E. Compute \((f \circ g)(x)\) and \((g \circ f)(x)\)

(a) \(f(x)=1-x, \ g(x)=1+x\)

(b) \(f(x)=x^2-3x-4, \ g(x)=2-3x\)

(c) \(f(x)=x/3, \ g(x)=1-x^4\)

(d) \(f(x)=2^x, \ g(x)=x^2+1\)

Write each function as a composition \(f \circ g\) of simpler functions \(f\) and \(g\).

F. (a) \(\sqrt[3]{3x+4}\)

(b) \(|2x-3|\)

(c) \((ax+b)^5\)

(d) \(1/\sqrt{x}\)

G. (a) \(\sqrt[2]{2x+1}\)

(b) \(1/x^2\)

(c) \(2x^2+1\)

(d) \(2\sqrt[3]{x}+1\)

H. Write each function as a composition \(f \circ g \circ h\) of three simpler functions \(f, \ g\) and \(h\). \((3x-4)^4\)

Answers

A. (a) \(x^2-x-7\)

(b) \(-x^2+5x+5\)

(c) \(5\)

B. (a) \(x^2-2x-8\)

(b) \(-x^2+2x+8\)

C. (a) \(-4x-2\)

(b) \(-4x-2\)

(c) \(-4\)

D. (a) \(-6x-14\)

(b) \(-74\)

(c) \(-6x-7\)

(d) \(-67\)

E. (a) \(-x, \ 2-x\)

(b) \(9x^2-3x-6, \ -3x^2+9x+14\)

(c) \(\frac{1}{3}(1-x^4), \ 1-\frac{1}{81}x^4\)

(d) \(2^{x^2+1}, \ 2^{2x+1}\)

F. (a) \(f(x) = \sqrt[3]{x}, \ g(x) = 3x+4\)

(b) \(f(x) = |x|, \ g(x) = 2x-3\)

(c) \(f(x) = x^5, \ g(x) = ax+b\)

(d) \(f(x) = 1/x, \ g(x) = \sqrt{x}\)

G. (a) \(f(x) = \sqrt[3]{x}, \ g(x) = 2x+1\)

(b) \(f(x) = 1/x, \ g(x) = x^2\)

(c) \(f(x) = 2x+1, \ g(x) = x^2\)

(d) \(f(x) = 2x+1, \ g(x) = \sqrt[3]{x}\)

H. \(f(x) = x^4, \ g(x) = x-4, \ h(x) = 3x\)
Math 140   Hw  6    Worked examples of selected recommended problems.

\[ f(x) = 2x - 1, \ g(x) = x^2 - 3x - 6, \ h(x) = x^3, \ k(x) = 2, \ m(x) = x^2 - 9. \]

D. \( f(x) = 3x + 1, \ g(x) = -2x - 5 \)

(c) \((g \circ f)(x)\)

\[ (g \circ f)(x) = g(f(x)) = g(3x + 1) = -2(3x + 1) - 5 \]
\[ = -6x - 2 - 5 = -6x - 7 \]
Answer: \(-6x - 7\)

(d) \((g \circ f)(10)\)

\[ (g \circ f)(10) = g(f(10)) = g(3 \cdot 10 + 1) = g(31) \]
\[ = -2(31) - 5 = -62 - 5 = -67 \]
Answer: \(-67\)

E. Compute \((f \circ g)(x)\) and \((g \circ f)(x)\)

(a) \(f(x) = 1 - x, \ g(x) = 1 + x\)

\[ (f \circ g)(x) = f(g(x)) = f(1 + x) = 1 - (1 + x) = -x \]
\[ (g \circ f)(x) = g(f(x)) = g(1 - x) = 1 + (1 - x) = 2 - x \]
Answer: \(-x, \ 2 - x\)

(c) \(f(x) = x/3, \ g(x) = 1 - x^4\)

\[ (f \circ g)(x) = f(g(x)) = f(1 - x^4) = (1 - x^4)/3 \]
\[ (g \circ f)(x) = g(f(x)) = g(x/3) = 1 - (x/3)^4 \]
Answer: \(\frac{1}{3}(1 - x^4), \ 1 - \frac{1}{81}x^4\)

(d) \(f(x) = 2^x, \ g(x) = x^2 + 1\)

\[ (f \circ g)(x) = f(g(x)) = f(x^2 + 1) = 2^{(x^2+1)} \]
\[ (g \circ f)(x) = g(f(x)) = g(2^x) = (2^x)^2 + 1 = 2^{2x} + 1 \]
Answer: \(2^{x^2+1}, \ 2^{2x} + 1\)

Write each function as a composition of simpler functions \(f \circ g\) of simpler functions \(f\) and \(g\).

F. (a) \(\sqrt[3]{3x + 4}\)

The largest inner function if \(g(x) = 3x + 4\)
After \(3x + 4\), what remains to be done is taking the cube root. Thus the outer function is \(f(x) = \sqrt[3]{x}\).
Check: \(f(g(x)) = f(3x + 4) = \sqrt[3]{3x + 4}\)
Answer: \(f(x) = \sqrt[3]{x}, \ g(x) = 3x + 4\)

(b) \(|2x - 3|\)

Inner function: \(g(x) = 2x - 3\)
g(x) = 2x is also an inner function but not the largest.
Outer function: \(f(x) = |x|\)
Check: \(f(g(x)) = f(2x - 3) = |2x - 3|\)
Answer: \(f(x) = |x|, \ g(x) = 2x - 3\)

(c) \((ax + b)^5\)

Inner: \(g(x) = ax + b\)
Outer: \(f(x) = x^5\)
Answer: \(f(x) = x^5, \ g(x) = ax + b\)

(d) \(1/\sqrt{x}\)

Inner: \(g(x) = \sqrt{x}\)
Outer: \(f(x) = \frac{1}{x}\)
Answer: \(f(x) = 1/x, \ g(x) = \sqrt{x}\)

G. (a) \(\sqrt[3]{2x + 1}\)

Inner: \(g(x) = 2x + 1\)
Outer: \(f(x) = \sqrt[3]{x}\)
Answer: \(f(x) = \sqrt[3]{x}, \ g(x) = 2x + 1\)

(b) \(1/x^2\)

Inner: \(g(x) = x^2\)
Outer: \(f(x) = 1/x\)
Answer: \(f(x) = 1/x, \ g(x) = x^2\)

(c) \(2x^2 + 1\)

Inner: \(g(x) = x^2\)
Outer: \(f(x) = 2x + 1\)
Answer: \(f(x) = 2x + 1, \ g(x) = x^2\)

(d) \(2\sqrt{x} + 1\)

Inner: \(g(x) = \sqrt{x}\)
Outer: \(f(x) = 2x + 1\)
Answer: \(f(x) = 2x + 1, \ g(x) = \sqrt{x}\)
Inverse functions

**DEFINITION.** $f^{-1}$, the *inverse* of $f$, is the function, if any, such that,

\[ f(f^{-1}(x)) = x \quad \text{when } f^{-1}(x) \text{ is defined} \]

\[ f^{-1}(f(x)) = x \quad \text{when } f(x) \text{ is defined}. \]

This says that $f$ and $f^{-1}$ undo each other:

\[ f^{-1} \text{ undoes what } f \text{ does and gives you back } x. \]

- **$f(x) = 2x$, $f^{-1}(x) = \frac{1}{2}x$.** Verify: $f(f^{-1}(x)) = x$ & $f^{-1}(f(x)) = x$.

- **$g(x) = x + 3$, $g^{-1}(x) = x - 3$.**

- **$h(x) = 2x + 3$, $h^{-1}(x) = (x - 3)/2 = \frac{1}{2}x - \frac{3}{2}$.** Verify that $h^{-1}(h(x)) = x$. To undo a sequence of operations, you must undo them in the reverse order: the inverse of $g(f(x))$ is $f^{-1}(g^{-1}(x))$.

\[
\begin{array}{c}
\text{THEOREM.} \\
y = f^{-1}(x) \text{ iff } f(y) = x.
\end{array}
\]

To find $f^{-1}(x)$ for complicated functions:

- **Start with $f(y) = x$.**
- **Solve for $y$ to get $y = f^{-1}(x)$.

- **$f(x) = x^3$, find $f^{-1}(x)$.**

\[
\begin{align*}
\text{Let } f(y) & = x \\
\therefore y & = x \\
\therefore y & = \sqrt[3]{x} \quad \text{since } f^{-1}(x) = \sqrt[3]{x}
\end{align*}
\]

**WARNING.** $f^{-1}(x)$ and $(f(x))^{-1}$ are not the same.

- $(f^{-1}(x))^{-1}$ is the reciprocal $= 1/f(x)$.
- $f(x) = x^3$;
  \[
  \begin{align*}
  f^{-1}(x) & = \sqrt[3]{x} \\
  (f^{-1}(x))^{-1} & = 1/\sqrt[3]{x} \quad (f(0))^{-1} = \text{undefined}.
  \end{align*}
  \]

- **$f(x) = \frac{x+1}{x-1}$. Find $f^{-1}(x)$.**

\[
\begin{align*}
\text{Let } f(y) & = x \\
y & = \frac{y+1}{y-1} = x \\
y + 1 & = xy - x \\
y - xy & = -x - 1 \\
y(1-x) & = -x - 1 \\
y & = \frac{-x-1}{1-x} = \frac{x+1}{x-1} \\
\therefore f^{-1}(x) & = \frac{x+1}{x-1}
\end{align*}
\]

- **$g(x) = x^2 + 7x$ for $x \leq -\frac{7}{2}$. Find $g^{-1}(x)$.**

\[
\begin{align*}
y^2 + 7y & = x \quad \text{for } y \leq -\frac{7}{2} \\
y^2 + 7y - x & = 0 \\
a y^2 + b y + c & = 0 \quad \text{for } a = 1, b = 7, c = -x.
\end{align*}
\]

\[
y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7 \pm \sqrt{49 + 4x}}{2} \\
y = \frac{-7 - \sqrt{49 + 4x}}{2}. \quad \text{Choose } -\text{ not } + \text{ since } y \leq -\frac{7}{2}.
\]

\[
\therefore g^{-1}(x) = \frac{-7 - \sqrt{49 + 4x}}{2}
\]

- **If $f(x) = x + 3$ then $f^{-1}(x) = x - 3$.**
- **If $g(x) = x/2$ then $g^{-1}(x) = 2x$.**
- **If $h(x) = \sqrt{x}$ then $h^{-1}(x) = x^2$ for $x \geq 0$.**

Note how the graph of $f$ is related to the graph of $f^{-1}$.

By the Theorem, $y = f^{-1}(x)$ iff $f(y) = x$. Thus the graph of $y = f^{-1}(x)$ is the graph of $f(y) = x$ which is just the graph of $f(x) = y$ with $x$ and $y$ interchanged. Interchanging $x$ and $y$ reflects the plane around the major diagonal $y = x$.

Hence

**THEOREM.** The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across the major diagonal $y = x$.

For each function, draw the three graphs $y = f(x)$, $y = x$, $y = f^{-1}(x)$ on the same coordinate system.

- **$f(x) = x^3$**
- **$f(x) = -x^3$**

**DEFINITION.** $f$ is 1-1 (“one-to-one”) iff $x \neq y$ implies $f(x) \neq f(y)$.

- **$f(x) = 3x$ is 1-1**
- **$f(x) = x^2$ is not 1-1** but $(-1)^2 = 1^2$.

**THEOREM.** The following are equivalent:

- $f$ has an inverse
- $f$ is 1-1
- no horizontal line intersects its graph more than once.

- **Which of the following functions has an inverse?**

\[
\begin{array}{c}
\text{THEOREM.} \\
\text{The domain of } f^{-1} \text{ is the range of } f. \text{ The range of } f^{-1} \text{ is the domain of } f.
\end{array}
\]

**Proof.** The reflection around the major diagonal which carries the graph of $f$ to the graph of $f^{-1}$ also carries the domain of $f$ to the range of $f^{-1}$ and the range of $f$ to the domain of $f^{-1}$.

Stated in full, the inverse is the *compositional inverse.*

For addition, the inverse in negation.

For multiplication, the inverse is the reciprocal.
1(6). \( f(x) = \frac{2-x}{1-2x} \), find \( f^{-1}(x) \).

\[
\frac{2-y}{1-2y} = x \\
2 - y = x(1 - 2y) \\
2 - y = x - 2xy \\
-y + 2xy = x - 2 \\
y = \frac{x+2}{2x-1} \\
f^{-1}(x) = \frac{x+2}{2x-1}
\]

1(6). \( f(x) = x^2 + 3x \) for \( x \geq -3/2 \), find \( f^{-1}(x) \). Hint, use the quadratic formula.

\[
y^2 + 3y = x \quad \text{for} \quad y \geq -3/2 \\
y^2 + 3y + (-x) = 0 \\
y = \frac{-3 \pm \sqrt{9 - 4(1)(-x)}}{2(1)} \\
f^{-1}(x) = \frac{-3 + \sqrt{9 + 4x}}{2}
\]

2(4). In (a), (b), the dotted line is the main diagonal, the solid line is the graph of \( f(x) \). If \( f^{-1}(x) \) exists, draw its graph. If not, draw a horizontal line which proves it is not 1-1.
2. In (a), (b), the dotted line is the main diagonal, the solid line is the graph of \( f(x) \). If \( f^{-1}(x) \) exists, draw its graph. If not, draw a horizontal line which proves it is not 1-1.

Here are two examples from the lecture.

- \( f(x) = \frac{x+1}{x-1} \). Find \( f^{-1}(x) \).
  
  \[
  f(y) = x \\
  \frac{y+1}{y-1} = x \\
  y + 1 = x(y - 1) \\
  y + 1 = xy - x \\
  y - xy = -x - 1 \\
  y(1 - x) = -x - 1 \\
  y = \frac{-x-1}{1-x} = \frac{x+1}{x-1} \\
  \therefore f^{-1}(x) = \frac{x+1}{x-1}
  \]

- \( g(x) = x^2 + 7x \) for \( x \leq -\frac{7}{2} \). Find \( g^{-1}(x) \).
  
  \[
  g(x) = y \\
  x^2 + 7x = y \\
  y = \frac{-7 \pm \sqrt{49+4x}}{2} \\
  \]
  
  \[
  \therefore g^{-1}(x) = \frac{-7 - \sqrt{49+4x}}{2} \\
  \]

Note, if \( g^{-1}(x) \) doesn’t involve \( x \), the answer is wrong. If it has a \( \pm \), it is also wrong.
1(2). \( g(t) = \frac{1}{t} + 1 \).

(a) Find \( g^{-1}(t) \)

(b) Find the inverse \( g^{-1}(2) \)
and the reciprocal \( 1/g(2) \)

2(3). \( f(x) = \frac{1}{3}x - 2 \).

(a) Find \( f^{-1}(x) \).

(b) Verify that \( f(f^{-1}(x)) = x \)
and that \( f^{-1}(f(x)) = x \)

(c) Graph, \( y = f(x), y = x, y = f^{-1}(x) \) on same coordinate system.

3(3). \( f(x) = \frac{1}{x} \).

(a) Find \( f^{-1}(x) \)

(b) Verify that \( f(f^{-1}(x)) = x \)
and that \( f^{-1}(f(x)) = x \)

(c) Graph, \( y = f(x), y = f^{-1}(x) \) on same coordinate system.

4(1). \( f(x) = \frac{2x-3}{x+4} \). Find \( f^{-1}(x) \). Omit the range & domain.

5(1). Given the graph of \( h(x) \), graph \( y = h^{-1}(x) \).

§2.8 230:7-12, 15,16.
In 6, 7, 8, graph the function. Circle “1-1” if 1-to-1; circle “not 1-1” if not.

6(1). \( y = \sqrt{x} \) \hspace{1cm} 1-1? \hspace{1cm} not 1-1?

7(1). \( y = |x| \) \hspace{1cm} 1-1? \hspace{1cm} not 1-1?

8(1). \( y = 1 - x^3 \) \hspace{1cm} 1-1? \hspace{1cm} not 1-1?
Math 140    Hw 7    Recommended problems, don’t turn this in.

A. \( f(x) = 3x - 1 \)
   (a) Find \( f^{-1}(x) \)
   (c) Graph, \( y = f(x) \), \( y = x \), \( f = f^{-1}(x) \) on same coordinate system.

B. \( f(x) = \sqrt{x - 1} \).
   (a) Find \( f^{-1}(x) \).
   (c) Graph, \( y = f(x) \), \( y = x \), \( f = f^{-1}(x) \) on same coordinate system.

C. \( f(x) = \frac{x + 2}{x - 3} \).
   (a) Find the domain and range of \( f \).
   (b) Find \( f^{-1}(x) \)
   (c) Find the domain and range of \( f^{-1} \).

D. \( f(x) = 2x^3 + 1 \). Find \( f^{-1}(x) \).

E. Given the graph of \( g(x) \), graph
   \[ g(x) \]
   \[ y = x \]
   (a) \( y = g^{-1}(x) \)
   (b) \( y = g^{-1}(x) - 1 \)
   (c) \( y = g^{-1}(x - 1) \)
   (d) \( y = g^{-1}(-x) \)
   (e) \( y = -g^{-1}(x) \)
   (f) \( y = -g^{-1}(-x) \)

In E, F, G, graph the function. Circle “1-1” if 1-to-1; circle “not 1-1” if not.

F. \( y = x^2 + 1 \)  1-1?  not 1-1?

G. \( y = \frac{1}{x} \)  1-1?  not 1-1?

H. \( y = x^3 \)  1-1?  not 1-1?

Answers
A. \( f^{-1}(x) = \frac{1}{3}(x + 1) \)
B. \( f^{-1}(x) = x^2 + 1 \) for \( x \geq 0 \)
C. (a) dom: \((-\infty, 3) \cup (3, \infty)\), rng: \((-\infty, 1) \cup (1, \infty)\)
   (b) \( f^{-1}(x) = (3x + 2)/(x - 1) \)
   (c) dom: \((-\infty, 1) \cup (1, \infty)\), rng: \((-\infty, 3) \cup (3, \infty)\)
D. \( f^{-1}(x) = \sqrt[3]{2}(x - 1) \)
E. Not shown.
F. Not 1-1.
G. Is 1-1
H. Is 1-1
A. \( f(x) = 3x - 1 \)
   (a) Find \( f^{-1}(x) \)
   Switch \( x \) and \( y \) in \( f(x) = y \) to get
   \( f(y) = x \) iff
   \[ 3y - 1 = x \]
   \[ 3y = x + 1 \]
   \[ y = \frac{1}{3}(x + 1) \]
   Answer: \( f^{-1}(x) = \frac{1}{3}(x + 1) \)
   (c) Graph, \( y = f(x), y = x, y = f^{-1}(x) \) on same coordinate system.
   Answer:

B. \( f(x) = \sqrt{x - 1} \).
   (a) Find \( f^{-1}(x) \).
   \( f(y) = x \) iff
   \[ \sqrt{y - 1} = x \]
   Note that \( x \) is \( \geq 0 \) since square roots \( \geq 0 \).
   \[ y - 1 = x^2 \]
   \[ y = x^2 + 1 \] for \( x \geq 0 \)
   Answer: \( f^{-1}(x) = x^2 + 1 \) for \( x \geq 0 \)
   (c) Graph, \( y = f(x), y = x, y = f^{-1}(x) \) on same coordinate system.

C. \( f(x) = \frac{x+2}{x-3} \).
   (b) Find \( f^{-1}(x) \)
   \( f(y) = x \) iff
   \[ \frac{y+2}{y-3} = x \]
   \[ y + 2 = (y - 3)x \]
   \[ y + 2 = xy - 3x \]
   \[ y - xy = -3x - 2 \]
   \[ y(1 - x) = -3x - 2 \]
   \[ y = (3x + 2)/(1 - x) = (3x + 2)/(x - 1) \]
   Answer: \( f^{-1}(x) = (3x + 2)/(x - 1) \)

D. \( f(x) = 2x^3 + 1 \). Find \( f^{-1}(x) \).
   \( f(y) = x \) iff
   \[ 2y^3 + 1 = x \]
   \[ 2y^3 = x - 1 \]
   \[ y^3 = \frac{1}{2}(x - 1) \]
   \[ y = \sqrt[3]{\frac{1}{2}(x - 1)} \]
   Answer: \( f^{-1}(x) = \sqrt[3]{\frac{1}{2}(x - 1)} \)

E. Given the graph of \( g(x) \), (a) graph \( g^{-1}(x) \)

In F and G, graph the function. Circle “1-1” if 1-to-1; circle “not 1-1” if not.
F. \( y = x^2 + 1 \)
   A horizontal crosses it twice, \( \therefore \) it is not 1-1

G. \( y = 1/x \)
   No horizontal line crosses the graph twice, \( \therefore \) it is 1-1.
Math 140  Lecture 8

**Definition.** A quadratic function is a degree-2 polynomial $y = ax^2 + bx + c$, $a \neq 0$.

The graph is a parabola.
- If $a > 0$, the horns point up.
- If $a < 0$, the horns point down.
- If $|a| > 1$, the parabola is narrower than $y = x^2$.
- If $|a| < 1$, the parabola is wider than $y = x^2$.

Find the roots by factoring or using the quadratic formula:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$ No roots if $b^2 - 4ac < 0$.

**Completing the Square Theorem.** Every quadratic function may be written in the form: $y = a(x - x_0)^2 + y_0$ where $(x_0, y_0)$ is the vertex (nose) of the parabola.

**Proof.** Factor the $a$ out of the $ax^2 + bx$ part of $ax^2 + bx + c$.
Complete the square. Anything which is added must also be subtracted to preserve equality.

**Examples**
- Find the roots (they are the $x$-intercepts).
- Write in completed square form: $y = a(x - x_0)^2 + y_0$.
- Graph. On the graph list both coordinates of the vertex.

- $y = -\frac{1}{2}(x + 1)^2$
  Roots: $x = -1$
  vertex: $(-1, 0)$

- $y = 2x^2 - 2x$
  Roots: $x = 0, 1$
  vertex: $(1/2, -1/2)$

- $y = x^2 + 2x - 1$
  Roots: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{1 + 4}}{2} = -1 \pm 1$
  $y = (x^2 + 2x + 1) - 1 - 1$
  $y = (x + 1)^2 - 2$

**Word problems**
- Draw the picture. Indicate the variables in the picture.
- Write the given equations which relate the variables.
- Solve for the wanted quantities. List Given and Answer.

- The perimeter of a rectangle is 10 feet. Express the area $A$ in terms of the width $x$.

  **Picture:**
  - $A$ = area
  - $P$ = perimeter
  
  **Given:**
  - $10 = 2x + 2y$
  - $A = xy$
  
  **Answer:** $A = x(5 - x)$ square feet.

- The corner of a triangle lies on the line $y = 4 - x$. Express the area $A$ and perimeter $P$ of the triangle in terms of the base $x$.

  **Given:**
  - $y = 4 - x$
  - $A = \frac{xy}{2}$
  - $P = x + y + z$
  - $z^2 = x^2 + y^2$

  **Answer:** $A = \frac{x(4-x)}{2}$
  
  **Want $A$ in $x$, need $y$ in $x$, need an equation in $x$ and $y$:**
  
  **Answer:** $P = 4 + \sqrt{2x^2 - 8x + 16}$.

- The area of an isosceles triangle is 16. Express the height of the triangle in terms of its width $x$.

  **Given:** $\frac{1}{2}xh = 16$

  **Answer:** $h = \frac{32}{x}$
485(8). Write the distance between the point \((1, 0)\) and a point \((x, y)\) on the curve \(y = x^3\) in terms of \(x\).

You get credit for drawing the picture indicating the variables, for listing the given facts (explicit and implicit) and for the answer.

**Picture:** Draw the picture. On the picture indicate your variables.

Given: Write the equations which relate the variables.

\[ d = \sqrt{(x - 1)^2 + (y - 0)^2} \]
\[ y = x^3 \]

**Answer:**

Want \(d\) in \(y\).

Have \(d\) in \(x\) and \(y\).

Need \(y\) in \(x\) (see given).

\[ y = x^3 \]
\[ d = \sqrt{(x - 1)^2 + y^2} \]
\[ = \sqrt{(x^2 - 2x + 1) + (x^3)^2} \]
\[ = \sqrt{x^6 + x^2 - 2x + 1} \]

3(8). \(y = x^2 - 2x - 15\). Write in completed-square form.

Find the vertex, intercepts, graph.

The vertex must be an order pair with ()'s, e.g., \((3, 4)\) not just 3, 4.

The completed-square form must be \(y = a(x - b)^2 + c\) not, for example, \(4(x + 3)^2 - 3\).

\[(x^2 - 2x + 1) - 15 - 1 \]
\[y = (x - 1)^2 + (-15)\]
\[(1, -16)\]
\[x = -3, 5\]
\[-15\]

**completed square:**

vertex =

\[x\text{-intercepts} = \]
\[y\text{-intercept} = \]

draw the graph.
3. \( y = 2(x + 4)^2 - 3 \). Rewrite in the completed-square form \( a(x - x_0)^2 + y_0 \). Hint, \( x_0, y_0 \) can be negative \( \underline{} \)/1

3. \( y = 2x^2 + 8x + 3 \). Find the vertex, intercepts, graph.

Do the “horns” of the parabola point up \( \cup \) or down \( \cap \)?

Leave the constant 3 alone. Factor the 2 out of \((2x^2 + 8x)\) then complete the square.

If your equation looks like \( a(x + x_0)^2 - y_0 \), rewrite it in the completed-square form \( a(x - x_0)^2 + y_0 \). \( \underline{} \)/1

vertex = \( \underline{} \)/1

You must use “( )”. E.g., vertex=(3,4), not vertex = 3,4. 7 symbols.

x-intercept(s)? Set \( y = 0 \).

Either factor \( 2x^2 + 8x + 3 \) or use the quadratic formula \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \). No roots if the radical is undefined. 8 symbols counting \( \pm \) as 1 symbol.

y-intercept? Equation has 3 symbols

Draw the graph. \( \underline{} \)/2
§2.5 201:5-18.
- Write the function in the form: \( y = a(x - h)^2 + k \).
  E.g., rewrite \( y = (x + 1)^2 \) as \( y = (x - (-1))^2 + 0 \).
- List the roots. One has no roots, one has 1, two have 2.
- Graph. On the graph mark the vertex and give both coordinates.

1(3). \( y = -(x + 2)^2 \)  \( y = \)
Roots:

2(3). \( y = 2(x + 2)^2 + 4 \)  \( y = \)
Roots:

3(3). \( y = x^2 + 6x - 1 \)  \( y = \)
Radical roots
Roots:

4(3). \( y = -3x^2 + 12x \)  \( y = \)
Roots:

§2.6 210: 1-12.
- In 5 and 6, draw the picture and indicate your variables on the picture.
- Write the given equations which relate the variables.
- Solve for the wanted quantities.

5(4). A rectangle is inscribed in a circle of diameter 12.
(a) Express the perimeter \( P \) as a function of the width \( x \).
(b) Express the area \( A \) as a function of the width \( x \).
\( P \) has 11 or 12 symbols, \( A \) has 8.

Given:

\[ \begin{align*}
\text{P}= & \text{perimeter} \\
\text{A}= & \text{area}
\end{align*} \]

(a) \( P = \)
(b) \( A = \)

6(4). The top of a right triangle lies on the curve \( y = \sqrt{x} \) as shown.
(a) Express the area \( A \) of the triangle in terms of \( x \).
(b) Express the perimeter \( P \) of the triangle in terms of \( x \).
\( A \) has 5 symbols, \( P \) has 10.

Given:

\[ \begin{align*}
\text{y}= & \\
\text{x}= & \\
\text{P}= & \text{perimeter} \\
\text{A}= & \text{area}
\end{align*} \]

(a) \( A = \)
(b) \( P = \)

7(2). The product of two numbers is 16. Express the sum \( S \) of the squares of the two numbers as a function of one of the numbers. \( S \) has 9 symbols.

\( S = \)
- Write the function in the form: \( y = a(x - h)^2 + k. \)
  E.g., rewrite \( y = (x + 1)^2 \) as \( y = (x - (-1))^2 + 0. \)
- List the roots.
- Graph. On the graph mark the vertex and give both coordinates.

A. \( y = x^2 - 4x \)

\[
\begin{align*}
\text{Graph:} & \\
\text{Roots:} & \\
\text{Vertex:} (2, -4)
\end{align*}
\]

B. \( y = 1 - x^2 \)

\[
\begin{align*}
\text{Graph:} & \\
\text{Roots:} & \\
\text{Vertex:} (0, 1)
\end{align*}
\]

C. \( y = x^2 - 2x - 3 \)

\[
\begin{align*}
\text{Graph:} & \\
\text{Roots:} -1, 3 \\
\text{Vertex:} (1, -4)
\end{align*}
\]

D. \( y = -x^2 + 6x + 2 \)

\[
\begin{align*}
\text{Graph:} & \\
\text{Roots:} & \\
\text{Vertex:} (3, 11)
\end{align*}
\]

- Draw the picture and indicate your variables on the picture.
- Write the given equations which relate the variables.
- Solve for the wanted quantities.

E. (a) The perimeter of a rectangle is 16. Express the area of the rectangle in terms of the width & x. 
   (b) The area of a rectangle is 85. Express the perimeter of the rectangle in terms of the width & x.

F. \( P = (x, y) \) is a point on the curve \( y = x^2 + 1. \)
   (a) Express the distance of \( P \) from \((0,0)\) as a function \( d(x) \) of \( x. \)
   (b) Express the slope of the line segment from \((0,0)\) to \( P \) as a function \( m(x) \) of \( x. \)

G. A piece of wire is \( \pi y \) inches long. It is first bent into a circle.
   (a) The wire is bent into a circle. Express the area \( A(y) \) of the circle in terms of \( y. \)
   (b) The wire is bent into a square. Express the area \( A(y) \) of the square in terms of \( y. \)

H. The sum of two numbers is 16.
   (a) Express the product of the two numbers in terms of a single variable.
   (b) Express the sum of the squares of the two numbers in terms of a single variable.

Answers

A. \((x - 2)^2 - 4\), roots: 0,4, vertex: (2,-4).  
B. \(- (x - 0)^2 + 1\), roots: ±1, vertex: (0,1).  
C. \((x - 1)^2 - 4\), roots: -1,3, vertex: (1,-4).  
D. \(- (x - 3)^2 + 11\), roots: \(3 \pm \sqrt{11}\), vertex: (3,11).  
E. (a) \(8x - x^2\) (b) \(2x^2 + 170/x\)  
F. (a) \(\sqrt{x^4 + 3x^2 + 1}\) (b) \((x^2 + 1)/x\)  
G. (a) \(\pi y^2/4\) sq. inches (b) \(\pi^2 y^2/16\) sq. inches  
H. (a) \(16x - x^2\) (b) \(2x^2 - 32x + 256\)
Math 140     Hw 8     Worked examples of selected recommended problems.

- Write the function in the form: \( y = a(x - h)^2 + k \).
  
E.g., rewrite \( y = (x + 1)^2 \) as \( y = (x - (-1))^2 + 0 \).

- List the roots.

- Graph. On the graph mark the vertex and give both coordinates.

C. \( y = x^2 - 2x - 3 \)

\[
y = (x - 3)(x + 1)
\]

roots: -1, 3

\[
y = (x - 2)x - 3
\]

\[-2 \rightarrow \frac{-2}{2}(-2) = -1 \rightarrow (-1)^2 = 1
\]

\[
y = (x^2 - 2x + 1) - 1 = (x - 1)^2 - 4
\]

\[
y = (x - 1)^2 + (-4)
\]

vertex: (1,4)

Answer: \((x - 1)^2 + (-4), \) roots: -1,3, vertex: (1,-4).

D. \( y = -x^2 + 6x + 2 \)

\[
y = \frac{-6 \pm \sqrt{6^2 - 4(-1)(2)}}{2(-1)} = \frac{-6 \pm \sqrt{44}}{-2} = 3 \pm \sqrt{11}
\]

roots:

\[
y = -(x^2 - 6x) + 2
\]

\[-6 \rightarrow \frac{6}{2}(-6) = -3 \rightarrow (-3)^2 = 9
\]

\[
y = -(x^2 - 6x + 9) + 2 + 9
\]

\[
y = -(x - 3)^2 + 11
\]

vertex: (3,11)

Answer: \(-(x - 3)^2 + 11,\) roots: \(-3 \pm \sqrt{11} \), vertex: (3,11).

- In 2 and 4, draw the picture and indicate your variables on the picture.

- Write the given equations which relate the variables.

- Solve for the wanted quantities.

F. \( P = (x, y) \) is a point on the curve \( y = x^2 + 1 \).

(a) Express the distance of \( P \) from \((0,0)\) as a function \( d(x) \) of \( x \).

(b) Express the slope of the line segment from \((0,0)\) to \( P \) as a function \( m(x) \) of \( x \).

\[
\text{Given} \quad y = x^2 + 1
\]

\[
d = \sqrt{(x - 0)^2 + (y - 0)^2}
\]

\[
m = \frac{y}{x}
\]

(a) Want \( d \) in \( x \), need \( y \) in \( x \) but we have \( y = x^2 + 1 \)

\[
d = \sqrt{x^2 + (x^2 + 1)^2} = \sqrt{x^2 + x^4 + 2x^2 + 1}
\]

\[
d = \sqrt{x^2 + 3x^2 + 1}
\]

Answer: \( d = \sqrt{x^2 + 3x^2 + 1} \)

(b) Want \( m \) in \( x \)

\[
m = \frac{x^2 + 1}{x}
\]

Answer: \( m = (x^2 + 1)/x \)

G. A piece of wire is \( \pi y \) inches long. It is first bent into a circle.

(a) The wire is bent into a circle. Express the area \( A(y) \) of the circle in terms of \( y \).

(b) The wire is bent into a square. Express the area \( S(y) \) of the square in terms of \( y \).

\[
\text{Answer:} \quad A = \pi \frac{y^2}{4} \quad \text{sq. inches}
\]

(b) The sum of two numbers is 16.

(a) Express the product of the two numbers in terms of a single variable.

(b) Express the sum of the squares of the two numbers in terms of a single variable.

Let \( x \) and \( y \) be the two numbers. Let \( P \) be the product and \( S \) the sum of the squares.

Given: \( x + y = 16 \)

\[
P = xy
\]

\[
S = x^2 + y^2
\]

(a) Want \( P \) in \( x \), need \( y \) in \( x \).

\[
x + y = 16
\]

\[
y = 16 - x
\]

\[
P = xy = x(16 - x)
\]

Answer: \( P = 16x - x^2 \)

(b) Want \( S \) in \( x \), need \( y \) in \( x \).

\[
S = x^2 + y^2 = x^2 + (16 - x)^2 = x^2 + 256 - 32x + x^2
\]

Answer: \( S = 2x^2 - 32x + 256 \)
A 6' × 6' tarp forms the top of a pup tent. Write the height \( h \) of a pup tent as a function of the floor area \( A \).

Given: \( A = 6b, h^2 + \left(\frac{b}{2}\right)^2 = 3^2 \)
Want \( A \) in \( h \). Need \( b \) in \( h \).
\[
\left(\frac{b}{2}\right)^2 = 3^2 - h^2, \quad b = 2\sqrt{3^2 - h^2}
\]
\[
A = 6b = 6(2\sqrt{9 - h^2}) = 12\sqrt{9 - h^2}.
\]

The height of a can (right circular cylinder) is three times the radius.

\[
h \quad S = \text{curved surface area}
\]
\[
2\pi r
\]

Given: \( h=3r \), \( S=2\pi rh \)
(a) Express the curved surface area as a function of the radius. 4 symbols
\[
S = 6\pi r^2
\]
(b) Express the radius as a function of the curved surface area. 5 symbols
\[
r^2 = \frac{S}{6\pi}, \quad r = \sqrt{\frac{S}{6\pi}}.
\]

Three sides of a 500 square foot rectangle are fenced. Express the length of the fence as a function of side \( x \).

\[
x \quad y
\]
Area = 500
\[
f = \text{length of fence}
\]
Given: \( xy = 500 \)
\[
f = 2x + y
\]
Want \( f \) in \( x, \cdot \) need \( y \) in \( x, \cdot \) need an equation in \( x \) and \( y \).
\[
y = \frac{500}{x}, \quad f = 2x + \frac{500}{x} \text{ feet}.
\]

The iClicker quiz and the classwork for Lecture 10 is hard. Study both in advance.

**Graphing polynomials**
A polynomial graph is smooth: no breaks, no sharp corners.

For an expanded polynomial \( ax^n + ... + c \) with \( ax^n \) the term of highest degree: \( ax^n \) is the leading term, \( a \) is the leading coefficient, and \( n \) is the degree. The \( y \)-intercept is the constant term \( c \).

- \( y = -3x^4 + x^2 - 5 \), leading term = \(-3x^4\), leading coeff. = -3, degree = 4, constant term = -5.
- \( y = x - x^3 \), leading term = \(-x^3\), leading coeff. = -1, degree = 3, constant term = 0.

For large \( x \) (near \( \pm\infty \)), graph looks like the leading term \( ax^n \).
As \( x \) goes to \( \infty \), \( y \) goes to \( +\infty \) if \( a > 0 \), to \( -\infty \) if \( a < 0 \).
Graphs of odd degree go to \( +\infty \) in one direction, \( -\infty \) in the other, like \( y = x^3 \), \( y = -x^3 \).
Graphs of even degree either go to \( +\infty \) in both directions or to \( -\infty \) in both directions, like \( y = x^2 \), \( y = -x^2 \).

Use the factored form to get the roots and their degrees (the degree of a root is the exponent of its factor).
- At roots of degree 1, the graph crosses \( x \)-axis like \( y = x \) or \( y = -x \).
- At roots of odd degrees 3, 5, 7, ..., the graph crosses the \( x \)-axis like \( y = x^3 \) or \( y = -x^3 \).
- At roots of even degrees, 2, 4, 6, ..., the graph touches but doesn’t cross the \( x \)-axis, like \( y = x^2 \) or \( y = -x^2 \).

At a root, the graph looks like \( \pm \) the graph of the root’s factor. The sign of the other factors determines the + or -. To get the constant term of a factored polynomial, replace each factor by its leading term.

To get the constant term (\( y \)-intercept), set \( x = 0 \). When graphing, find the \( x \) and \( y \)-intercepts and also calculate a key value in each of the key intervals before, after, and between roots.

**Graph** \( y = (x + 2)^3(4 - x)(2x - 1)^2 \).
-2 is a root of degree 3, 4 is a root of degree 1, \( \frac{1}{2} \) has degree 2.
lead term = \((x)^3(-x)(2x)^2\) = \(-4x^6\), degree = 6, lead coeff. = -4.
constant (\( y \)-intercept): set \( x = 0 \), \( y = (2)^3(4)(-1)^2 = 32 \).

Key values in the key intervals \((-\infty,-2], [-2,\frac{1}{2}], [\frac{1}{2},4], [4,\infty)\):
\( f(-3) = -343 \), \( f(-1) = 45 \), \( f(2) = 1152 \), \( f(5) = -27783 \).

The dotted lines indicate the portion of the graph around \( f(2) = 1152 \) which exceeds our coordinate limits.
485(10). A square is inscribed inside a circle. Express the area of the inscribed square as a function of the area of the circle.

**Picture:**

![Diagram of a square inscribed in a circle with labels: A = circle area, S = square area.]

Given: \( A = \pi r^2 \)

\[
S = x^2 \\
x^2 + x^2 = (2r)^2
\]

Want \( S \) in \( A \). Have \( S \) in \( x \). Need \( x \) in \( r \) and then \( r \) in \( A \).

\[
2x^2 = 4r^2 \\
x^2 = 2r^2 \\
r^2 = A/\pi \\
S = x^2 = 2r^2 = \frac{2A}{\pi}
\]

Answer: \( S = \frac{2A}{\pi} \)

485(12). Four pieces of a length \( x \) are cut from a 4 ft length of wire. The four pieces are arranged to form the four sides of square. The remaining piece of wire is bent into a circle. Write the area of the square as a function of the area of the circle.

**Picture:**

![Diagram of a square and a circle with labels: S = square area, A = circle area.]

Given: Write the equations which relate the variables.

\[
S = x^2 \\
4 - 4x = 2\pi r \\
A = \pi r^2
\]

Want \( S \) in \( A \).

Have \( S \) in \( x \).

Need \( x \) in \( r \) and \( r \) in \( A \).

\[
4x = 4 - 2\pi r \\
x = 1 - \frac{1}{2} \pi r \\
r^2 = A/\pi \\
r = \sqrt{A/\pi} \\
S = x^2 = (1 - \frac{1}{2} \pi r)^2 \\
= (1 - \frac{1}{2} \pi \sqrt{A/\pi})^2 \\
= (1 - \frac{1}{2} \sqrt{\pi^2 A/\pi})^2
\]

Answer:

\[
S = (1 - \frac{1}{2} \sqrt{\pi A})^2 \text{ sq. ft.}
\]

A common error involves solving for the wrong variable. If \( 2x = 3y \) and you are to write \( x \) in terms of \( y \), the answer is \( x = (3/2)y \) not \( y = (2/3)x \).
Classwork 10 is hard, work on it before class.

4&5. The volume of a cone is \( \frac{1}{3} \pi r^2 h \). The curved surface (or side) area of the cone is \( \pi r \sqrt{r^2 + h^2} \). Suppose the volume is 2 cubic inches. Write the curved surface area as a function of the radius \( r \).

Picture: Draw the picture. On the picture indicate your variables.
(Is the volume a variable? Should have three variables.)

Given: Write the equations which relate the variables.
(Should have two given equations.)

Answer: Is your answer a formula in \( h \) or \( r \)? It should involve \( \pi \), 36, a radical and a fraction. Remember the units. 11 to 14 symbols after “=”. 
2.6 210: 13-18, 23b, 24a, 25a, 26a, 29a, 30a, 31a, 33a, 34a.

§2.6 210: 13-18,23b,24a,25a,26a,29a,30a,31a,33a,34a.

Draw the picture; indicate your variables on the picture.
Write the equations (the “given”) which relate the variables.
Solve for the wanted quantities.

1(3). Let 2s be the length of the side of an equilateral triangle.
(a) Express the height of the triangle as a function of s. 3 symbols
(b) Express the area of the triangle as a function of s. 4 symbols

2(3). The height of a right circular cylinder is twice the radius. Express the volume as a function of the radius 4 symbols

3(3). For the rectangle pictured, express the area as a function of x.

A=area
y=10-x^2
(x,y)

Given:

Want A in x. 7 symbols

4(4). For the shaded triangle below, find the area A and the perimeter P.

A=area
P=perimeter

Given:

(a) Want A in x, need y in x. 8 symbols
(b) Want P in x, have y in x. 9 symbols

5(3). Three sides of a rectangle are fenced with 500 feet of fencing. Express the area of the rectangle as a function of x.

A = area

Given:

Want A in x, need y in x. 8 symbols

§3.1 262:11-22. Graph and on the graph mark the x and y-intercepts. Label the degrees of the roots. No credit if the graph isn’t smooth. E.g., the graph of |x| isn’t smooth.

6(2). y = -(x + 2)^3

7(2). y = (x - 3)(x + 2)(x + 1)

Check your graphs at https://www.desmos.com/calculator
Math 140    Hw 9    Recommended problems, don't turn this in.

Draw the picture; indicate your variables on the picture. Write the equations which relate the variables. Solve for the wanted quantities.

A(3). Express the area of an equilateral triangle as a function of a side $x$.

B(3). The height of a right circular cylinder is twice the radius. Express the radius as a function of the volume.

C. The volume of a right circular cylinder is $12\pi$. 
(a) Express the height as a function of the radius.

(b) Express the total surface area as a function of the radius.

D. The volume $V$ and the surface area $S$ of a sphere of radius $r$ are given by the formulas $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. Express $V$ as a function of $S$.

E. The hypotenuse of a right triangle is 20 cm. Express the area of the triangle as a function of the length $x$ of one of the legs.

F. In the figure. Express $AB$ as a function of $x$.

Graph and on the graph mark the $x$ and $y$-intercepts. Label the degrees of the roots. No credit if the graph isn't smooth. E.g., the graph of $|x|$ isn't smooth.

G. $y = (x - 4)^3 - 2$

H. $y = -2(x + 5)^4$

I. $y = \frac{1}{2}(x + 1)^5$

J. $y = -(x - 1)^3 - 1$

K. $y = (x - 2)(x - 1)(x + 1)$

L. $y = 2x(x - 2)(x - 1)$

Answers
A. $A = \sqrt{3}x^2/4$
B. $r = \frac{3}{2}\sqrt{V/2\pi}$
C. (a) $h(r) = 12/r^2$ (b) $S(r) = 2\pi r^2 + 24\pi/r$
D. $V(S) = S(\sqrt{S}/(6\sqrt{\pi})$
E. $A(x) = \frac{1}{2}x\sqrt{400 - x^2}$
F. $d(x) = (x + 4)\sqrt{x^2 + 25}/x$

See Hw 9 worked examples for remaining answers.
D. The volume $V$ and the surface area $S$ of a sphere of radius $r$ are given by the formulas $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. Express $V$ as a function of $S$.

Given: $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$
Want $V$ in $S$, need $r$ in $S$.

$\frac{S}{4\pi} = r^2$
$r = \sqrt{\frac{S}{4\pi}}$
$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \left(\sqrt{\frac{S}{4\pi}}\right)^3$
Answer: $V(S) = S\sqrt{S/6\pi}$

E. The hypotenuse of a right triangle is 20 cm. Express the area of the triangle as a function of the length $x$ of one of the legs.

$y \quad 20$
$x$

Given
$A = \frac{1}{2}xy$
$y^2 + x^2 = 20^2$

Want $A$ in $x$, need $y$ in $x$

$y^2 = 400 - x^2$
$y = \sqrt{400 - x^2}$
$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{400 - x^2}$
Answer: $A(x) = \frac{1}{2}x\sqrt{400 - x^2}$

F. In the figure. Express $AB$ as a function of $x$.

Let $h$ be the height pictured.

Given: $(AB)^2 = (x + 4)^2 + h^2$  Pythagorean Theorem
$\frac{h}{x+4} = \frac{5}{x}$  Similar triangles

Want $AB$ in $x$, need $h$ in $x$.

$h = \frac{5(x+4)}{x}$
$AB = \sqrt{(x + 4)^2 + h^2} = \sqrt{(x + 4)^2 + \frac{25(x+4)^2}{x^2}}$
$= (x + 4)\sqrt{1 + \frac{25}{x^2}} = (x + 4)\sqrt{\frac{x^2+25}{x^2}}$
Answer: $d(x) = (x + 4)\sqrt{x^2 + 25 /x}$

G. $y = (x - 4)^3 - 2$
Start with the graph of $x^3$, shift right 4 units, then down 2 units. The $y$-intercept is -66. There is one root $x = 4 + \sqrt[3]{2}$ of degree 1.

H. $y = -2(x + 5)^4$. Start with $x^4$ (looks like $x^2$), shift left 5 units, reflect vertically around the $x$-axis, stretch vertically away from the $x$-axis by a factor of 2.

I. $y = \frac{1}{2}(x + 1)^5$. Start with $x^5$ (looks like $x^3$), shift left 1 unit, compress vertically toward the $x$-axis.

J. $y = -(x - 1)^3 - 1$. Start with $x^3$, shift right 1 unit, reflect vertically around the $x$-axis, shift down 1 unit.

K. $y = (x - 2)(x - 1)(x + 1)$
The $y$-intercept is 2, the roots are $x = 2, 1, -1$, all of degree 1.

L. $y = 2x(x - 2)(x - 1)$
The $y$-intercept is 0, the roots are $x = 0, 2, 1$, all of degree 1.
**Math 140  Lecture 10**  

**Rational functions and their graphs**  

**Definition.** A rational function is a ratio of two polynomials. It is *reduced* if the top and bottom have no common factors.

Like polynomials, rational functions have smooth graphs. But they may have asymptotes.

- In the graphs below, \( x=0 \) (the \( y \)-axis) is the vertical asymptote, \( y=0 \) (the \( x \)-axis) is the horizontal asymptote. Let \( \text{v.a.} \) and \( \text{h.a.} \) abbreviate vertical and horizontal asymptotes.

\[
\begin{array}{ccc}
\text{v.a.} & \text{h.a.} & \text{v.a.} \\
\hline
y=1/x & y=0 & y=-1/x \\
x=0 & y=0 & x=0
\end{array}
\]

- For odd degree vertical asymptotes, one side goes to \( +\infty \), the other to \( -\infty \). See \( 1/x \) and \( -1/x \).

- For even degree vertical asymptotes, both sides go to \( +\infty \) or both go to \( -\infty \). See \( 1/x^2 \) and \( -1/x^2 \).

**Definition.** For rational functions, the *leading term* is the reduced ratio of the leading terms of the top and bottom.

Recall, to get the leading term of a factored polynomial, replace each factor to its leading term and then simplify.

- **Rational functions:**

  \[
  \frac{\frac{x}{1-x}}{\frac{x^2+1}{3x^2}} = \frac{(x-1)(5x+2)}{(x-3)(1-2x)^2} = \frac{1-2x}{2x-6} \\
  \text{Hor. asymptote: } y = -1, y = \frac{1}{3}, y = 0 \text{ none}
  \]

For a reduced rational function:

- **\( x \)-intercepts** (roots) occur where the top is 0. If the root has degree \( n \), the \( x \)-intercept looks like that of \( y=x^n \) or \( y=-x^n \).

- **\( x \)-intercepts** (roots) occur where the top is 0. If the factor has degree \( n \), the vertical asymptote looks like that of \( y=1/x^n \) or \( y=-1/x^n \).

- As \( x \to \pm\infty \), the graph resembles the graph of the leading term which is either a constant \( b \), a fraction \( a/x^n \) or a term \( ax^n \) of positive degree.

1. If a constant \( b \), then \( y=b \) is a horizontal asymptote.
2. If it is \( a/x^n \), then \( y=0 \) is a horizontal asymptote.
3. If it is \( ax^n \), there is no horizontal asymptote.

**Graph.** On the graph mark the \( x \) and \( y \)-intercepts. Mark the vertical and horizontal asymptotes with their equations (\( y=a \) or \( x=a \)).

**Key intervals** lie before, between and after key numbers. In each key interval, calculate a *key value*.

- \( y = \frac{2x^2-6}{2x^3-8x^2} \)

  - reduce and factor: \( \frac{x^2(x-4)}{x^3-4x^2} = \frac{x-3}{x^2(x-4)} \)
  - y-intercept: none
  - x-intercept: 3 (deg 1)
  - vertical asymptotes: \( x=0 \) (deg 2), \( x=4 \) (deg 1)
  - lead term: \( \frac{2x^2}{2x^2} = \frac{1}{x^2} \)
  - horizontal asymptote: \( y=0 \)
  - key values: \( f(-1)=4/5 \), \( f(1)=2/3 \), \( f(7/2)=-4/49 \), \( f(5)=2/25 \)

- \( y = \frac{2x^3-8x^2}{2x-6} \)

  - reduce and factor: \( \frac{x^2(x-4)}{2x-6} \)
  - y-intercept: 0
  - x-intercepts: 0 (deg 2), 4 (deg 1)
  - vertical asymptote: \( x=3 \) (deg 1)
  - lead term: \( \frac{2x^3}{2x} = x^2 \)
  - horizontal asymptote: none
  - key values: \( f(-1)=5/4 \), \( f(1)=3/2 \), \( f(7/2)=-49/49 \), \( f(5)=25/2 \)
6(a)(11) \( y = \frac{x+1}{x(1-x)} \). Graph, intercepts, asymptotes.

<table>
<thead>
<tr>
<th>x-intercept = -1</th>
<th>y-intercept = none</th>
</tr>
</thead>
<tbody>
<tr>
<td>v.a.: ( x = 0, x = 1 )</td>
<td></td>
</tr>
<tr>
<td>lead. term: (-1/x)</td>
<td>h.a.: ( y = 0 )</td>
</tr>
</tbody>
</table>

Draw the graph. Indicate the asymptotes with dotted lines.

Graph pieces have been moved and exaggerated.
The center piece has been moved down to be visible.
The vertical scale of the left piece has been exaggerated.
On tests and homework, you must show all pieces.
If you use a graphing calculator for homework, you may have to shrink the coordinate system to see all pieces.

(b)(12) \( y = \frac{x(1-x)}{x+1} \). Graph. List intercepts, asymptotes.

<table>
<thead>
<tr>
<th>x-intercept = 0, 1</th>
<th>y-intercept = 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>v.a.: ( x = -1 )</td>
<td></td>
</tr>
<tr>
<td>lead. term: (-x)</td>
<td>h.a.: ( y = \text{none} )</td>
</tr>
</tbody>
</table>

Draw the graph. Indicate any horizontal/vertical asymptote.
6(a) \[ y = \frac{(x-2)^2}{x^2+2x} = \frac{(x-2)^2}{x(x+2)}. \]

<table>
<thead>
<tr>
<th>Table: Key Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>x-intercepts: ( x = )</td>
</tr>
<tr>
<td>y-intercept: ( y = )</td>
</tr>
<tr>
<td>Vertical asymptotes: ( x = )</td>
</tr>
<tr>
<td>( x = )</td>
</tr>
<tr>
<td>Leading Term:</td>
</tr>
<tr>
<td>Horizontal asymptote: ( y = )</td>
</tr>
<tr>
<td>Key values: ( f(x) = \frac{(x-2)^2}{x(x+2)} )</td>
</tr>
<tr>
<td>values, ( f(-3) = )</td>
</tr>
<tr>
<td>not just +,-</td>
</tr>
<tr>
<td>( f(__) = )</td>
</tr>
<tr>
<td>( f(3) = \frac{1}{15} )</td>
</tr>
<tr>
<td>Four values, one for each key interval.</td>
</tr>
</tbody>
</table>

**Draw the graph.**
Indicate the horizontal asymptote and two vertical asymptotes with dotted lines.
Label the asymptotes with their equations.
Label the root with its \( x \)-coordinate.
The graph has three pieces.
It crosses its horizontal asymptote once.
Graphs never cross vertical asymptotes but they can cross horizontal asymptotes.
Since there is only one root and it has even degree, the graph touches but does not cross the \( x \)-axis.
§3.1 262:5-10, 23-30.
Graph and on the graph mark the x and y-intercepts. Label the degrees of the roots. No credit if the graph isn’t smooth.
1(2). \( y = x^3 - 9x \)
   There are two turning points.
   roots:
   key values:
   
2(2). \( y = (x - 1)(x - 4)^2 \)
   There are two turning points.
   roots:
   key values:
   
§3.6 313:11-14, 33-46.
Graph. On the graph mark the x and y-intercepts. Mark the vertical and horizontal asymptotes with their equations (y=a or x=a). Just writing "a" won’t do. Graphs must be smooth.
3(4). \( y = \frac{x-1}{x+1} \)
   One y-intercept; one x-intercept.
   roots:
   vert. asym:
   leading term:
   hor. asym:
   key values:
   
4(4). \( y = \frac{-1}{(x-2)^2} \)
   There is just one intercept.
   roots:
   vert. asym:
   leading term:
   hor. asym:
   key values:
   
5(4). \( y = \frac{x}{(x+1)(x-3)} \)
   Two vertical, one horizontal asymptote. One intercept.
   roots:
   vert. asym:
   leading term:
   hor. asym:
   key values:
   
Check your graphs at https://www.desmos.com/calculator
Graph and on the graph mark the x and y-intercepts. Label the degrees of the roots. No credit if the graph isn't smooth.

A. \(y = x^3 - 4x^2 - 5x\)

B. \(y = x^3 + 3x^2 - 4x - 12\)

C. \(y = x^3(x + 2)\)

D. \(y = 2(x - 1)(x - 4)^3\)

E. \(y = (x + 1)^2(x - 1)(x - 3)\)

Graph. On the graph mark the x and y-intercepts. Mark the vertical and horizontal asymptotes with their equations \((y = a \text{ or } x = a)\). The graphs must be smooth.

F. \(y = -2/(x - 3)\)
   - roots:
   - vert. asym:
   - leading term:
   - hor. asym:

G. \(y = (x - 3)/(x - 1)\)
   - roots:
   - vert. asym:
   - leading term:
   - hor. asym:
   - values:

H. \(y = (4x - 2)/(2x + 1)\)
   - roots:
   - vert. asym:
   - leading term:
   - hor. asym:
   - values:

I. \(y = 1/(x - 2)^2\)
   - roots:
   - vert. asym:
   - leading term:
   - hor. asym:
   - values:

J. \(y = 3/(x + 2)^2\)
   - roots:
   - vert. asym:
   - leading term:
   - hor. asym:
   - values:

K. \(y = 1/(x + 2)^3\)
   - roots:
   - vert. asym:
   - leading term:
   - hor. asym:
   - values:

L. \(y = -4/(x + 5)^3\)
   - roots:
   - vert. asym:
   - leading term:
   - hor. asym:
   - values:

M. \(y = -x/[(x + 2)(x - 2)]\)
   - roots:
   - vert. asym:
   - leading term:
   - hor. asym:
   - values:

Answers
For answers to B33, E39, G15, J21, M27, see Hw 10 worked examples. For the others, see the next 2 pages. Continued on next two pages.
Graph and on the graph mark the x and y-intercepts. Label the degrees of the roots. No credit if the graph isn't smooth.

A. \( y = x^3 - 4x^2 - 5x \)

C. \( y = x^3(x + 2) \)

D. \( y = 2(x - 1)(x - 4)^3 \)

H. \( y = (4x - 2)/(2x + 1) \)

I. \( y = 1/(x - 2)^2 \)

Graph. On the graph mark the x and y-intercepts. Mark the vertical and horizontal asymptotes with their equations \((y = a \text{ or } x = a)\). The graphs must be smooth.

F. \( y = -2/(x - 3) \)

roots: none

vert. asym: \( x = 3 \)

leading term: \(-2/x\)

hor. asym: \( y = 0 \)

Continued on next page.
K. $y = \frac{1}{(x + 2)^3}$
   
   roots: none
   
   vert. asym: $x = -2$
   
   leading term: $\frac{1}{x^3}$
   
   hor. asym: $y = 0$

L. $y = \frac{-4}{(x + 5)^3}$
   
   roots: none
   
   vert. asym: $x = -5$
   
   leading term: $\frac{-4}{x^3}$
   
   hor. asym: $y = 0$
Math 140  Hw  10  Worked examples of selected recommended problems.

Graph and on the graph mark the x and y-intercepts. Label the degrees of the roots. No credit if the graph isn't smooth.

B. \( y = x^3 + 3x^2 - 4x - 12 \)
For large \( x \), graph looks like \( x^3 \).
y-intercept = constant term = -12
Factored form: \( y = (x-2)(x+2)(x+3) \)
Roots: \( x = -3, -2, 2 \) each with degree 1.

E. \( y = (x+1)^2(x-1)(x-3) \)
lead term = \((x)^2(x)(x) = x^4\)
constant term = \((1)^2(-1)(-3) = 3\)
roots: \( x = -1 \) deg 2, 1 deg 1, 3 deg 1.

Graph. On the graph mark the x and y-intercepts. Mark the vertical and horizontal asymptotes with their equations (y=a or x=a). The graphs must be smooth.

G. \( y = (x-3)/(x-1) \)
roots: \( x = 3 \)
vert. asym: \( x = 1 \)
leading term: 1
hor. asym: \( y = 1 \)
values: \( f(0)=3, f(2)=-1, f(4)=1/3 \)
**Math 140  Lecture 11**

### Exponential functions

**Definition.** An exponential function is of the form \( y = b^x \) with the base \( b > 0 \).

- \( b^0 = 1 \), \( b^1 = b \), \( b^2 = b \cdot b \), ...
- \( b^{-n} = 1/b^n \)
- \( b^{1/n} = \sqrt[n]{b} \), the \( n \)th root of \( b \)
- \( b^{\rho/\eta} = \left( b^\rho \right)^{1/\eta} = (b^{1/\eta})^\rho \)

### Exponent rules

| \((b^n)^m = b^{nm}\) | \((5^2)^3 = (5^2)(5^2)(5^2) = 5^6\) |
| \(b^n b^m = b^{n+m}\) | \(5^2 5^3 = (5 \cdot 5)(5 \cdot 5 \cdot 5) = 5^5\) |
| \(b^n/b^m = b^{n-m}\) | \(5^3/5 = 5^{-4} = 1/5^4 = 5^7/5^3 = 5^4\) |
| \((ab)^n = a^n b^n\) | \(2^5 3^5 = 6^5\) |
| \((a/b)^n = a^n b^{-n}\) | \((2/3)^5 = 2^5/3^5\) |

Simplify to an integer or a single exponent \( b^n \).

- \( \sqrt{2}^{1/2} \sqrt[3]{2}^{3/2} = \sqrt{2}^{1/2 + 3/2} = (2^{1/2})^{4/2} = 2^{1/2} = 2^1 = 2 \)
- \( (2^{1/3})^{1/5} = 2^{1/3 \cdot 1/5} = 2^{3/15} = 2^{1/5} = 2^3 = 8 \)
- \( (3^{2/3} \sqrt[5]{3})(3^{2/3} \sqrt[5]{3}) = 3^{2/3 \cdot 2/5 + 2/5} = 3^4 = 81 \)
- \( 8^{1-\pi} = 8^{(1-\pi)-(1+\pi)} = 8^{1-\pi-1-\pi} = 8^{-2\pi} \)

**Property.** If \( b \neq 1 \), \( b^x = b^y \iff x = y \). Hence \( b^x \) is 1-1.

### Gateway problems with the variable in the exponent: put both sides above a common base then equate exponents.

- \( 27^x = 9 \)
  - \((3^3)^x = 3^2\)
  - \(3^{3x} = 3^2\)
  - \(3x = 2\)
  - \(x = 2/3\)
- \( 8^{x+1} = 32/\sqrt{2} \)
  - \(8^{x+1} = 2^x 2^{-1/2} \)
  - \((2^{3})^{x+1} = 2^{5 \cdot 1} \)
  - \(2^{3x+3} = 2^{9/2} \)
  - \(3x + 3 = 9/2 \)
  - \(6x + 6 = 9 \)
  - \(6x = 3 \)
  - \(x = 1/2\)
- \( 2^x = 32/\sqrt{2} \)
  - \(2^x = 2^{5/2} \)
  - \(2^x = 2^{5 \cdot 1/2} \)
  - \(2^x = 2^{5 \cdot 1/2} \)
  - \(z = 1/2\)

Note, this is different, the variable is in the base rather than the exponent. Raise both sides to the 1/8 power.

### Estimate \( 2^{50} \) and \( 2^{-20} \)

- \(2^{50} = (2^{10})^5 \approx (10^3)^5 = 10^{15}\)
- \(2^{-20} = (2^{10})^{-2} \approx (10^3)^{-2} = 10^{-6}\)

The graph of \( y = 1^x \) is the horizontal line \( y = 1 \).
Otherwise, the graph of \( y = b^x \)
- has y-intercept 1 but no x-intercept,
- it goes to \( \infty \) in one direction,
- it has the horizontal asymptote \( y = 0 \) in the other.

For \( b > 1 \), the graph of \( b^x \) is like the graph of \( 2^x \) as below. For \( 0 < b < 1 \), the graph is like \((1/2)^x\).

**The function \( e^x \)**

**Definition.** \( e^x \) is the unique exponential function \( b^x \) whose the tangent at \((0, 1)\) has slope 1.

**Fact:** \( e \approx 2.7 \) Again, \( \approx \) means approximately equal.

Thus \( 2 < e < 3 \). Hence the graph of \( e^x \) lies between the graphs of \( 2^x \) and \( 3^x \).

Similarly, \( e^{-x} \) lies between \( 2^{-x} \) and \( 3^{-x} \).

**True or false?**

- \( \sqrt{e} < 1 \) false since \( \sqrt{e} \approx \sqrt{2.7} > \sqrt{1} = 1 \)
- \( e^2 < 9 \) true since \( e^2 \approx 2.7^2 < 3^2 = 9 \)
9(3). Simplify to at most 3 symbols.

(a) \( \left( \sqrt{5} \cdot \sqrt{2} \right)^{\sqrt{2}} = \)
\( = (\sqrt{5} \cdot \sqrt{2})^{\sqrt{2}} = (\sqrt{5} \cdot 2)^{\sqrt{2}} = 5 \)

(b) \( \sqrt[5]{\frac{1 + \sqrt{2}}{1 - \sqrt{2}}} = \)
\( = \sqrt[5]{\frac{1 + \sqrt{2}}{1 - \sqrt{2}}} \cdot \sqrt[5]{\frac{1 - \sqrt{2}}{1 - \sqrt{2}}} \)
\( = \sqrt[5]{(1 + \sqrt{2})^{1 - \sqrt{2}}} \)
\( = \sqrt[5]{1 + \sqrt{2} + \sqrt{2}} = \sqrt[5]{2} \sqrt[5]{2} \)
\( = ((\sqrt[5]{2})^2)^{\sqrt{2}} = 5 \sqrt{2} \)

7(2). Estimate as a power of 10. \( 2^{-80} \)

\( 2^{-80} = (2^{10})^{-8} \)
\( \approx (10^3)^{-8} \)
\( = 10^{-24} \)

10(5). Graph \( y = 5 - 10^{-x} \). Find, if any, the horizontal and vertical asymptotes.

Asymptotes must be equations, not numbers. E.g. \( x=3, y=3 \), not just 3.

horizontal asymptote \( y=5 \)
vertical asymptote \( none \)

Graph

\[ 10^x \rightarrow 10^{x-2} \rightarrow -10^{x-2} \rightarrow -10^{x-2} + 5 \]
7. Estimate as a power of 10: $2^{50} \approx \sqrt{2}$
First write $2^{50}$ as a power of $2^{10}$, e.g. $(2^{10})^n$.
Then use the approximation $2^{10} \approx 10^3$.

9. Simplify to at most 3 symbols.
   
   (a) $\left(\sqrt[3]{3} \sqrt[5]{5}\right)^{\sqrt[3]{5}} = \ldots$

   Initially, leave the radical $\sqrt[3]{3}$ alone.
   Recall that nested exponents multiply.
   1st get rid of the radical in the exponent.
   2nd get rid of the radical $\sqrt[3]{3}$ in the base.
   The answer is a two-digit integer.

   (b) $\frac{\sqrt[3]{3} \sqrt[5]{5}}{\sqrt[3]{3} \sqrt[5]{3}} = \ldots$

   Bring the denominator upstairs by negating its exponent.
   Simplify the combined exponent.
   Get rid of the radical $\sqrt[3]{3}$.
   The answer has an integer in the base, a radical in the exponent.
   3 symbols.

Gateway problems

Solve for $x$

First write $\frac{121}{\sqrt{11}}$ as a power of 11. Hint: $\sqrt[11]{11} = 11^{1/3}$, $11^2 = 121$, $\sqrt{11} = 11^{1/2}$.

A. $\left(\sqrt[11]{11}\right)^{2x-2} = \frac{121}{\sqrt{11}} \ldots$

   4 symbol fraction.

B. $\left(\sqrt{11}\right)^4 = \frac{121}{\sqrt{11}} \ldots$

   5 symbols.
§3.6 313:47-56. Graph. On the graph mark the x and y-intercepts. Mark the vertical and horizontal asymptotes with their equations (y = a or x = a). The graphs must be smooth.

1(4). \( y = \frac{2x}{(x+1)^2} \)  
1 point graph, 1 root, 1 v.a., 1 h.a.

roots:
vert. asym:
lead term:
hor. asym:

2(1). Estimate as a power of 10.  \( 2^{90} \approx \)
4 symbols

§1.2 21:31-70

In 3, 4, 5: simplify to an integer or single exponent \( b^n \).

*9. \( ((\sqrt[3]{5})^{\frac{1}{2}})^2 = (5^{1/2})^{1/2} = 5^{1/4} \)

3(1). \( ((\sqrt[3]{5})^{\sqrt{2}})^{\sqrt{2}} \)
single digit

4(1). \( \frac{10^{x+2}}{10^{x-2}} \)
3 symbols

5(1). \( ((\sqrt[3]{3})^x)^4 \)
2 symbols

6(1). Solve for \( t \): \( 2t = \frac{1}{4} \)
2 symbols

7(1). Solve for \( y \): \( 2^{3y+1} = \sqrt{2} \)
4 symbol negative fraction.

§4.1 337:19-32.

8(3). Graph and on the graph mark the x-intercept, y-intercept, and horizontal asymptote.
\( y = -3^x + 3 \)

9(3). Graph and on the graph mark the x-intercept, y-intercept, and horizontal asymptote.
\( y = 3^{-x} - 3 \)

10(3). Graph and on the graph mark, if any, the x-intercept, y-intercept, and horizontal asymptote.
\( y = -e^{-x} \)

Check your graphs at https://www.desmos.com/calculator
A. Estimate as a power of 10:
   (a) $2^{30}$  (b) $2^{50}$

In B-E: simplify to a number of the form $b^n$.
B. $(5^{\sqrt{3}})^{\sqrt{3}}$
C. $(4^{1+\sqrt{2}})(4^{1-\sqrt{2}})$
D. $\frac{2^{4+\pi}}{2^{1+\pi}}$
E. $(\sqrt{5}^{\sqrt{2}})^2$

F. Solve for $y$: $3^{1-2y} = \sqrt{3}$

G. Solve for $z$: $3^z = 9\sqrt{3}$

**Answers**
A(a). $2^{30} \approx 10^9$  (b) $2^{50} \approx 10^{15}$
B. 125  C. 16
D. 8  E. $5\sqrt{2}$
F. $y = 1/4$  G. $z = 5/2$

H. Graph and on the graph mark the $x$-intercept, $y$-intercept, and horizontal asymptote.
$y = -2^x + 1$

I. Graph and on the graph mark the $x$-intercept, $y$-intercept, and horizontal asymptote.

J. Graph and on the graph mark the $x$-intercept, $y$-intercept, and horizontal asymptote.
$y = 3^{-x} + 1$

K. Graph and on the graph mark the $x$-intercept, $y$-intercept, and horizontal asymptote.
$y = -e^{x-2}$

**Answers**
See Hw 11 worked examples.
A. Estimate as a power of 10:
(a) \(2^{30} \approx 10^9\)

In B-E: simplify to a number of the form \(b^n\):

B. \((5 \sqrt[3]{7})^3\)
\((5 \sqrt[3]{7})^3 = 5 \cdot 7 = 35 = 125\)

C. \((4 + \sqrt{2})(4 - \sqrt{2})\)
\((4 + \sqrt{2})(4 - \sqrt{2}) = 4^2 - (\sqrt{2})^2 = 16 - 2 = 14\)

D. \(\frac{2^{4+\pi}}{2^{1+\pi}}\)
\(\frac{2^{4+\pi}}{2^{1+\pi}} = 2^{(4+\pi)-(1+\pi)} = 2^{3-\pi} = 8\)

E. \((\sqrt{5} \sqrt[2]{2})^2\)
\((\sqrt{5} \sqrt[2]{2})^2 = (\sqrt{5})^2 = (\sqrt{2})^2 = 5\sqrt{2}\)

F. Solve for \(y\): \(3^{1-2y} = \sqrt{3}\)
\(3^{1-2y} = \sqrt{3} \iff 3^{1-2y} = 3^{1/2}\)
\(1 - 2y = 1/2 \iff 2 - 4y = 1 \iff -4y = -1 \iff y = 1/4\)

G. Solve for \(z\): \(3^z = 9 \sqrt{3}\)
\(3^z = 9 \sqrt{3} \iff 3^z = 3^{3/2}\)
\(z = 5/2\)

H. Graph and on the graph mark the \(x\)-intercept, \(y\)-intercept, and horizontal asymptote.
\(y = -2^x + 1\)
Start with \(2^x\), reflect around \(x\)-axis, shift up 1.

I. Graph and on the graph mark the \(x\)-intercept, \(y\)-intercept, and horizontal asymptote.
\(y = 3^{-x} + 1\)
Start with \(3^x\), reflect across \(y\)-axis, shift up 1.

J. Graph and on the graph mark the \(x\)-intercept, \(y\)-intercept, and horizontal asymptote.
\(y = 2^{x-1}\)
Start with \(2^x\) and shift right 1.

K. Graph and on the graph mark the \(x\)-intercept, \(y\)-intercept, and horizontal asymptote.
\(y = -e^{x-2}\)
Start with \(e^x\), shift right 2 units, reflect across \(x\)-axis.
Math 140  Lecture 12
Exam 2 covers Lectures 7 -12. Study the recommended exercises.
Review area, circumference, volume formulas - inside front cover.
RECALL. The graphs of \( e^x \) and \( e^{-x} \).

Graph \( y = e^{x-1} - 1 \).

Give the domain, range, intercepts and asymptotes.

- **x-intercept:** \( x = 1 \)
- **y-intercept:** \( y = \frac{1}{e} \approx -0.367 \)
- **hor. asym.:** \( y = -1 \)
- **domain:** \( (-\infty, \infty) \)  
  **range:** \( (-1, \infty) \).

Logarithms

Assume \( b > 0, b \neq 1 \). Thus \( b^x \) is 1-1 and it has an inverse.

**Definition.** \( \log_b(x) \), the log of \( x \) to the base \( b \), is the inverse of the exponential function \( b^x \).

\[ \log_b(x) \text{, the natural logarithm, } = \log_e(x) = \text{the inverse of } e^x. \]

Note, “ln” is “el-n” not “one-n”, not “eye-n”.

Inverses act in opposite directions and inverses cancel.

\[ y = \log_b(x) \text{ iff } b^y = x. \]
\[ y = \ln(x) \text{ iff } e^y = x. \]

\[ \log_b(b^x) = x, \quad \ln(e^x) = x. \]

\[ b^{\log_b(x)} = x, \quad e^{\ln(x)} = x. \]

If we have \( \ln(b^x) \) instead of \( \ln(e^x) \), then \( \ln \) and \( b^x \) don’t completely cancel and \( \ln(e^x) = x \) becomes:

\( \ln(b^x) = x \ln b \). The exponent comes down to the outside.

**FACT.** \( e^0 = 1 \Rightarrow 0 = \ln(1) \).

\( e^1 = e \Rightarrow 1 = \ln(e) \)

- **Simplify to a rational.**
  \begin{align*}
  \log_5 \sqrt[5]{5} &= \log_5 5^{1/5} = \log_5 5^{1/2} = \log_5 5 = \frac{3}{2}.
  \\
  \log_2 \frac{1}{8} &= \log_2 2^{-3} = -3.
  \\
  \log_8 2 &= 1st \ solve \ 8^x = 2. \ (2^3)^x = 2^1, \ 3x = 1, \ x = \frac{1}{3}.
  \\
  \log_8 2 &= \log_8 8^{1/3} = 1/3.
  \\
  \end{align*}

\[ 2^3 = 8 \quad \text{and} \quad 2^2 = 4 \]

Since 2' and \( \log_2(x) \) act in opposite directions,

\[ 3 = \log_2(8) \quad \log_2(x) \text{ sends } 3 \text{ to } 8. \]

\[ 5^2x-1 = 6 \]

Take the log of both sides.

\[ \log_5(5^{2x-1}) = \log_5(6) \]

\[ 2x - 1 = \log_5(6) \]

**Solve for \( x \), write the answer using logarithms.**

\[ 5x^2 = 4 \]

\[ x^2 = \log_5(4) \]

\[ x = \pm \sqrt{\log_5(4)} \]

Leave answers in exact form, no decimals.

\[ e^{3x+1} = 8 \]

\[ 3t + 1 = \ln(8) \]

\[ 3t = \ln(8) - 1 \]

\[ t = \frac{1}{3}(\ln(8) - 1) \]

\[ 3^{2x} = 5^{x+1} \]

Use natural logarithms and method 2.

\[ \ln(3^{2x}) = \ln(5^{x+1}) \]

\[ 2x \ln 3 = (x + 1) \ln 5 \]

\[ x(2 \ln 3 - \ln 5) = \ln 5 \]

\[ x = \frac{\ln 5}{(2 \ln 3 - \ln 5)} = \frac{\ln 5}{\ln 9/5} \]

See Practice Exam 2, prob. 11b.

Since they are inverses, the graph of \( \log_b(x) \) is the reflection of \( b^x \) across the major diagonal and \( \ln(x) \) is the reflection of \( e^x \).

**Write in log form.** Take the appropriate log of both sides and cancel inverses. Or use the fact that exponentiation and logarithms act in opposite directions.

- **Graph \( y = \ln(x+1) + 1 \).**
  - **x-intercept:** \( x = -1 \), refl. across y, right 1, refl. across x, up 1.
  - **domain:** \( (-1, \infty) \)
  - **range:** \( (-\infty, \infty) \).
9(2). Simplify to at most 3 symbols.

\[ \log_8 \sqrt{2} = \]
\[ = \log_8 8^{1/3} \]
\[ = \log_8 (8^{1/3})^{1/2} \]
\[ = \log_8 8^{1/6} = 1/6 \]

11(6). \( 5^{x-1} = 2^{x+1} \). Solve for \( x \) using natural logarithms.

\[ \ln(5^{x-1}) = \ln(2^{x+1}) \]
\[ (x-1) \ln 5 = (x+1) \ln 2 \]
\[ x \ln 5 - \ln 5 = x \ln 2 + \ln 2 \]
\[ x \ln 5 - x \ln 2 = \ln 2 + \ln 5 \]
\[ x(\ln 5 - \ln 2) = \ln 2 + \ln 5 \]
\[ x = \frac{\ln 2 + \ln 5}{\ln 5 - \ln 2} = \frac{\ln 10}{\ln 5/2} \]

10(8). \( y = 3 - \ln(2 - x) \). For an asymptote, you must give an equation, not just a number.

<table>
<thead>
<tr>
<th>y-intercept = 3 - ln(2)</th>
<th>domain = (-\infty, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical asymptote: ( x = 2 )</td>
<td></td>
</tr>
<tr>
<td>Horizontal asymptote: none</td>
<td></td>
</tr>
</tbody>
</table>

List the sequence of reflections and shifts which gets you from \( \ln(x) \) to \( 3 - \ln(2 - x) \)
Start with the rewritten value \( \ln(2 - x) = \ln(-x + 2) = \ln(-(x - 2)) \), work forward to \( 3 - \ln(2 - x) \) and backward to \( \ln(x) \).

\[ \ln(x) \rightarrow \ln(-x) \rightarrow \ln(-(x - 2)) \rightarrow -\ln(-(x - 2)) \rightarrow 3 - \ln(-(x - 2)) \]
reflect horizontally in \( y \) right 2 reflect vertically in \( x \) up 3

Draw the graph. Indicate the asymptote with a dotted line.
9. Simplify to at most 3 or 4 symbols.
   (a) \( \log_9 \left( \frac{1}{3} \right) = \)
   (b) \( \log_9 (\sqrt{3}) = \)

Hint for (a): write the argument as a power of the base: solve \( 9^x = \frac{1}{3} \).

Replace the argument of the logarithm
   with this power of its base: \( \log_9 \left( \frac{1}{3} \right) = \log_9 (9^x) \)

Use the fact that \( \log_b b^x = x \). Do (b) similarly.

11. \( 3^{x-1} = 2^{x+4} \). Solve for \( x \) using natural logarithms.

Take the natural logarithm of both sides.

\[ \ln 3^{x-1} = \ln 2^{x+4} \]

Bring the exponents inside the logarithm outside.
They become coefficients on the outside.
Use \( \log x^n = n \log x \).

Get terms involving \( x \) on the left, everything else (the constants) on the right.

Factor out \( x \) and then divide to solve for \( x \). 16 symbol fraction.

Use the following properties to simplify the answer to a ratio of two logarithms, e.g., \( \frac{\ln 40}{\ln(5/3)} \), 12 symbols.
Exam 2 next week.
§4.1 337:33-38. Graph. List the intercepts, the asymptotes and the domain and range. Write “none” if they don’t exit.
1(2). \( y = e^{x+1} \) Two “none” ’s, three “\( \infty \)” symbols.

\[
\begin{array}{ccc}
\text{x-intercept} & \text{y-intercept} \\
\text{hor. asym.} & \text{vert. asym.} \\
\text{domain} & \text{range} \\
\end{array}
\]

In 2, 3, and 4, circle true, or false.

2(1). \( e^2 < 4 \) \hspace{1cm} true? false?
3(1). \( e^3 < 27 \) \hspace{1cm} true? false?
4(1). \( e^0 = 1 \) \hspace{1cm} true? false?

§4.2 349:3-14. Rewrite each equation (do not solve).
One answer has a fraction.

5(1). \( \frac{1}{125} = 5^{-3} \) Rewrite using logarithms.
6(1). \( \log_5 25 = x \) Rewrite using exponents.

8(1). \( e^{3t} = 8 \) Rewrite using logarithms.
7(1). \( \ln 3t = 8 \) Rewrite using exponents.

§4.2 349:15-24. Simplify to a rational number. Two fractions, one digit.

9(1). \( \log_{10} \left( \frac{1}{54} \right) \)
10(1). \( \log_{10} (10) \)
11(1). \( \log_2 (8 \sqrt{2}) \)

§4.2 350:41-46, 49-56. Graph. List the intercepts, asymptotes, domain and range.
12(2). \( y = -\ln(x) \) Two “none” ’s, three “\( \infty \)” symbols.

\[
\begin{array}{ccc}
\text{x-intercept} & \text{y-intercept} \\
\text{hor. asym.} & \text{vert. asym.} \\
\text{domain} & \text{range} \\
\end{array}
\]

13(2). \( y = \ln(-x) \)

\[
\begin{array}{ccc}
\text{x-intercept} & \text{y-intercept} \\
\text{hor. asym.} & \text{vert. asym.} \\
\text{domain} & \text{range} \\
\end{array}
\]

Simplify to a single-digit integer.

14(1). \( \ln(e) \)
15(1). \( \ln(e^{-2}) \)
16(1). \( (\ln(e))^{-2} \)

§4.4 366:1-22. Solve for \( x \). Write the exact answer (no decimal answers) using a logarithm with the appropriate base. Two answers have fractions, all have a logarithm

17(1). \( 10^{2x-1} = 145 \) \hspace{1cm} \( x = \)
18(1). \( (10^x)^2 = 40 \) \hspace{1cm} \( x = \)
19(1). \( e^{t-1} = 16 \) \hspace{1cm} \( t = \)

20(2). Solve for \( x \) using \( \ln \). \( 7^{-4x} = 2^{1+3x} \) \hspace{1cm} \( x = \)
Fractional answer with 3 ln’s.
Rewrite each equation using logarithms rather than exponentials (no decimal answers).

A.  \(9 = 3^2\)

B.  \(1000 = 10^3\)

C.  \(7^3 = 343\)

D.  \(\sqrt{2} = 2^{1/2}\)

Simplify to a rational number.

E.  \(\log_9 27\)

F.  \(\log_4 (1/32)\)

G.  \(\log_5 5\sqrt{5}\)

Graph. List the intercepts, asymptotes, domain and range.

H.  \(y = -\log_2 (-x)\)  For answer, see Hw 12 worked examples.

\[\text{x-intercept} \quad \text{y-intercept} \quad \text{hor. asym.} \quad \text{vert. asym.} \quad \text{domain} \quad \text{range}\]

J.  \(y = \ln(x + e)\)

\[\text{x-intercept} \quad \text{y-intercept} \quad \text{hor. asym.} \quad \text{vert. asym.} \quad \text{domain} \quad \text{range}\]

Simplify.

K(a).  \(\ln(e^4)\)

(b).  \(\ln(1/e)\)

(c).  \(\ln(\sqrt{e})\)

Solve for \(x\). Write the exact answer (no decimal answers) using a logarithm with the appropriate base.

L.  \(10^x = 25\)  \(x = \)  

M.  \(10^{x^2} = 40\)  \(x = \)

N.  \(e^{2t+3} = 10\)  \(t = \)

Answers

A. \(\log_3 9 = 2\)

B. \(\log_{10} 1000 = 3\)

C. \(\log_7 343 = 3\)

D. \(\log_2 \sqrt{2} = 1/2\)

E. 3/2

F. -5/2

G. 3/2

K(a). 4  (b). -1  (c). 1/2

L. \(\log_{10} 25\)  M. \(\pm \sqrt{1 + \log_{10} 4} \) or \(\pm \sqrt{\log_{10} 40}\)

N. \((-3 + \ln 10)/2\)
Rewrite each equation using logarithms rather than exponentials (no decimal answers).

A. \( 9 = 3^2 \)
   If exponential to the base 3 sends 2 to 9, then its inverse, \( \log_3 \), does the opposite: it sends 9 to 2.
   Answer: \( \log_3 9 = 2 \)
   The basic principle here is:
   \[ y = b^x \text{ iff } \log_b y = x \]

B. \( 1000 = 10^3 \)
   Answer: \( \log_{10} 1000 = 3 \)
   An alternate way to do these problems is to take the logarithm of both sides and then use the fact that \( \log_b b^x = x \)

C. \( 7^3 = 343 \)
   Take \( \log_7 \) of both sides.
   \( \log_7 7^3 = \log_7 343 \)
   \( \therefore 3 = \log_7 343 \)
   Answer: \( \log_7 343 = 3 \)

D. \( \sqrt{2} = 2^{1/2} \)
   \( \log_2 \sqrt{2} = \log_2 2^{1/2} \)
   Answer: \( \log_2 \sqrt{2} = 1/2 \)

Simplify to a rational number.

For these, use the principle: \( \log_b b^x = x \).

E. \( \log_9 27 \)
   \( \log_9 27 = \log_9 3^3 = \log_9 (9^{1/2})^3 = \log_9 9^{3/2} = 3/2 \)
   Answer: \( 3/2 \)

F. \( \log_4 (1/32) \)
   \( \log_4 (1/32) = \log_4 2^{-5} = \log_4 (4^{1/2})^{-5} = \log_4 4^{-5/2} = -5/2 \)
   Answer: \( -5/2 \)

G. \( \log_5 \sqrt{5} \)
   \( \log_5 \sqrt{5} = \log_5 5^{1/2} = \log_5 5^{3/2} = 3/2 \)
   Answer: \( 3/2 \)

Solve for \( x \). Write the exact answer (no decimal answers) using a logarithm with the appropriate base.

L. \( 10^x = 25 \)
   \( \log(10^x) = \log(25) \)
   Answer: \( x = \log_{10} 25 \)

M. \( 10^{x^2} = 40 \)
   \( \log(10^{x^2}) = \log(40) \)
   \( x^2 = \log(40) \)
   Answer: \( \pm \sqrt{\log_{10} 40} \)

N. \( e^{2t+3} = 10 \)
   \( \ln(e^{2t+3}) = \ln(10) \)
   \( 2t + 3 = \ln(10) \)
   \( 2t = -3 + \ln(10) \)
   Answer: \( t = (-3 + \ln 10)/2 \)
Proofs. (In these proofs, the “iff”s are required, -1 if missing.)
\[ \log_b x = \log_b x + \log_b y \quad \text{iff} \quad b^{\log_b x} = b^{\log_b x + \log_b y} \]
\[ xy = b^{\log_b x + \log_b y} \quad \text{iff} \quad xy = xy \quad \text{true.} \]

Likewise for \( \log_b \frac{x}{y} = \log_b x - \log_b y \).
\[ \log_b x = n \log_b y \quad \text{iff} \quad b^{\log_b x} = b^{n \log_b y} \quad \text{iff} \]
\[ x^n = b^{\log_b x} \quad \text{iff} \quad x^n = [b^{\log_b x}]^n \quad \text{iff} \]
\[ x^n = x^n \quad \text{true.} \]

Combine into a single logarithm.
\[ 2 \log_{10} x + \log_{10} y = \log_{10} x^2 + \log_{10} y = \log_{10} x^2 y \]
\[ \log_2 x - 4 \log_2 y = \log_2 x - 4 \log_2 y^4 = \log_2 (x^{\frac{x}{y^4}}) \]
\[ \ln(x^2 + y^2) - \ln(y^3 + a) = \ln \left( \frac{x^2 + y^2}{y^3 + a} \right) \]

Write as a sum/difference/multiple of the simplest possible logarithms.
\[ \log_b \sqrt[3]{\frac{2y}{y^3 + a}} = \log_b \left[ \left( \frac{2y}{y^3 + a} \right)^{\frac{1}{3}} \right] = \frac{1}{4} \log_b (2y) - \log_b (y^3 + a) \]
\[ \ln \left( \frac{1}{\sqrt{x^2 + y^2}} \right) = \ln \left( \frac{1}{x^2 + y^2} \right) = -\frac{1}{2} \ln(x^2 + y^2) \]

Note: \( \log_b (x \cdot y) = \log_b x + \log_b y \neq \log_b (x + y) \). The last term cannot be broken into simpler pieces.

Solving for \( x \).
\begin{itemize}
  \item Get terms involving \( x \) on the left, the rest on the right.
  \item Combine into a single logarithm.
  \item Exponentiate both sides to the base of the logarithm.
  \item Solve.
  \item Delete solutions that give undefined logarithms.
\end{itemize}
1(2). Combine into a single logarithm:
\[ \ln(xy) - x \ln(2) - \frac{1}{2} \ln(x - y) \]

\[ = \ln(xy) - \ln(2^x) - \ln \sqrt{x - y} \]

\[ = \ln \left( \frac{xy}{2^x \sqrt{x - y}} \right) \]

2(2). Write as a sum and/or difference of multiples of \( \log_5 x, \log_5(y - 1), \log_5(x + 1) \):
\[ \log_5 \left( \frac{(y - 1)^3}{\sqrt{x(x + 1)^4}} \right) \]

\[ = \log_5 (y - 1)^3 - \log_5 x^{1/2} - \log_5 (x + 1)^4 \]

\[ = 3 \log_5 (y - 1) - \frac{1}{2} \log_5 x - 4 \log_5 (x + 1) \]

3(6). Solve for \( x \). Show your work. The solution(s) must be valid.
Write "none" if neither solution is valid.

\[ \ln(x - 4) - \ln 6 = -\ln(1 + x) \]

\[ \ln[(x - 4)(1 + x)] = \ln 6 \]

\[ (x - 4)(1 + x) = 6 \]

\[ x + x^2 - 4 - 4x = 6 \]

\[ x^2 - 3x - 10 = 0 \]

\[ (x - 5)(x + 2) = 0 \]

\[ x = 5, -2 \]

At \( x = 5 \) the equation
\[ \ln(5 - 4) - \ln 6 = -\ln(1 + 5) \]
holds with all terms defined
\[ \therefore 5 \text{ is valid} \]

At \( x = -2 \) the equation
\[ \ln(-2 - 4) - \ln 6 = -\ln(1 - 2) \]
has undefined terms
\[ \therefore -2 \text{ is invalid} \]

\[ \text{Ans. } x = 5 \]

4(3). Express in terms logarithms to the base 2: \( \ln 13 \)
\[ \frac{\log_2 13}{\log_2 e} \]
Here is an example from the lecture. Your problem is in the next column.

- Solve for $x$: $\log_2 4x - \log_2 3 = \log_2 (x + 2)$

$\log_2 4x - \log_2 (x + 2) = \log_2 3$

$\log_2 \frac{4x}{x+2} = \log_2 3$

$2^{\log_2 \frac{4x}{x+2}} = 2^{\log_2 3}$

$\frac{4x}{x+2} = 3$

$4x = 3x + 6$
$x = 6$

Validity check for $x = 6$
Substitute 6 into the original equation.
$\log_2 4x - \log_2 3 = \log_2 (x + 2)$
This gives
$\log_2 (4 \cdot 6) - \log_2 3 = \log_2 (6 + 2)$
All logarithms have positive arguments. Hence they are defined and the solution is valid.

3. Solve for $x$: $\ln(1-x) - \ln 6 = -\ln(2-x)$. Show your work. The solution(s) must be valid. Write "none" if neither solution is valid.

Get the terms involving $x$ on the left, everything else (the $\ln 6$) on the right.

Combine the logarithms into a single logarithm. ___/2

Exponentiate both sides to eliminate the logarithms.

Multiply out. For this problem, the result should be a quadratic equation. To solve, get everything on the left, 0 on the right. Then factor. ___/2

There should be two solutions: a positive digit and a negative digit. Determine the validity of each solution. Substitute each solution into the original equation.
$\ln(1-x) - \ln 6 = -\ln(2-x)$
Check if the logarithms all have positive arguments. If so, the solution is valid. If not, the solution is invalid. ___/2
1. \(\ln e^3 - \ln e\)
   Single digit

2. \(e^{\ln 3} + e^{\ln 2} - e^{\ln e}\)
   3 symbols

§4.3 357:39-46. Combine into a single logarithm.
3. \(2 \log_{10} x - 3 \log_{10} y\)
   10 symbols

4. \(\ln (x^3 - 1) - \ln (x^2 + x + 1)\)
   13 symbols, 7 if simplified

§4.3 357:13-34. Write as a sum / difference / multiple of the simplest possible logarithms.
5. \(\log_b \sqrt[3]{\frac{x+3}{x}}\)
   20 symbols

6. \(\ln \frac{1}{\sqrt{x^2 + x + 1}}\)
   14 symbols

§4.3 357:49-52. Express \(\log_2 b\) in terms of \(\log_{10}\).
7. 13 symbol fraction

8. Express \(\log_2 10\) in terms of natural logarithms, \(\ln\).
   8 symbol fraction

§4.4 367:35-50. Solve for \(x\).
9. \(\ln x + \ln (x + 1) = \ln 12\)
   Warning: only the positive 1-digit solution is in the domain of \(\ln\).

10. \(\ln (x + 1) = 2 + \ln (x - 1)\)
    9 symbol fraction
Simplify.
A. \( \log_{10} 70 - \log_{10} 7 \)
B. \( \log_{7} \sqrt{7} \)
C. \( \log_{3} 108 + \log_{3}(3/4) \)
D. \( -\frac{1}{2} + \ln \sqrt{e} \)
E. \( 2^{\log_5 5} - 3 \log_5 \sqrt[3]{5} \)

Combine into a single logarithm.
F. \( \log_{10} 30 + \log_{10} 2 \)
G. \( \log_{5} 6 + \log_{5}(1/3) + \log_{5} 10 \)
H. \( \log_{b} 4 + 3[\log_{b}(1 + x) - \frac{1}{2} \log_{b}(1 - x)] \)

Write as a sum / difference / multiple of the simplest possible logarithms.
I. \( \log_{10} \frac{x^2}{1+x^2} \)
J. \( \ln \frac{x^2}{\sqrt{1+x^2}} \)
K. \( \log_{10} \sqrt{9-x^2} \)
L. \( \ln \frac{\sqrt{4-x^2}}{(x-1)(x+1)^{3/2}} \)

M. Express \( \log_{b} 2 \) in terms of \( \log_{10} \).

Express in terms of natural logarithms \( \ln \).
N. \( \log_{10} 6 \).
O. \( \log_{10} e \)

P. Solve for \( x \) using \( \ln \). \( 5^x = 3^{2x-1} \).

Solve for \( x \).
Q. \( \log_{10}(2x + 4) + \log_{10}(x - 2) = 1 \)
R. \( \log_{10}(x + 3) - \log_{10}(x - 2) = 2 \)
S. \( \log_{10}(x + 1) = 2 \log_{10}(x - 1) \)

Answers
A. 1
B. 1/2
C. 4
D. 0
E. 4
F. \( \log_{10} 60 \)
G. \( \log_{5} 20 \)
H. \( \log_{b} \left(\frac{4(1+x)^3}{(1-x)^{3/2}} \right) \)
I. \( 2 \log_{10} x - \log_{10}(1 + x^2) \)
J. \( 2 \ln x - \frac{1}{2} \ln(1 + x^2) \)
K. \( \frac{1}{2} \log_{10}(3 + x) + \frac{1}{2} \log_{10}(3 - x) \)
L. \( \frac{1}{2} \ln(2 + x) + \frac{1}{2} \ln(2 - x) - \ln(x - 1) - \frac{3}{2} \ln(x + 1) \)
M. \( \frac{\log_{10}^2}{\log_{10} 6} \)
N. \( \frac{\ln 6}{\ln 10} \)
O. \( \frac{1}{\ln 10} \)

P. \( x = (\ln 3)/(2 \ln 3 - \ln 5) \)
Q. \( x = 3 \)
R. \( x = 203/99 \)
S. \( x = 3 \)
Simplify.
A. \( \log_{10} 70 - \log_{10} 7 \)
   \( = \log_{10} \frac{70}{7} = \log_{10} 10 = 1 \)
   Answer: 1

B. \( \log_{7} \sqrt{7} \)
   \( \log_{7} \sqrt{7} = \log_{7} 7^{1/2} = 1/2 \)
   Answer: 1/2

C. \( \log_{3} 108 + \log_{3}(3/4) \)
   \( = \log_{3} 108 + \log_{3}(3/4) = \log_{3} 27 \cdot 3 = \log_{3} 3^{4} = 4 \)
   Answer: 4

D. \( \frac{-1}{2} + \ln \sqrt{e} \)
   \( = -\frac{1}{2} + \ln e^{1/2} = -\frac{1}{2} + \frac{1}{2} = 0 \)
   Answer: 0

E. \( 2 \log_{2}^{5} - 3 \log_{5}^{3/5} \)
   \( = 5 - 3 \log_{5}^{5^{1/3}} = 5 - 3(1/3) = 4 \)
   Answer: 4

Combine into a single logarithm.
F. \( \log_{10} 30 + \log_{10} 2 \)
   \( = \log_{10} 30 \cdot 2 = \log_{10} 60 \)
   Answer: \( \log_{10} 60 \)

G. \( \log_{5} 6 + \log_{5}(1/3) + \log_{5} 10 \)
   \( = \log_{5} (6 \cdot \frac{1}{3} \cdot 10) = \log_{5} 20 \)
   Answer: \( \log_{5} 20 \)

H. \( \log_{b} 4 + 3[\log_{b}(1 + x) - \frac{1}{2} \log_{b}(1 - x)] \)
   \( = \log_{b} 4 + 3 \log_{b}(1 + x) - \frac{3}{2} \log_{b}(1 - x) \)
   \( = \log_{b} 4 + \log_{b}(1 + x)^{3} - \log_{b}(1 - x)^{3/2} \)
   \( = \log_{b} \left[ \frac{4 + (1 + x)^{3}}{(1 - x)^{3/2}} \right] \)
   Answer: \( \log_{b} \frac{4 + (1 + x)^{3}}{(1 - x)^{3/2}} \)

Write as a sum / difference / multiple of the simplest possible logarithms.
I. \( \log_{10} \frac{x^{2}}{1 + x^{2}} \)
   \( = \log_{10} x^{2} - \log_{10} (1 + x^{2}) \)
   \( = 2 \log_{10} x - \log_{10} (1 + x^{2}) \)
   Answer: \( 2 \log_{10} x - \log_{10} (1 + x^{2}) \)

J. \( \ln \frac{x^{2}}{1 + x^{2}} \)
   \( = \ln x^{2} - \ln (1 + x^{2})^{1/2} \)
   \( = 2 \ln x - \frac{1}{2} \ln (1 + x^{2}) \)
   Answer: \( 2 \ln x - \frac{1}{2} \ln (1 + x^{2}) \)

K. \( \log_{10} \sqrt{9 - x^{2}} \)
   \( = \log_{10} (9 - x^{2})^{1/2} = \log_{10} ((3 - x)(3 + x))^{1/2} \)
   \( = \frac{1}{2} \log_{10} (3 - x)(3 + x) = \frac{1}{2} \log_{10} (3 - x) + \frac{1}{2} \log_{10} (3 + x) \)
   Answer: \( \frac{1}{2} \log_{10} (3 - x) + \frac{1}{2} \log_{10} (3 + x) \)

L. \( \ln \frac{1}{(x-1)(x+1)^{3/2}} \)
   \( = \ln(4 - x^{2})^{1/2} - \ln(x - 1) - \ln(x + 1)^{3/2} \)
   \( = \frac{1}{2} \ln(2 + x)(2 - x) - \ln(x - 1) - \frac{3}{2} \ln(x + 1) \)
   Answer: \( \frac{1}{2} \ln(2 + x) + \frac{1}{2} \ln(2 - x) - \ln(x - 1) - \frac{3}{2} \ln(x + 1) \)

M. Express \( \log_{b} 2 \) in terms of \( \log_{10} \).
   \( \log_{b} 2 = \frac{\log_{10} 2}{\log_{10} b} \)

Express in terms of natural logarithms \( \ln \).
N. \( \log_{10} 6 = \frac{\ln 6}{\ln 10} \)

O. \( \log_{10} e = \frac{\ln e}{\ln 10} = \frac{1}{\ln 10} \)

Solve for \( x \).
P. Solve for \( x \) using \( \ln \). \( 5^{x} = 3^{2x-1} \)
   \( 5^{x} = 3^{2x-1} \) iff \( \ln 5^{x} = \ln 3^{2x-1} \) iff
   \( x \ln 5 = (2x - 1) \ln 3 \) iff \( x \ln 5 = 2x \ln 3 - \ln 3 \) iff
   \( x \ln 5 - 2x \ln 3 = -\ln 3 \) iff \( x(\ln 5 - 2\ln 3) = -\ln 3 \) iff
   \( x = -\ln 3 / [\ln 5 - 2\ln 3] \)
   Answer: \( x = (\ln 3) / (2 \ln 3 - \ln 5) \)

Q. \( \log_{10} (2x + 4) + \log_{10} (x - 2) = 1 \)
   iff \( \log_{10} (2x + 4)(x - 2) = 1 \)
   iff \( 10^{\log_{10} (2x + 4)(x - 2)} = 10^{1} \) iff
   \( (2x + 4)(x - 2) = 10 \) iff \( 2x^{2} - 8 = 10 \) iff
   \( 2x^{2} - 18 = 0 \) iff \( x^{2} = 9 \) iff
   \( x = -3, 3 \)
   -3 is not valid since \( x = -3 \) implies \( x - 2 = -5 \) and
   so \( \log_{10} (x - 2) = \log_{10} (-5) = \text{undefined} \)
   Answer: \( x = 3 \)

S. \( \log_{10} (x + 1) = 2 \log_{10} (x - 1) \)
   iff \( \log_{10} (x + 1)^{2} = \log_{10} (x - 1)^{2} \) iff
   \( \log_{10} (x + 1)/(x - 1)^{2} = 0 \)
   iff \( (x + 1)/(x - 1)^{2} = 10^{0} \) iff \( 10^{x+1} / (x-1)^{2} = 1 \)
   iff \( x + 1 = (x - 1)^{2} \) iff \( x + 1 = x^{2} - 2x + 1 \)
   \( x^{2} - 3x = 0 \) iff \( x = 0, 3 \)
   \( x = 0 \) is invalid since \( x = 0 \) implies \( x - 1 = -1 \) and
   so \( \log_{10} (x - 1) = \log_{10} (-1) = \text{undefined} \)
   Answer: \( x = 3 \)
Math 140  Lecture 14

Exponential growth
A bacteria colony starts with 10 bugs. Each bug splits into two bugs every hour. How many bugs are there after $t$ hours?

<table>
<thead>
<tr>
<th>Number of hours</th>
<th>Number of bugs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>$10^2$</td>
</tr>
<tr>
<td>2</td>
<td>$(10^2)2 = 10^{2+2}$</td>
</tr>
<tr>
<td>3</td>
<td>$(10^2)2 = 10^{2+2}$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$t$ hours</td>
<td>$10^{2t}$</td>
</tr>
</tbody>
</table>

Exponential functions measure the size of a growing population, the amount of money in a compound interest account, the number of atoms left after radioactive decay, etc. $N(t)$ = the amount at time $t$.

**Base-e Form Lemma.** Every exponential function can be written in the form $N(t) = N_0 e^{kt}$.

- $N_0 = N(0)$ is the initial amount.
- $k$, the coefficient of $t$, is the growth constant.
- If $k > 0$, $N(t)$ measures exponential growth.
- If $k < 0$, $N(t)$ measures exponential decay.

**Fact:** Every $a > 0$ is a power of $e$: $a = e^{\ln a}$.

Example: $2 = e^{\ln 2}$.

- Write the bug population $N(t) = 10 \cdot 2^t$ in base-e form.
  - $N(t) = 10 \cdot 2^t = 10 \cdot (e^{\ln 2})^t = 10 \cdot e^{(\ln 2)t}$
  - Hence, in base-e form, $N(t) = 10e^{(\ln 2)t}$.
  - Thus $N_0 = 10$, the initial amount
  - and $k = \ln(2)$, the natural log of the initial base.

- If you deposit $P$ dollars at 5% interest per year. How much is there after one year?
  - Answer: $P + (.05)P = P(1 + .05) = P \cdot 1.05$

- You put $4,000 in an account at 5% interest compounded annually. Write the amount $N(t)$ of money in the account after $t$ years in base-e form.

<table>
<thead>
<tr>
<th>Number of years</th>
<th>Amount $N(t)$ in account</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4,000</td>
</tr>
<tr>
<td>1</td>
<td>$4,000$</td>
</tr>
<tr>
<td>2</td>
<td>$4,000 \cdot (1.05)$</td>
</tr>
<tr>
<td>3</td>
<td>$4,000 \cdot (1.05)^3$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$t$ years</td>
<td>$4,000 \cdot (1.05)^t$</td>
</tr>
</tbody>
</table>

Hence $N(t) = 4000(1.05)^t$ dollars

- $8e^{(\frac{1}{2}\ln(\frac{1}{2}))} = 8e^{\frac{1}{2} \ln(\frac{1}{2})}$
- $\ln(\frac{1}{2}) = -\ln(2)$

- This is a decay since $\ln(1/2) = \ln(2^{-1}) = -\ln(2)$ is negative.
A compound interest account starts with $4000. 6 years later it is $6000.
Show work; only 3 points per part without work. Remember the units (-1 point if omitted).

\[ N_0 = 4000 \]
\[ N(6) = 6000 \]
\[ N(t) = N_0 e^{kt} \]
\[ N(t) = 4000 e^{kt} \]
\[ 4000e^{k6} = N(6) = 6000 \]
\[ e^{6k} = 6000/4000 = 3/2 \]
\[ 6k = \ln(3/2) \]
\[ k = \frac{1}{6} \ln \frac{3}{2} \]
\[ N(t) = 4000e^{\left(\frac{1}{6} \ln \frac{3}{2}\right)t} \]

(a) Determine the growth constant \( k \).
\[ k = \frac{1}{6} \ln \frac{3}{2} \]

(b) How much will there be in the account after 12 years?
\[ N(12) = 4000e^{\left(\frac{1}{6} \ln \frac{3}{2}\right)12} \]
\[ = 4000e^{2 \ln \frac{3}{2}} \text{ dollars (stop here, don’t need to continue to } 4000e^{\ln \frac{9}{4}} = 4000(\frac{9}{4}) = 9000) \]

(c) When will the account have $8000?
Let \( t \) = time when amount is $8000
\[ N(t) = 8000 \]
\[ 4000e^{\left(\frac{1}{6} \ln \frac{3}{2}\right)t} = 8000 \]
\[ e^{\left(\frac{1}{6} \ln \frac{3}{2}\right)t} = 2 \]
\[ \left(\frac{1}{6} \ln \frac{3}{2}\right)t = \ln 2 \]
\[ t = \ln 2/(\frac{1}{6} \ln \frac{3}{2}) = 6 \ln 2 / \ln \frac{3}{2} \text{ years} \]
Here is an example from the lecture.
Your problem is in the next column.

- A bacteria colony starts with $10^3$ bugs. Four hours later it has $5 \times 10^3$ bugs.

First find $N(t) =$ the number of bugs after $t$ hours.

Given:

- Initial amount: $N_0 = 10^3$
- Exponential equation: $N(t) = N_0 e^{kt}$

Find $k$. Translate “four hours later the pop. is $5 \times 10^3$” into an equation about $k$.

$N(4) = 5 \cdot 10^3$

and $N(t) = 10^3 e^{kt}$ implies

$10^3 e^{4k} = N(4) = 5 \cdot 10^3$

Thus $10^3 e^{4k} = 5 \cdot 10^3$

- Solve for $k$.

\[ e^{4k} = 5 \]

\[ \ln(e^{4k}) = \ln 5 \]

\[ 4k = \ln 5 \]

\[ k = \frac{\ln 5}{4} \]

Base-e form: $N(t) = N_0 e^{kt} = 10^3 e^{\frac{\ln 5}{4} t}$

Find $N(t)$. $N(t) = N_0 e^{kt} =$

\[ \underline__/1 \]

How much is there after 6 years? $N(6) =$

\[ 9 \text{ symbols + the units, e.g., inches, feet, lbs, dollars, seconds, ...} \]

\[ \underline__/1 \]

When will there be $6000$ in the account? Let $t =$ time when amount is $6000$.

- Translate “the amount at time $t$ is $6000$” into an equation in $t$.

\[ \underline__/1 \]

- Solve for $t$. 8 symbols followed by the units.

\[ \underline__/1 \]

5. A compound interest account starts with $2000$. 3 years later it has $4000$.

- The initial amount $N_0 =$

- $N(t) = N_0 e^{kt}$ since this is an exponential problem.

Find $k$.

- Translate “3 years later it has $4000$” into an equation about $k$.

\[ \underline__/1 \]

- Solve for $k$. 5 symbols.

\[ \underline__/1 \]

Find $N(t)$. $N(t) = N_0 e^{kt} =$

\[ \underline__/1 \]

How much is there after 6 years? $N(6) =$

\[ 9 \text{ symbols + the units, e.g., inches, feet, lbs, dollars, seconds, ...} \]

\[ \underline__/1 \]

When will there be $6000$ in the account? Let $t =$ time when amount is $6000$.

- Translate “the amount at time $t$ is $6000$” into an equation in $t$.

\[ \underline__/1 \]

- Solve for $t$. 8 symbols followed by the units.

\[ \underline__/1 \]
§4.5 379:5-22.

Write \( N(t) \) in base \( e \) form. Then solve the problem.

Except for 2, give exact answers, not approximate decimal answers.

1(8). Initially, \( 2 \times 10^4 \) bacteria are present in a colony.

Eight hours later there are \( 3 \times 10^4 \).

(2) \( N(t) = \)

(a) Determine the growth constant \( k \).

(b) What is the population two hours after the start?

(c) How long will it take for the population to triple?

2(3). In 1995, China, India, and the US had 1218, 930, and 263 million respectively. Their relative growth rates were 1.1%, 1.9%, and .7% respectively. Estimate their populations in the year 2050. “Relative growth rate” is another name for “growth rate constant”. Change %’s to decimals, e.g., 1.1% to .011. For this problem only, give decimal answers. Round to the nearest million, e.g., 1810 million. 4, 4, 3 symbols.

(a) How many grams will remain after 1 year?

(b) 10 years?

4(6). The half-life of radium-226 is 1620 years. Initially the sample has 2 grams.

(2) \( N(t) = \)

(a) How many grams will remain after 100 years?

(b) Find the time required for 80% of the 2-grams same to decay? Hint, at this time only .4 grams are left.
**Math 140    Hw 14    Recommended problems, don't turn this in.**

Write \( N(t) \) in base \( e \) form. Then solve the problem.

A. A colony of bacteria starts with 2000 bugs. Two hours later it has 3800.

\[
N(t) = 2000 e^{\frac{t}{2} \ln \left( \frac{19}{10} \right)}
\]

(a) How many grams will remain after 48 hours?

(b) How long will it be until only one gram remains?

B. The half-life of iodine-131 is 8 days. Initially a sample has 1 gram.

\[
N(t) = e^{\frac{t}{8} \ln \left( \frac{1}{2} \right)}
\]

(a) How many grams will remain after 5 years?

(b) How long will it take for 90\% of the sample to decay? Hint, when 90\% has decayed, 10\% remains.

C. The half-life of sodium-24 is 15 hours. Initially the sample has 40 grams.

\[
N(t) = 40 e^{\frac{t}{15} \ln(1/2)}
\]

(a) 40e \( \frac{t}{15} \ln(1/2) \) g

(b) 15 ln(1/40)/ln(1/2) hr

D. The half-life of plutonium-241 in 13 years. Initially a sample has 2 grams.

\[
N(t) = 2 e^{\frac{t}{13} \ln(1/2)}
\]

(a) 2e \( \frac{t}{13} \ln(1/2) \) g

(b) 13 ln(0.1)/ln(1/2) yr

---

Answers

A. \( N(t) = 2000 e^{\frac{t}{2} \ln \left( \frac{19}{10} \right)} \)

(a) \( k = \frac{1}{2} \ln \left( \frac{19}{10} \right) \)

(b) \( 2000 e^{\frac{5}{2} \ln \left( \frac{19}{10} \right)} \)

C. \( N(t) = 40 e^{\frac{t}{15} \ln(1/2)} \)

(a) 40e \( \frac{t}{15} \ln(1/2) \) g

(b) 15 ln(1/40)/ln(1/2) hr
Write $N(t)$ in base $e$ form. Then solve the problem.

A. A colony of bacteria starts with 2000 bugs. Two hours later it has 3800.

Given: $N_0 = 2000$
$N(2) = 3800$
$N(t) = N_0e^{kt}$

Let $N(2) = 2000e^{k(2)}$
$2000e^{2k} = 3800$

$\therefore e^{2k} = 3800/2000 = 38/20 = 19/10$
$2k = \ln(19/10)$

$k = \frac{1}{2}\ln(19/10)$

$N(t) = 2000e^{\frac{1}{2}\ln(19/10)t}$

(a) Determine the growth constant $k$.

Answer: $k = \frac{1}{2}\ln(19/10)$

(b) What is the population 5 hours after the start?

$N(5) = 2000e^{\frac{1}{2}\ln(19/10)\cdot 5}$
Answer: $2000e^{\frac{5}{2}\ln(19/10)}$

(c) How long will it take to reach 10,000?

$N(t) = 10000$ iff
$2000e^{\frac{1}{2}\ln(19/10)\cdot t} = 10000$ iff
$e^{\frac{1}{2}\ln(19/10)\cdot t} = 5$ iff
$\ln(e^{\frac{1}{2}\ln(19/10)\cdot t}) = \ln 5$ iff
$\frac{1}{2}\ln(19/10)\cdot t = \ln 5$

Answer: $t = 2\ln(5)/\ln(19/10)$ hr

B. The half-life of iodine-131 is 8 days. Initially a sample has 1 gram.

Given: $N(0) = 1$
$N(8) = 1/2$
$N(t) = N_0e^{kt}$

$\therefore N(8) = 1e^{k(8)}$
$e^{8k} = \frac{1}{2}$
$8k = \ln \frac{1}{2}$
$k = \frac{1}{8}\ln \frac{1}{2}$

$N(t) = e^{\frac{1}{8}\ln(\frac{1}{2})t}$

How many grams will remain after 7 days?

$N(7) = e^{\frac{1}{8}\ln(\frac{1}{2})\cdot 7}$
Answer: $e^{\frac{7}{8}\ln(\frac{1}{2})}$ g

C. The half-life of sodium-24 is 15 hours. Initially the sample has 40 grams.

Given: $N(0) = 40$
$N(15) = 20$
$N(t) = N_0e^{kt}$

$\therefore N(15) = 40e^{k(15)}$
$\therefore 40e^{15k} = 20$
$\therefore e^{15k} = \frac{1}{2}$
$\therefore 15k = \ln \frac{1}{2}$
$k = \frac{1}{15}\ln \frac{1}{2}$

$N(t) = 40e^{\frac{1}{15}\ln(\frac{1}{2})t}$

(a) How many grams will remain after 48 hours?

$N(48) = 40e^{\frac{48}{15}\ln(\frac{1}{2})}$
Answer: $40e^{\frac{16}{5}\ln(\frac{1}{2})}$ g

(b) How long will it be until only one gram remains?

$N(t) = 1$ iff
$40e^{\frac{1}{15}\ln(\frac{1}{2})t} = 1$ iff
$e^{\frac{1}{15}\ln(\frac{1}{2})t} = \frac{1}{40}$ iff
$\frac{1}{15}(\ln \frac{1}{2})t = \ln \frac{1}{40}$
Answer: $15\ln(\frac{1}{40})/\ln(\frac{1}{2})$ hr

D. The half-life of plutonium-241 in 13 years. Initially a sample has 2 grams.

Given: $N(0) = 2$
$N(13) = 1$
$N(t) = N_0e^{kt}$

$\therefore N(13) = 2e^{k(13)}$
$\therefore 2e^{13k} = 1$
$\therefore e^{13k} = \frac{1}{2}$
$\therefore 13k = \ln \frac{1}{2}$
$k = \frac{1}{13}\ln \frac{1}{2}$

$N(t) = 2e^{\frac{1}{13}\ln(\frac{1}{2})t}$

(a) How many grams will remain after 5 years?

$N(5) = 2e^{\frac{5}{13}\ln(\frac{1}{2})}$
Answer: $2e^{\frac{5}{13}\ln(\frac{1}{2})}$ g

(b) How long will it take for 90% of the sample to decay?

Hint: when 90% has decayed, 10% remains.

$N(t) = (0.1)2$
$2e^{\frac{1}{13}\ln(\frac{1}{2})t} = 0.2$
$e^{\frac{1}{13}\ln(\frac{1}{2})t} = \frac{1}{10}$
$\frac{1}{13}(\ln \frac{1}{2})t = \ln \frac{1}{10}$
$t = 13\ln \frac{1}{10}/\ln \frac{1}{2}$

Answer: $13\ln(\frac{1}{10})/\ln(\frac{1}{2})$ yr
Angles, arcs, and radians

RECALL. A circle of radius \( r \) has circumference \( 2\pi r \).
\( \pi \approx 3.14 \). Unit circle circumference = \( 2\pi r = 2\pi \times 1 = 2\pi \).
\( \theta \) and \( \omega \) are the Greek letters “theta” and “omega”.

DEFINITION. Suppose the vertex of an angle is at the center of a circle of radius \( r \). Let \( s \) be the length of the arc the angle intercepts on the circle. Then
\[ \theta = \frac{s}{r} \]
is the radian measure of the angle. For unit circles, the radius \( r = 1 \) and radian measure equals arc length: \( \theta = s \).

- Radian and degree measures on the unit circle.

Clockwise angles are negative.

CONVERSION FORMULAS. \( 180^\circ = \pi \) radians. Thus
\[ 1^\circ = \frac{\pi}{180} \] radians; \( 1 \) radian = \( \frac{180^\circ}{\pi} \) degrees.

- Convert \( 100^\circ \) to radians.
\[ 100^\circ = 100 \times \frac{\pi}{180} = \frac{5\pi}{9} \approx 1.74 \] radians
- Convert \( \pi/6 \) radians to degrees.
\[ \frac{\pi}{6} \text{ radians} = \frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ. \]

When \( \theta \) is in radians, we can solve \( \theta = \frac{s}{r} \) for arclength:
\[ s = \theta \cdot r \]

- Find the length of a \( 30^\circ \) arc on a circle of radius 12 inches.
First convert to radians. By the above \( 30^\circ = \pi/6 \) radians.
\[ s = \theta \cdot r = \frac{\pi}{6} \times 12 = 2\pi \text{ inches} \]

- Find the degree measure of an angle which intercepts a 5 inch arc on a circle of radius 12 inches.
\[ \theta = \frac{s}{r} = \frac{5}{12} \times \frac{180^\circ}{\pi} = \frac{75^\circ}{\pi} \]

Speed

DEFINITION. If an object travels a distance \( d \) in time \( t \), its linear speed is \( \frac{d}{t} \).

If an object rotates through an angle \( \theta \) in time \( t \), its rotational speed is \( \omega = \frac{\theta}{t} \).

THEOREM. If a point rotates around a circle of radius \( r \) with rotational speed \( \omega \), then its linear speed is \( \omega r \).

Proof. If a point rotates through an angle \( \theta \) on a circle of radius \( r \) in time \( t \), then \( \omega = \frac{\theta}{t} \) and the distance \( d \) it travels = the length of the arc it traces = \( \theta r \). \( \therefore \) its linear speed = \( \frac{d}{t} = \frac{\theta r}{t} = \frac{\theta}{t} = \omega r \). We assume \( \theta \) is in radians.

- A point revolves around a circle of radius 3 feet at 10 revolutions per minute.

(a) What is its rotational speed (in radians)?
\[ 1 \text{ revolution} = 2\pi \text{ radians}, \therefore \omega = 10 \text{ revs/min} = 10 \times \frac{2\pi \text{ radians}}{60 \text{ min}} = \frac{\pi}{3} \text{ radians/minute}. \]

(b) What is its linear speed? Its linear speed
\[ = \omega r = (\frac{\pi}{3} \text{ radians/minute}) \times 3 \text{ feet} = \frac{\pi}{3} \text{ feet/minute}. \]

Radian and degree measures on the unit circle.

THEOREM. If a point rotates around a circle of radius \( r \), the linear speed \( v \) is
\[ v = \omega r \]

TRIGONOMETRIC FUNCTIONS

The unit circle has radius one and center \((0,0)\). There are four quadrants I, II, III, IV as pictured.

An angle is in standard position if its vertex is the origin \((0,0)\) and its initial side is the positive x-axis. The other side of the angle is the terminal side. The terminal side intersects the unit circle at a point \(P_0\).

The six trigonometric functions of \( \theta \) are:

- The six trigonometric functions of \( \theta \) are:

<table>
<thead>
<tr>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \cot \theta )</th>
<th>( \sec \theta )</th>
<th>( \csc \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>( x )</td>
<td>( \frac{\sin \theta}{\cos \theta} )</td>
<td>( \frac{\cos \theta}{\sin \theta} )</td>
<td>( \frac{1}{\cos \theta} )</td>
<td>( \frac{1}{\sin \theta} )</td>
</tr>
</tbody>
</table>

- Draw an angle of \( \pi/2 \) radians in standard position. Find the six trigonometric functions.

\[ \sin(\pi/2) = 1, \quad \csc(\pi/2) = 1 \]
\[ \cos(\pi/2) = 0, \quad \sec(\pi/2) = \text{undefined} \]
\[ \tan(\pi/2) = \text{undefined}, \quad \cot(\pi/2) = 0 \]

- A point \((x,y)\) on the unit circle is in the second quadrant and \(y = \frac{3}{4}\). Find the six trig functions for \( \theta \).

\[ (x,y) \text{ on the unit circle} \Rightarrow x^2 + y^2 = 1. \]
\[ \therefore x^2 = 1 - y^2 = 1 - \left(\frac{3}{4}\right)^2 = 1 - \frac{9}{16} = \frac{7}{16}. \]
\[ x = \pm \sqrt{7}/4. \quad (x,y) \text{ in the second quadrant} \Rightarrow x \text{ is negative}. \]
\[ x = -\sqrt{7}/4. \quad \sin \theta = \frac{3}{4}, \quad \csc \theta = 4/3 \]
\[ \cos \theta = -\sqrt{7}/4, \quad \sec \theta = -4/\sqrt{7} \]
\[ \tan \theta = -3/\sqrt{7}, \quad \cot \theta = -\sqrt{7}/3 \]
9. (a)(2) Find the length of the arc intercepted by an angle of 40° on a circle of radius 3 feet. Remember units

\[ 40° = 40 \frac{\pi}{180} \]
\[ = \frac{2\pi}{9} \text{ radians} \]
length= \( \theta r = \)
\[ \frac{2\pi}{9} \cdot 3 = 2\pi/3 \text{ ft.} \]

(b)(2) Find the number of degrees in an angle which intercepts a 4 foot arc on a circle of radius 12 feet.

\[ = \frac{4}{12} = \frac{1}{3} \text{ radians} = \frac{1}{3} \cdot \frac{180}{\pi} = \frac{60°}{\pi} \]

10. A point rotates around a circle of radius 10 inches at 3 revolutions per second.

(a)(2) Find its angular speed in radians per second. Give the exact answer using \( \pi \). Remember to include the units.

3 revs/sec
\[ = 3 \cdot 2\pi = \]
6\( \pi \) radians/sec

(b)(2) Find its linear speed. Exact answer using \( \pi \), include units.

\[ = \omega r = 6\pi \cdot 10 \]
\[ = 60\pi \text{ inches/sec} \]
9. (a) Find the length of the arc intercepted by an angle of 30° on a circle of radius 50 inches. /1

First convert the angle to radians.
Recall 180° = π radians, hence 1° = \( \frac{\pi}{180} \) radians.

\[ \theta = 30° = \]

Now use the formula

length = \( \theta r \) =

Reduce the fraction, remember the units, 5 symbols.

(b) Find the number of degrees in an angle which intercepts a 30 inch arc on a circle of radius 50 inches. /1

First find the angle in radians using the formula

\[ \theta = \frac{s}{r} = \]

Now convert the radian measure to degrees. 5 symbols
Since 180° = π radians, 1 radian = \( \frac{180°}{\pi} \) degrees.

10. A point rotates around a circle of radius 50 inches at 30 revolutions per second.
   (a) Find its angular speed in radians per second. Give the exact answer using \( \pi \). Remember to the units. /1

First convert 30 revolutions per second to radians per second.
1 revolution = 2\( \pi \) radians.

Hence 30 revolutions / second = _____ \times _____ radians / second = ______ radians / second

(b) Find its linear speed. Exact answer using \( \pi \), include units. /1

Use the formula:
linear speed = \( \omega r \) =

Remember the units, 5 symbols.
Remember to include any needed units, e.g., mi/hr, rad/sec. Except for 1,2, give only exact answers. No decimals.


1. Convert 30° to radian measure
   
   exact answer using \( \pi \): 3 symbols
   
   two-place decimal answer: 3 symbols

2. Convert 150° to radian measure
   
   exact answer using \( \pi \): 4 symbols
   
   two-place decimal answer: 4 symbols

3. Convert 3\( \pi \) radians to degrees
   
   4 symbols

4. Convert 3\( \pi /2 \) radians to degrees
   
   4 symbols

$\frac{11}{5.2} 416:3-8, 23-26.$

9. Find the six trigonometric functions of -\( \pi /2 \).

<table>
<thead>
<tr>
<th>( \sin(-\pi /2) )</th>
<th>( \csc(-\pi /2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \cos(-\pi /2) )</th>
<th>( \sec(-\pi /2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \tan(-\pi /2) )</th>
<th>( \cot(-\pi /2) )</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>

10. Find the six trigonometric functions of 4\( \pi \).

<table>
<thead>
<tr>
<th>( \sin(4\pi) )</th>
<th>( \csc(4\pi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
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</table>

<table>
<thead>
<tr>
<th>( \cos(4\pi) )</th>
<th>( \sec(4\pi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
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</table>

<table>
<thead>
<tr>
<th>( \tan(4\pi) )</th>
<th>( \cot(4\pi) )</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
</tbody>
</table>

11. A point \((x, y)\) on the unit circle and on the terminal side of an angle \( \theta \) is in the fourth quadrant.

Find the six trigonometric functions if \( x = 1/3 \).

<table>
<thead>
<tr>
<th>( \sin \theta )</th>
<th>( \csc \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \cos \theta )</th>
<th>( \sec \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \tan \theta )</th>
<th>( \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
</tr>
</tbody>
</table>

$\frac{1}{5.1} 406:13-18.$

12. A point \((x, y)\) on the unit circle and on the terminal side of an angle \( \theta \) is in the first quadrant.

Find the six trigonometric functions if \( x = 3/5 \).

Rational answers, give exact answers, not decimals.

Use improper fractions, not mixed fractions. E.g. 3/2, not 1\( \frac{1}{2} \).

<table>
<thead>
<tr>
<th>( \sin \theta )</th>
<th>( \csc \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \cos \theta )</th>
<th>( \sec \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \tan \theta )</th>
<th>( \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>=</td>
<td>=</td>
</tr>
</tbody>
</table>

13. Complete the table. Mark “+” where the functions are positive, “-” where they are negative.

<table>
<thead>
<tr>
<th>( \cos \theta, \sec \theta )</th>
<th>( \sin \theta, \csc \theta )</th>
<th>( \tan \theta, \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Quad I</th>
<th>Quad II</th>
<th>Quad III</th>
<th>Quad IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>-</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A. Convert $45^\circ$ to radian measure
   exact answer using $\pi$:
   two-place decimal answer:

B. Convert $90^\circ$ to radian measure
   exact answer using $\pi$:
   two-place decimal answer:

C. Convert $\pi/3$ radians to degrees

D. Convert $5\pi/3$ radians to degrees

E. Find the length of an arc intercepted by a $4\pi/3$ radian angle on a circle of radius 3 ft.? Exact answer.

F(a). A point rotates around a circle of radius 12 cm at 6 revolutions/sec.
   Find it's angular speed $\omega$. Give the exact answer using $\pi$.
   \[ \omega = \frac{12\pi}{6} \]
   (b). A point rotates around a circle of radius 12 cm at 6 revolutions/sec.
   Find it's linear speed. Give the exact answer using $\pi$.

G. Find the six trigonometric functions of $-3\pi/2$.
   \[
   \begin{align*}
   \sin(-3\pi/2) &= 1 \\
   \csc(-3\pi/2) &= \text{undef} \\
   \cos(-3\pi/2) &= 0 \\
   \sec(-3\pi/2) &= \text{undef} \\
   \tan(-3\pi/2) &= \text{undef} \\
   \cot(-3\pi/2) &= 0
   \end{align*}
   \]

H. Find the six trigonometric functions of 0.
   \[
   \begin{align*}
   \sin(0) &= 0 \\
   \csc(0) &= \text{undef} \\
   \cos(0) &= 1 \\
   \sec(0) &= 1 \\
   \tan(0) &= 0 \\
   \cot(0) &= \text{undef}
   \end{align*}
   \]

I. \[ \begin{align*}
   \sin \theta &= 2\sqrt{2}/3 \\
   \cos \theta &= 1/3 \\
   \tan \theta &= 2\sqrt{2} \\
   \csc \theta &= 3/(2\sqrt{2}) \\
   \sec \theta &= 3 \\
   \cot \theta &= 1/(2\sqrt{2})
   \end{align*} \]

J. \[ \begin{align*}
   \sin \theta &= -4/5 \\
   \cos \theta &= -3/5 \\
   \tan \theta &= 4/3 \\
   \csc \theta &= -5/4 \\
   \sec \theta &= -5/3 \\
   \cot \theta &= -3/4
   \end{align*} \]
A. Convert $45^\circ$ to radian measure.

$$\theta = 45^\circ \times \frac{\pi}{180} = \frac{\pi}{4} \approx 0.79$$

- exact answer using $\pi$: $\frac{\pi}{4}$
- two-place decimal answer: 0.79

C. Convert $\frac{\pi}{3}$ radians to degrees.

$$\theta = \frac{\pi}{3} \times \frac{180}{\pi} = 60^\circ$$

D. Convert $5\frac{\pi}{3}$ radians to degrees.

$$\theta = 5 \times \frac{\pi}{3} \times \frac{180}{\pi} = 300^\circ$$

E. Find the length of an arc intercepted by a $4\frac{\pi}{3}$ radian angle on a circle of radius 3 ft. Exact answer.

$$s = r\theta = (3)(\frac{4\pi}{3})$$

**Answer**: $4\pi$ ft.

F(a). A point rotates around a circle of radius 12 cm at 6 revolutions/sec.

Find its angular speed $\omega$. Give the exact answer using $\pi$.

1 rev = $2\pi$ radians
6 revs/sec = $6(2\pi)$rad/sec = $12\pi$ rad/sec

**Answer**: $12\pi$ rad/sec

(b). A point rotates around a circle of radius 12 cm at 6 revolutions/sec.

Find its linear speed. Give the exact answer using $\pi$.

By part (a), $\omega = 12\pi$
linear speed = $\omega r = (12\pi)(12) = 144\pi$

**Answer**: $144\pi$ cm/sec

G. Find the six trigonometric functions of $-3\pi/2$.

The unit-circle point with angle $-3\pi/2$ is $(0, 1)$.

<table>
<thead>
<tr>
<th>$\sin(-3\pi/2)$</th>
<th>$\csc(-3\pi/2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\cos(-3\pi/2)$</td>
<td>$\sec(-3\pi/2)$</td>
</tr>
<tr>
<td>0</td>
<td>$\text{undef}$</td>
</tr>
<tr>
<td>$\tan(-3\pi/2)$</td>
<td>$\text{undef}$</td>
</tr>
<tr>
<td>0</td>
<td>$\cot(-3\pi/2)$</td>
</tr>
<tr>
<td>$\text{undef}$</td>
<td>0</td>
</tr>
</tbody>
</table>

H. Find the six trigonometric functions of $0$.

The unit-circle point with angle $0$ is $(1, 0)$.

<table>
<thead>
<tr>
<th>$\sin(0)$</th>
<th>$\csc(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\text{undef}$</td>
</tr>
<tr>
<td>$\cos(0)$</td>
<td>$\sec(0)$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\tan(0)$</td>
<td>$\cot(0)$</td>
</tr>
<tr>
<td>0</td>
<td>$\text{undef}$</td>
</tr>
</tbody>
</table>

I. A point $(x,y)$ on the unit circle and on the terminal side of an angle $\theta$ is in the first quadrant.

Find the six trigonometric functions if $x = 1/3$.

First find $y$.

$x^2 + y^2 = 1$
$y^2 = 1 - x^2$
y = $\pm \sqrt{1 - x^2}$. It is the + since $(x, y)$ is in quadrant I.
y = $\sqrt{1 - (1/3)^2} = \sqrt{1 - 1/9} = \sqrt{8/9} = \sqrt{2} / 3$
y = $2\sqrt{2} / 3$, since $8 = 4 \cdot 2$, $\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$

**Answer**:

<table>
<thead>
<tr>
<th>$\sin \theta$</th>
<th>$\csc \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2\sqrt{2}/3$</td>
<td>$(2\sqrt{2})/3$</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>$\sec \theta$</td>
</tr>
<tr>
<td>$1/3$</td>
<td>3</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>$\cot \theta$</td>
</tr>
<tr>
<td>$2\sqrt{2}$</td>
<td>$1/(2\sqrt{2})$</td>
</tr>
</tbody>
</table>

J. A point $(x,y)$ on the unit circle and on the terminal side of an angle $\theta$ is in the third quadrant.

Find the six trigonometric functions if $x = -3/5$.

First find $y$.

$y^2 = 1 - x^2$
y = $\pm \sqrt{1 - x^2}$. It is - since $(x, y)$ is in quadrant III.
y = $-\sqrt{1 - (-3/5)^2} = -\sqrt{1 - 9/25} = -\sqrt{16/25} = -4/5$

**Answer**:

<table>
<thead>
<tr>
<th>$\sin \theta$</th>
<th>$\csc \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-4/5$</td>
<td>$-5/4$</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>$\sec \theta$</td>
</tr>
<tr>
<td>$-3/5$</td>
<td>$-5/3$</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>$\cot \theta$</td>
</tr>
<tr>
<td>$4/3$</td>
<td>$3/4$</td>
</tr>
</tbody>
</table>
For the other angles, we have a right triangle whose hypotenuse is a radius of length 1.

\( \pi/6 = 30^\circ \): This is a 30°–60° right triangle. Thus the small leg is \( \frac{1}{2} \) since it is half the hypotenuse which is 1. Let \( x \) be the third side.

\[
\begin{align*}
\text{Then} & \quad x^2 + (1/2)^2 = 1^2 \\
x^2 & \quad + 1/4 = 1 \\
x^2 & \quad = 3/4 \\
x & \quad = \sqrt{3}/2 \\
\therefore \quad & \quad \sin(\pi/6) = \sin(30^\circ) = 1/2 \text{ and } \cos(\pi/6) = \sqrt{3}/2
\end{align*}
\]

\( \pi/3 = 60^\circ \): This is the 30° triangle again but with the 30° and 60° angles swapped. Swapping sides 1/2 and \( \sqrt{3}/2 \) gives: \( \sin(\pi/3) = \sin(60^\circ) = \sqrt{3}/2 \) and \( \cos(\pi/3) = 1/2. \)

\( \pi/4 = 45^\circ \): If one angle of a right triangle is 45°, so is the other acute angle and the triangle is isosceles.

\[
\begin{align*}
\text{Let } x \text{ be the side length.} & \quad \therefore \quad \sin(\pi/4) = \sin(45^\circ) = 1/\sqrt{2} = \cos(\pi/4)
\end{align*}
\]

**THEOREM.** The sin and cos of \( \theta \) equals the sin and cos of its reference angle except for the sign which is determined by \( \theta \)'s quadrant.

- **Rewrite in terms of reference angles, then evaluate.**
  \[
  \begin{align*}
  \cos(120^\circ) & \quad = -\cos(60^\circ) = -1/2 \quad \text{quadrant II} \\
  \sin(120^\circ) & \quad = \sin(60^\circ) = \sqrt{3}/2 \quad \text{quadrant II} \\
  \cos(3\pi/4) & \quad = -\cos(\pi/4) = -1/\sqrt{2} \quad \text{quadrant II} \\
  \sin(3\pi/4) & \quad = \sin(\pi/4) = 1/\sqrt{2} \quad \text{quadrant II} \\
  \cos(-3\pi/4) & \quad = -\cos(\pi/4) = -1/\sqrt{2} \quad \text{quadrant II} \\
  \sin(-3\pi/4) & \quad = -\sin(\pi/4) = -1/\sqrt{2} \quad \text{quadrant II}
  \end{align*}
  \]

- **List three angles (in radian measure) whose cos is \( \frac{1}{2} \).**
  \( \pi/3, -\pi/3, 5\pi/3 \)

**DEFINITION.** For any angle in standard position (vertex = the origin, initial side = the positive x-axis), its *reference angle* is the acute positive angle (in \([0, \pi/2]\)) between the terminal side and the x-axis (positive or negative, whichever is nearest).
11. Suppose $\sin \theta = 3/5$.

\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 \\
\cos^2 \theta + (3/5)^2 &= 1 \\
\cos^2 \theta &= 1 - 9/25 \\
\cos \theta &= \pm \sqrt{16/25} \\
\cos \theta &= \pm 4/5
\end{align*}
\]

(a)(3) Find $\tan \theta$ if $\pi/2 < \theta < \pi$.

$\pi/2 < \theta < \pi$

$\therefore \cos \theta = -4/5$

$\tan \theta = \sin \theta / \cos \theta$

$= \frac{3/5}{-4/5} = -\frac{3}{4}$

(b)(2) Find $\cos \theta$ if $0 < \theta < \pi/2$.

$0 < \theta < \pi/2$

$\therefore \cos \theta = \frac{4}{5}$

16(5). Simplify to 5 symbols: $\frac{4 \tan \theta - 5}{5 \cot \theta - 4}$

$\frac{4 \tan \theta - 5}{5 \cot \theta - 4}$

$= \frac{4 \tan \theta - 5}{5(1/ \tan \theta) - 4}$

$= \frac{4 \tan \theta - 5}{\tan \theta} \cdot \frac{\tan \theta}{5(1/ \tan \theta) - 4}$

$= \frac{4 \tan \theta - 5}{4 \tan \theta} \tan \theta$

$= -\frac{4 \tan \theta - 5}{-(4 \tan \theta - 5)} \tan \theta$

$= -\tan \theta$
11. Suppose \( \sin \theta = -\frac{1}{4} \).
   Find \( \tan \theta \) if \( \frac{3\pi}{2} < \theta < 2\pi \).

To find \( \tan \theta \), we need \( \sin \theta \) and \( \cos \theta \). \( \sin \theta \) is given. We need \( \cos \theta \). Use the fact that \( \cos \theta = \pm \sqrt{1 - \sin^2 \theta} \).

Use the inequality \( \frac{3\pi}{2} < \theta < 2\pi \) to determine if “\( \pm \)” is “+” or “-”. Then find \( \tan \theta = \sin \theta / \cos \theta \).

In the two figures write the sines and cosines respectively of the listed angles. Some have already been done. See Lecture 16.
Math 140 9 Hw 16 Name ____________________________ Score _____/21

Last day to withdraw. The Keller 419A secretary signs for me.
The parts of problems 1-4 are $\frac{1}{2}$ point each.
For each angle, sketch the reference angle, draw an arrow to indicate the angle's direction. A reference angle must be positive and $\leq \pi/2 = 90^\circ$.

1(1). (a) 300°

2(1). (a) $\frac{5\pi}{6}$

§5.1 407:33-50. Rewrite the functions in terms of their reference angles, then evaluate. E.g.,

\[
\cos(120^\circ) = \cos(60^\circ) = -\frac{1}{2}
\]

3(4).

\[
\begin{array}{c|c|c|c|c|c|c}
\cos 150^\circ & = & = & = & = & = & = \\
\cos(-150^\circ) & = & = & = & = & = & = \\
\sin 150^\circ & = & = & = & = & = & = \\
\sin(-150^\circ) & = & = & = & = & = & = \\
\end{array}
\]

4(4).

\[
\begin{array}{c|c|c|c|c|c|c}
\cos(2\pi/3) & = & = & = & = & = & = \\
\cos(-2\pi/3) & = & = & = & = & = & = \\
\sin(2\pi/3) & = & = & = & = & = & = \\
\sin(-2\pi/3) & = & = & = & = & = & = \\
\end{array}
\]

5(1). List three angles (radian measure) whose sin is $-\frac{1}{2}$

\[\theta_1 = \theta_2 = \theta_3 = \]

§7.1 533:1-10.

6(1). Simplify $-\cos^2 \theta \sin^2 \theta + (2 \sin \theta \cos \theta)^2$.

11 symbols

7(1). Simplify $(5-2\tan \theta)/(2\tan \theta - 5)$ to an integer.

8(1). Factor $3\sec^2 \beta + 2\sec \beta - 8$.

9(1). Factor $16\sin^3 B - 9\sin^2 B$.

§5.2 417:63-70. §6.2 484:17,18,21,22.

10(4). $\cos \theta = \frac{5}{13}$ and $3\pi/2 < \theta < 2\pi$.

Find $\sin \theta$
6 symbols

Find $\cot \theta$
5 symbols

11(2). $\sec \theta = -\sqrt{13}/2$ and $\sin \theta > 0$. Find $\tan \theta$.
4 symbols
Math 140    Hw 16    Recommended problems.

For each angle, sketch the reference angle, draw an arrow to indicate the angle’s direction. A reference angle must be positive and \( \leq \pi/2 = 90^\circ \).

A. (a) \( 110^\circ \)  
(b) \( 60^\circ \)

B. (a) \( 3\pi/4 \)  
(b) \( 7\pi/6 \)

Rewrite the functions in terms of their reference angles, then evaluate. E.g.,

\[
\cos(120^\circ) = -\cos(60^\circ) = -\frac{1}{2}
\]

C.

\[
\begin{array}{ccc}
\cos 210^\circ &=& \\
\cos(-210^\circ) &=& \\
\sin 210^\circ &=& \\
\sin(-210^\circ) &=& \\
\end{array}
\]

D.

\[
\begin{array}{ccc}
\cos(4\pi/3) &=& \\
\cos(-4\pi/3) &=& \\
\sin(4\pi/3) &=& \\
\sin(-4\pi/3) &=& \\
\end{array}
\]

E. List three angles (in radian measure) whose \( \cos \) is \(-\frac{1}{2}\)

\[
\theta_1 = \quad \theta_2 = \quad \theta_3 = 
\]

F. Simplify \( \frac{\sin \theta - \cos \theta}{\cos \theta - \sin \theta} \).

G. Factor \( \tan^2 \beta + 8 \tan \beta - 9 \).

H. Factor \( 4 \cos^2 B - 1 \).

I. Factor \( 9 \sec^2 B \tan^3 B + 6 \sec B \tan^2 B \)

J33. \( \sin \theta = -\frac{3}{5} \) and \( \pi < \theta < 3\pi/2 \). Find \( \cos \theta \)

Find \( \tan \theta \)

K. \( \sin t = \frac{\sqrt{3}}{4} \) and \( \pi/2 < t < \pi \). Find \( \tan t \).

Answers

A., B. See Hw 16 Worked examples

C. \(-\frac{\sqrt{3}}{2}, \quad -\frac{\sqrt{3}}{2}, \quad -\frac{1}{2}, \quad \frac{1}{2}\)

D. \(-\frac{1}{2}, \quad -\frac{1}{2}, \quad -\frac{\sqrt{3}}{2}, \quad \frac{\sqrt{3}}{2}\)

E. \(-\frac{2\pi}{3}, \quad \frac{2\pi}{3}, \quad \frac{4\pi}{3}\)

F. \(-1\)

G. \( (\tan \beta - 1)(\tan \beta + 9) \)

H. \((2 \cos B + 1)(2 \cos B - 1)\)

I. \(3 \sec B \tan^2 B(3 \sec B \tan B + 2)\)

J. \( \cos \theta = -\frac{4}{5}, \quad \tan \theta = \frac{3}{4}\)

K. \( \tan t = -\frac{\sqrt{3}}{13}\)
For each angle, sketch the reference angle, draw an arrow to indicate the angle’s direction. A reference angle must be positive and $\leq \pi/2 = 90^\circ$.

A. (a) $110^\circ$
(b) $60^\circ$

B. (a) $\pi/4$
(b) $7\pi/6$

Rewrite the functions in terms of their reference angles, then evaluate. E.g.,

C.

<table>
<thead>
<tr>
<th>$\cos 210^\circ$</th>
<th>$-\cos 30^\circ$</th>
<th>$-\sqrt{3}/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos(-210^\circ)$</td>
<td>$-\cos 30^\circ$</td>
<td>$-\sqrt{3}/2$</td>
</tr>
<tr>
<td>$\sin 210^\circ$</td>
<td>$-\sin 30^\circ$</td>
<td>$-1/2$</td>
</tr>
<tr>
<td>$\sin(-210^\circ)$</td>
<td>$\sin 30^\circ$</td>
<td>$1/2$</td>
</tr>
</tbody>
</table>

D.

<table>
<thead>
<tr>
<th>$\cos(4\pi/3)$</th>
<th>$-\cos(\pi/3)$</th>
<th>$-1/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos(-4\pi/3)$</td>
<td>$-\cos(\pi/3)$</td>
<td>$-1/2$</td>
</tr>
<tr>
<td>$\sin(4\pi/3)$</td>
<td>$-\sin(\pi/3)$</td>
<td>$-\sqrt{3}/2$</td>
</tr>
<tr>
<td>$\sin(-4\pi/3)$</td>
<td>$\sin(\pi/3)$</td>
<td>$\sqrt{3}/2$</td>
</tr>
</tbody>
</table>

E. List three angles (in radian measure) whose cos is $-\frac{1}{2}$

$$\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$
Recall. \( \sin^2 \theta + \cos^2 \theta = 1 \)  
\[ \tan \theta = \sin \theta / \cos \theta \quad \cot \theta = \cos \theta / \sin \theta \]  
\[ \sec \theta = 1 / \cos \theta \quad \csc \theta = 1 / \sin \theta \]

Simplify:  
- \( \cot \theta \sec \theta \cos \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{\sin \theta} \cos \theta = \cot \theta \)
- \[ \frac{\sin x - 1}{\sin x + 1} = \frac{(\sin x - 1) + \sin x}{(\sin x + 1) - \sin x} = 2 \frac{\sin x - 1}{1} = 2 \sin x - 1 \]

Definition. An identity is an equation which is always true.

- \( x^2 + y^2 = 1, y = 2x \) aren’t identities: they aren’t always true.  
  \( \sin^2 \theta + \cos^2 \theta = 1, xy = 2y \) are identities, i.e., always true.

You have two ways to prove an identity:  
(1) Start from one side (usually the more complicated side) and work your way to the other side.  
(2) Show that the equality is equivalent to “true”.  
The text uses (1); we’ll give examples of (2).

- Prove \( \cot^2 \theta = \csc^2 \theta - 1 \)
  \( \iff \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} - 1 \iff \cos^2 \theta = 1 - \sin^2 \theta \)
  \( \iff \sin^2 \theta + \cos^2 \theta = 1 \iff \text{true} \).

- Prove \( \frac{\cos \theta}{\csc \theta} = \frac{\sin \theta}{\sec \theta} \)
  \( \iff \cos \theta \sec \theta = \csc \theta \sin \theta \) (by cross multiplying)
  \( \iff \cos \theta \frac{1}{\cos \theta} = \frac{1}{\sin \theta} \sin \theta \iff 1 = 1 \iff \text{true} \).

Right triangles

The trig functions were defined on the unit circle where the hypotenuse of the right triangle had length 1. For the angle pictured, \( \sin \theta = y \). Any other right triangle with an angle \( \theta \) is similar to this unit-circle triangle. Hence the ratios of the corresponding sides are equal. Thus \( \sin \theta = y = y/1 = a/c \) = side opposite/hypotenuse.

Theorem. For any right triangle with an acute angle \( \theta \):
- \( \sin \theta = \text{opposite/hypotenuse} \)
- \( \cos \theta = \text{adjacent/hypotenuse} \)
- \( \tan \theta = \text{opposite/adjacent} \)

* More precisely, “length of the side opposite/length of the hypotenuse”. We’ll abbreviate this to just: opp/hyp.

In a right triangle, the two complementary small angles add up to 90°. If one is \( \theta \), the other is \( 90^\circ - \theta \) or \( \pi/2 - \theta \).

Theorem. \( \sin(\frac{\pi}{2} - \theta) = \cos \theta \quad \sin(90^\circ - \theta) = \cos \theta \)
\( \cos(\frac{\pi}{2} - \theta) = \sin \theta \quad \cos(90^\circ - \theta) = \sin \theta \)

Proof. In the picture below,

\[ \sin \theta \quad \cos(90^\circ - \theta) \] are both \( a/c \).  
\( \cos \theta \) and \( \sin(90^\circ - \theta) \) are both \( b/c \).

In Fig. 1, find \( \sin, \cos \) and \( \tan \) for \( \theta \) and \( 90^\circ - \theta \).

First find the third side.  
The hypotenuse \( = \sqrt{5^2 + 6^2} = \sqrt{61} \)
\[ \sin \theta = 6/\sqrt{61} \]
\[ \cos \theta = 5/\sqrt{61} \]
\[ \tan \theta = 6/5 \]
\[ \sin(90^\circ - \theta) = 5/\sqrt{61} \]
\[ \cos(90^\circ - \theta) = 6/\sqrt{61} \]
\[ \tan(90^\circ - \theta) = 5/6 \]

In Fig. 2, find \( \sin B, \cos B \) and \( \tan B \).

Let \( y \) be the third side. Then \( y^2 + (x - 1)^2 = (9x^2)^2 \)
\[ y^2 = 81x^4 - (x - 1)^2 = 81x^4 - (x^2 - 2x + 1) \]
\[ y^2 = 81x^4 - x^2 + 2x - 1 \]
\[ y = \sqrt{81x^4 - x^2 + 2x - 1} \]
\[ \sin B = y/9x^2 = \sqrt{81x^4 - x^2 + 2x - 1}/9x^2 \]
\[ \cos B = (x - 1)/(9x^2) \]
\[ \tan B = y/(x - 1) = \sqrt{81x^4 - x^2 + 2x - 1}/(x - 1) \]

\( \tan \theta = 2/3 \), find \( \sin \theta \) and \( \cos \theta \), where \( \theta \) is acute.

Since \( \tan = \text{opposite/adjacent} = 2/3 \), draw a right triangle with \( \text{opposite} = 2 \) and \( \text{adjacent} = 3 \) (the drawing need not be exact). Then find the third side.

\[ \sin \theta = 2/\sqrt{13} \]
\[ \cos \theta = 3/\sqrt{13} \]
10(4). In a right triangle, the side adjacent angle $\theta$ is $x + 2$ and the opposite side is $x - 2$.
Simplify to the specified number of symbols.

\[ y = \sqrt{2x^2 + 8} \]
Find $\tan(\theta)$. Simplify to a 7 symbol answer.

\[ \frac{x - 2}{x + 2} \]
Find $\sin\left(\frac{\pi}{2} - \theta\right)$. Simplify to a 10 symbol answer.

\[ \frac{x + 2}{\sqrt{2x^2 + 8}} \]

11(4) $\theta$ is acute and $\tan \theta = 5/3$. Find $\sin(\theta)$ and $\cos(\theta)$.

\[ x^2 = 3^2 + 5^2 \]
\[ x = \sqrt{9 + 25} = \sqrt{34} \]
\[ \sin(\theta) = \frac{5}{\sqrt{34}} \]
\[ \cos(\theta) = \frac{3}{\sqrt{34}} \]

12(3). Prove: $\tan \theta \sin \theta = \sec \theta - \cos \theta$

iff $\frac{\sin \theta}{\cos \theta} \sin \theta = \frac{1}{\cos \theta} - \cos \theta$

-- rewrite everything in terms of $\sin \theta$ and $\cos \theta$

iff $\sin^2 \theta = 1 - \cos^2 \theta$

iff $\sin^2 \theta + \cos^2 \theta = 1$ ✔

13(3). Prove: $\frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta} = 2 \sec \theta$

iff $\frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta} = \frac{2}{\cos \theta}$

-- rewrite everything in terms of $\sin \theta$ and $\cos \theta$

iff $\cos \theta \cos \theta + (1 + \sin \theta)(1 + \sin \theta) = 2(1 + \sin \theta)$

iff $\cos^2 \theta + (1 + 2 \sin \theta + \sin^2 \theta) = 2 + 2 \sin \theta$

iff $\sin^2 \theta + \cos^2 \theta + 1 + 2 \sin \theta = 2 + 2 \sin \theta$

iff $2 + 2 \sin \theta = 2 + 2 \sin \theta$ ✔

iff $\sin^2 \theta + \cos^2 \theta = 1$
11. \( \theta \) is acute and \( \tan \theta = 3/5 \). Find \( \sin(\theta) \), \( \cos(\theta) \), \( \sin(\frac{\pi}{2} - \theta) \) and \( \cos(\frac{\pi}{2} - \theta) \).

First draw a large right triangle with angle \( \theta \), and sides of lengths 3 and 5 in the correct positions.

In the triangle above, locate the angle which is complementary to \( \theta \). Label it \( \frac{\pi}{2} - \theta \) or 90\(^\circ\) - \( \theta \).

Find the length of the third side of the triangle.

\[
\sin(\theta) = \frac{3}{5} \]

\[
\cos(\theta) = \frac{4}{5} \]

\[
\sin(\frac{\pi}{2} - \theta) = \frac{4}{5} \]

\[
\cos(\frac{\pi}{2} - \theta) = \frac{3}{5} \]

13. Simplify: \( \frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} \). Simplify to a 5 symbol answer or 7 symbols if you include "( )".

Note: 2/tan\( \theta \) = 2cot\( \theta \), 3/cos\( \theta \) = 3sec\( \theta \), 5/sin\( \theta \) = 5csc\( \theta \).
1(1). Simplify: \( \sin \theta \csc \theta \tan \theta \)

2(2). Simplify: \((\sec A + \tan A)(\sec A - \tan A)\)

§7.1 533:25-50.
3(2). Prove: \( \tan^2 A + 1 = \sec^2 A \)

4(3). Prove: \( \frac{1}{\sin \theta} - \sin \theta = \cot \theta \cos \theta \)

5(3). Prove: \( \frac{\cot A - 1}{\cot A + 1} = \frac{1 - \tan A}{1 + \tan A} \)

§6.2 484:1-8.
6(3). Find sinA, cosA, tanA.

\[
\begin{array}{c}
2 \\
A \\
6
\end{array}
\]

\[
\sin A = \\
\cos A = \\
\tan A =
\]

7(3). Given \( x > 1 \), find sin\( \phi \), cos\( \phi \), tan\( \phi \).

\[
\begin{array}{c}
\phi \\
x^2 - 1 \\
x^2 + 1
\end{array}
\]

\[
\sin \phi = \\
\cos \phi = \\
\tan \phi =
\]

§6.2 484:19,20. The angles B and \( \theta \) below are acute.
8(2). cosB = 3/8, find sinB and tanB.

\[
\sin B = \\
\tan B =
\]

9(2). sin\( \theta \) = 3\( x \)/5, find cos\( \theta \) and tan\( \theta \).

\[
\cos \theta = \\
\tan \theta =
\]
Simplify.

A. \[ \frac{\sin^2 A - \cos^2 A}{\sin A - \cos A} \]
   Hint: factor the top.

B. \[ \sin^2 \theta \cos \theta \csc^3 \theta \sec \theta \]

C. \[ \cot B \sin^2 B \cot B \]

D. \[ \frac{\cos^2 A + \cos A - 12}{\cos A - 3} \]

E. \[ \frac{\tan \theta}{\sec \theta - 1} + \frac{\tan \theta}{\sec \theta + 1} \]

F. \[ \sec A \csc A - \tan A - \cot A \]

G. \[ \csc^2 \theta + \frac{\tan^2 \theta}{\sec^2 \theta} \]

Prove the identities.

H. \[ \sin \theta \cos \theta \sec \theta \csc \theta = 1 \]

I. \[ \cos A (\sec A - \cos A) = \sin^2 A \]

J. \[ (1 - \sin \theta)(\sec \theta + \tan \theta) = \cos \theta \]

K. \[ \sin A + \cos A = \frac{\sin A}{1 - \cot A} - \frac{\cos A}{\tan A - 1} \]

L. \[ \sin A \tan A = \frac{1 - \cos^2 A}{\cos A} \]

M. Find \( \sin A, \cos A, \tan A \).

N. Find \( \sin \theta, \cos \theta, \tan \theta \).
   \[ \sin \theta = \] \[ \cos \theta = \] \[ \tan \theta = \]

O. Find \( \sin \beta, \cos \beta, \tan \beta \).
   \[ \sin \beta = \] \[ \cos \beta = \] \[ \tan \beta = \]

Answers

A. \( \sin A + \cos A \)
B. \( \csc \theta \)
C. \( \cos^2 B \)
D. \( \cos A + 4 \)
E. \( 2 \csc \theta \)
F. 0
G. 1
M. \( \cos A = \frac{3}{\sqrt{13}}, \sin A = \frac{2}{\sqrt{13}}, \tan A = \frac{2}{3} \)
N. \( \sin \theta = \frac{\sqrt{4x^2 + 9}}{\sqrt{16x^4 - 9}}, \cos \theta = \frac{3}{\sqrt{4x^2 + 9}}, \tan \theta = \frac{2x}{3} \)
O. \( \sin \beta = \frac{\sqrt{3}}{2}, \cos \beta = \frac{3}{4}, \tan \beta = \frac{\sqrt{16x^4 - 9}}{3} \)
P. \( \sin B = \frac{3}{\sqrt{13}}, \tan B = \frac{\sqrt{3}}{3} \)
Q. \( \cos \theta = \frac{\sqrt{3}}{2}, \tan \theta = \frac{2}{\sqrt{13}} \)
R. \( \cos \theta = \frac{\sqrt{4 - x^2}}{2}, \tan \theta = \frac{x}{\sqrt{4 - x^2}} \)
S. \( \sin \theta = \sqrt{1 - x^4}, \tan \theta = \frac{\sqrt{1 - x^4}}{x^2} \)
Prove the identities.

E. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

F. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

G. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

H. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

I. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

J. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

K. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

L. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

M. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

N. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)

O. \( \text{cos } \theta = \text{sin } \theta \cdot \text{cot } \theta \)
Math 140 Lecture 18

Exam 3 covers lectures 13-18. Study the recommended exercises.

Definition. For any function \( f \): \( f \) is even iff \( f(-x)=f(x) \).
\[ f \] is odd iff \( f(-x)=-f(x) \).

Graphically, the left half (the half plane left of the y-axis) of an even function is the reflection of the right half across the y-axis. The left half of an odd function is the negative of this reflection.

- \( f(x) = x^2 \) is even since \( f(-x) = (-x)^2 = x^2 = f(x) \).
- \( g(x) = x^3 \) is odd since \( g(-x) = (-x)^3 = -x^3 = -g(x) \).

Instead of thinking of \( \sin \theta \) as a function of an angle \( \theta \), we can think of it as a function \( \sin(t) \) of a real variable \( t \).

**Minus Theorem.** \( \sin(-t) = -\sin(t) \), \( \cos(-t) = \cos(t) \), \( \tan(-t) = -\tan(t) \). Thus \( \cos(t) \) is even; \( \sin(t) \) and \( \tan(t) \) are odd functions.

**Proof.**

\[
\sin(-t) = \sin(t) \cos(-t) = -\sin(t) \cos(t) = -\tan(t).
\]

**Definition.** A function \( f \) is periodic with period \( p \) iff \( f(x+p) = f(x) \) for all \( x \). \( p \) must be the smallest such positive number.

**2π Theorem.** \( \sin(t+2\pi) = \sin(t) \), \( \cos(t+2\pi) = \cos(t) \), for integers \( n \), \( \sin(t+2n\pi) = \sin(t) \), \( \cos(t+2n\pi) = \cos(t) \).

**Proof.**

\[
\cos(t, \sin(t)) = (\cos(t+2\pi), \sin(t+2\pi))
\]

\[
(-\cos t, -\sin t) = (\cos(t+\pi), \sin(t+\pi))
\]

**π Theorem.** \( \sin(t+\pi) = -\sin(t) \), \( \cos(t+\pi) = -\cos(t) \), \( \tan(t+\pi) = \tan(t) \).

For odd \( n \), \( \sin(t+n\pi) = -\sin(t) \), \( \cos(t+n\pi) = -\cos(t) \).

**Proof.** See the picture for sin and cos. The terminal sides of \( t+\pi \) and \( t \) have the same slope, \( \therefore \tan(t+\pi) = \tan(t) \).

\[ \sin(3\pi - x) \]

Since 3 is odd, \( \cos(3\pi - x) = -\cos(-x) = -\cos(x) \)

Previously we found reference angles in \([0, \pi/2]\) geometrically. Now we can find them algebraically.

- Take out minus signs (use the Minus Theorem).
- Rewrite in terms of the nearest multiple of \( \pi \).
- Throw out multiples of \( \pi \) (use 2\( \pi \) and \( \pi \) Theorems).

**Rewrite with a reference angle in \([0, \pi/2]\), then evaluate.**

\[
\sin(-5\pi/6) = -\sin(5\pi/6).
\]

\[ 0 < 5\pi/6 < 5\pi/6 < 5\pi/6 = \pi. \]

\[ -\sin(5\pi/6) = -\sin(\pi - 5\pi/6) = -\sin(-5\pi/6) = -\sin(5\pi/6) = -1/2. \]

\[ \cos(4\pi/3). \]

\[ \pi = 3\pi/3 < 4\pi/3 < 6\pi/3. \]

\[ 4\pi/3 \text{ is nearest } 3\pi/3 = \pi. \]

\[ \cos(4\pi/3) = \cos(3\pi/3 + \pi/3) = \cos(\pi + \pi/3) = -\cos(\pi/3) = -1/2. \]

\[ \tan(-7\pi/4). \]

\[ \tan(-7\pi/4) = -\tan(7\pi/4). \]

\[ \pi = 4\pi/4 < 7\pi/4 < 8\pi/4 = 2\pi. \]

\[ 7\pi/4 \text{ is nearest } 8\pi/4 = 2\pi. \]

\[ \tan(7\pi/4) = -\tan(8\pi/4 - \pi/4) = -\tan(2\pi - \pi/4) = -\tan(-\pi/4) = \tan(\pi/4) = \sin(\pi/4) = \sqrt{2}/2. \]

**Pythagorean Identities.** \( \sin^2 t + \cos^2 t = 1 \), \( \tan^2 t + 1 = \sec^2 t \), \( \cot^2 t + 1 = \csc^2 t \).

**Proof.** We’ve proved the first. To prove the second, multiply both sides by \( \cos^2 t \). For the third, multiply both sides by \( \sin^2 t \).

**Simplify** \( \csc t + \csc t \cot^2 t \) \( \sec^2 t - \tan^2 t \)

\[
\frac{\csc t(1+\cot^2 t)}{(1+\tan^2 t) - \tan^2 t} = \frac{\csc t(\csc^2 t)}{1} = \csc^3 t
\]

**Simplify** \( \sin^2 t - \cos^2 t \) \( \sin^4 t - \cos^4 t \)

\[
\frac{\sin^2 t - \cos^2 t}{\sin^2 t + \cos^2 t} = 1
\]

**Prove** \( \cot \theta + \tan \theta + 1 = \frac{\cot \theta}{1-\tan \theta} + \frac{\tan \theta}{1-\cot \theta} \)

Let \( x = \tan \theta \), then \( \cot \theta = 1/\tan \theta = 1/x \).

Thus we must prove \( 1/x + x + 1 = \frac{1/x + x}{1-x} \)

iff \( 1/x + x + 1 = \frac{1/x + x}{1-1/x} \)

iff \( 1 + x^2 + x = (1 - x^2)(1 - x) \)

iff \( (x^2 + x + 1)(1 - x) = (1 - x^3) \) iff \( 1 - x^3 = 1 - x^3 \)

iff \( true \)
10. In a right triangle, the side adjacent angle \( \theta \) is \( x + 2 \) and the opposite side is \( x - 2 \).

Find \( \tan(\pi - \theta) \). Simplify to a 7 symbol answer.

\[
= \tan(-\theta) = -\tan \theta = -(\frac{x-2}{x+2}) = \frac{2-x}{2+x}
\]

11. \( \theta \) is acute and \( \tan \theta = 5/3 \). Find \( \sin(-\theta) \) and \( \cos(-\theta) \).

\[
\begin{align*}
x^2 &= 3^2 + 5^2 \\
x &= \sqrt{9 + 25} = \sqrt{34} \\
\sin(-\theta) &= -\sin \theta = -\frac{5}{\sqrt{34}} \\
\sin(\pi - \theta) &= \frac{5}{\sqrt{34}} \\
\cos(-\theta) &= \cos \theta = \frac{3}{\sqrt{34}} \\
\cos(\pi - \theta) &= -\frac{3}{\sqrt{34}}
\end{align*}
\]

12. Prove: \( \frac{1}{1-\sec x} + \frac{1}{1+\sec x} + 2\cot^2x = 0 \)

Any acceptable proof:

\[
\begin{align*}
\text{iff } & \frac{1}{1-\sec x} + \frac{1}{1+\sec x} = -2\cot^2 x \text{ clear the denominators, multiply by } (1-\sec x)(1+\sec x) \\
\text{iff } & (1-\sec x) + (1+\sec x) = -2\cot^2 x(1-\sec^2 x) \\
\text{iff } & 2 = -2\cot^2 x(1-(1-\tan^2 x + 1)) \\
\text{iff } & -2 = -2\cot^2 x(1-\tan^2 x - 1) \\
\text{iff } & -2 = -2\cot^2 x(-\tan^2 x) \\
\text{iff } & -2 = -2? \text{checkmark?}
\end{align*}
\]

14. (a)(4) Rewrite using a reference angle, then find the exact answer: \( \cos(-19\pi/6) \).

\[
\begin{align*}
= \cos(-19\pi/6) \\
3\pi &= \frac{18\pi}{6} < -\frac{19\pi}{6} < \frac{24\pi}{6} = 4\pi, 3\pi \text{ is the nearest multiple of } \pi. \\
= \cos(18\pi/6 + \pi/6) = \cos(3\pi + \pi/6) = -\cos(\pi/6) = -\frac{\sqrt{3}}{2}
\end{align*}
\]

(b)(4) Rewrite \( \sin(-40\pi/11) \) using a reference angle in \([0, \pi/2]\). Write using \( \sin \) or \( \cos \), don't give the decimal or radical answer.

\[
\begin{align*}
= -\sin(-40\pi/11) \\
3\pi &= \frac{33\pi}{11} < -\frac{40\pi}{11} < \frac{44\pi}{11} = 4\pi, 4\pi \text{ is the nearest multiple of } \pi. \\
= -\sin(4\pi - \frac{4\pi}{11}) = -\sin(-\frac{4\pi}{11}) = \sin\frac{4\pi}{11}
\end{align*}
\]

15. Simplify to a number: \( \frac{\cot^2t(\sec^2t-1)}{\sec^2t-\tan^2t+1} \)

\[
\begin{align*}
= \frac{\cot^2t(\tan^2t+1)-1}{(\tan^2t+1)-\tan^2t+1} = \frac{\cot^2t\tan^2t}{2} = 1/2
\end{align*}
\]
11. $\theta$ is acute and $\tan \theta = 3/10$.

Label the sides of the triangle in a way that represents the equation $\tan \theta = 3/10$. Find the third side, $x = \tan \theta = 3/10$.

\[
\cos(-\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{x}{\text{hypotenuse}}
\]

7 symbols.

\[
\sin(\theta + \pi) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{x}{\text{hypotenuse}}
\]

7 symbols.

\[
\cos(\pi - \theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{x}{\text{hypotenuse}}
\]

8 symbols.

14a. Rewrite $\sin\left(-\frac{22\pi}{7}\right)$ using a reference angle in $[0, \pi/2]$. Write using sin or cos, don't give the decimal answer. First take care of the minus sign "-".

Find the nearest multiples of $\pi$ above and below $\frac{22\pi}{7}$. Of these two multiples of $\pi$, circle the nearest one.

$\frac{\text{nearest multiple of } \pi \text{ above and below } \frac{22\pi}{7}}{1}$

Rewrite $\frac{22\pi}{7}$ as the nearest multiple of $\pi$ plus or minus an acute positive angle. This acute angle is the reference angle.

$\frac{\text{reference angle}}{2}$

6 or 8 symbols.

14b. Rewrite $\cos\left(-\frac{18\pi}{7}\right)$ using a reference angle in $[0, \pi/2]$. Write using sin or cos, don't give the decimal answer. 8 or 10 symbols.

$\frac{\text{reference angle}}{2}$
§6.3 496:9-32.

Exam 3 next.

For 1-12: (a) Rewrite in terms of an angle in $[0, \pi/2]$. (b) Find the exact answer, no decimals. See the example. 1/2 point each.

In 1-15, 4 answers are integers, 2 others are rational.

Example:  
\[
\sin(15\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}
\]

1(½). \(\cos(2\pi/3) = \)

2(½). \(\cos(-2\pi/3) = \)

3(½). \(\sin(2\pi/3) = \)

4(½). \(\sin(-2\pi/3) = \)

5(½). \(\cos(-13\pi/4) = \)

6(½). \(\sin(13\pi/4) = \)

7(½). \(\cos(9\pi/4) = \)

8(½). \(\sin(-9\pi/4) = \)

9(½). \(\tan(7\pi/4) = \)

10(½). \(\sin(17\pi/4) = \)

11(½). \(\cos(11\pi) = \)

12(½). \(\cos(53\pi/4) = \)

13(1). Simplify to an integer:  
\[
\frac{\sec^2 t - 1}{\tan^2 t}
\]

14(1). Simplify to an integer:  
\[
\frac{\csc^4 \theta - \cot^4 \theta}{\csc^2 \theta + \cot^2 \theta}
\]

§7.1 533:51-86.

15(2). Prove:  
\[
\sin^2 t - \cos^2 t = \frac{1 - \cot^2 t}{1 + \cot^2 t}
\]

16(2). Prove:  
\[
\frac{1 + \tan s}{1 - \tan s} = \frac{\sec^2 s + 2 \tan s}{2 - \sec^2 s}
\]
Math 140     Hw 18     Recommended problems, don't turn this in.

For A-D:  (a) Rewrite in terms of an angle in $[0,\pi/2]$.  (b) Find the exact answer, no decimals.  See the example.  ½ point each.

A(a).  $\cos(11\pi/6)$

A(b).  $\cos(-11\pi/6)$

A(c).  $\sin(11\pi/6)$

A(d).  $\sin(-11\pi/6)$

B(a).  $\cos(5\pi/4)$

B(b).  $\sin(-5\pi/4)$

C(a).  $\sec(5\pi/3)$

C(b).  $\csc(-5\pi/3)$

C(c).  $\tan(5\pi/3)$

C(d).  $\cot(-5\pi/3)$

D(a).  $\cos(\pi/4 + 2\pi)$

D(b).  $\sin(\pi/3 + 2\pi)$

D(c).  $\sin(\pi/2 - 6\pi)$

E.  Simplify:  $\frac{\sin^2 t + \cos^2 t}{\tan^2 t + 1}$

F.  Simplify:  $\frac{\sec^2 \theta - \tan^2 \theta}{1 + \cot^2 \theta}$

G.  Prove:  $\csc t = \sin t + \cot t \cos t$

H.  Prove:  $\frac{1}{1 + \sec s} + \frac{1}{1 - \sec s} = -2 \cot^2 s$

Answers

Aabcd.  $\frac{\sqrt{3}}{2}$,  $\frac{\sqrt{3}}{2}$,  $-\frac{1}{2}$,  $\frac{1}{2}$

Bab.  $-\frac{1}{\sqrt{2}}$,  $\frac{1}{\sqrt{2}}$

Cabod.  $2$,  $\frac{2}{\sqrt{3}}$,  $-\sqrt{3}$,  $\frac{1}{\sqrt{3}}$

Dabc.  $\frac{1}{\sqrt{2}}$,  $\frac{\sqrt{3}}{2}$,  $1$

E.  $\cos^2 t$

F.  $\sin^2 \theta$

G.  $\sin t + \cot t \cos t = \sin t + \frac{\cos t}{\sin t} \cdot \cos t$

\[
= \frac{\sin^2 t}{\sin t} + \frac{\cos^2 t}{\sin t} = \frac{\sin^2 t + \cos^2 t}{\sin t} = \frac{1}{\sin t} = \csc t
\]

H.  $\frac{1}{1 + \sec s} + \frac{1}{1 - \sec s} = \frac{(1 - \sec s) +(1 + \sec s)}{(1 + \sec s)(1- \sec s)}$

\[
= \frac{2}{1 - \sec^2 s} = \frac{2}{-\tan^2 s} = -2 \cot^2 s
\]
Math 140     Hw 18     Worked examples of selected recommended problems.

(a) Rewrite in terms of an angle in \([0, \pi/2]\).

(b) Find the exact answer, no decimals.

A(a). \(\cos(11\pi/6)\)
\[
\cos(11\pi/6) = \cos(12\pi/6 - \pi/6) = \cos(2\pi - \pi/6) = \cos(-\pi/6) = \cos(\pi/6) = \frac{\sqrt{3}}{2}
\]

A(d). \(\sin(-11\pi/6)\)
\[
\sin(-11\pi/6) = -\sin(11\pi/6) = -\sin(12\pi/6 - \pi/6) = -\sin(2\pi - \pi/6) = -\sin(-\pi/6) = \sin(\pi/6) = 1/2
\]

B(a). \(\cos(5\pi/4)\)
\[
\cos(5\pi/4) = \cos(\pi/4) = -\cos(\pi/4) = -1/\sqrt{2}
\]

B(b). \(\sin(-5\pi/4)\)
\[
\sin(-5\pi/4) = -\sin(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}
\]

C(a). \(\sec(5\pi/3)\)
\[
\sec(5\pi/3) = \sec(6\pi/3 - \pi/3) = \sec(-\pi/3) = \sec(\pi/3) = 2
\]

C(c). \(\tan(5\pi/3)\)
\[
\tan(5\pi/3) = \tan(6\pi/3 - \pi/3) = \tan(-\pi/3) = -\tan(\pi/3) = -\frac{\sqrt{3}/2}{1/2} = -\sqrt{3}
\]

E. \(\frac{\sin^2t + \cos^2t}{\tan^2t + 1}\)
\[
= \frac{1}{\sec^2t} = \cos^2t
\]

F. \(\frac{\sec^2\theta - \tan^2\theta}{1 + \cot^2\theta}\)
\[
= \frac{(\tan^2\theta + 1) - \tan^2\theta}{\csc^2\theta} = \frac{1}{\csc^2\theta} = \sin^2\theta
\]

G. \(\csc t = \sin t + \cot t \cos t\)

Proof. \(\sin t + \cot t \cos t = \sin t + (\cos t/\sin t) \cos t = \sin t + \cos^2 t/\sin t = \frac{\sin^2 t + \cos^2 t}{\sin t} = \frac{1}{\sin t} = \csc t\)

Alternate proof. \(\sin t + \cot t \cos t = \sin t + \frac{\cos t}{\sin t} \cos t = \frac{\sin^2 t}{\sin t} + \frac{\cos^2 t}{\sin t} = \frac{\sin^2 t + \cos^2 t}{\sin t} = \frac{1}{\sin t} = \csc t\)

H. \(\frac{1}{1 + \sec s} + \frac{1}{1 - \sec s} = -2 \cot^2 s\)
\[
\frac{1}{1 + \sec s} + \frac{1}{1 - \sec s} = \frac{(1 - \sec s) + (1 + \sec s)}{(1 + \sec s)(1 - \sec s)} = \frac{2}{1 - \sec^2 s} = \frac{2}{-\tan^2 s} = -2 \cot^2 s
\]
Math 140  Lecture 19

Graphs of sin and cos

RECALL. sin and cos have period $2\pi$:
\[
\sin(x+2\pi) = \sin(x), \quad \cos(x+2\pi) = \cos(x).
\]
We often graph periodic functions only over one period, e.g., $[0, 2\pi]$. Before and after this interval, they repeat. Graph $\sin(x)$.

\[
\begin{array}{c}
\text{Min} \uparrow \text{root} \uparrow \text{Max} \downarrow \text{root} \downarrow \text{Min} \uparrow \\
x\text{-intercept: } x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots \\
\text{Max value: } 1 \text{ at } x = \ldots, -3\pi/2, \pi/2, 5\pi/2, \ldots \\
\text{Min value: } -1 \text{ at } x = \ldots, -5\pi/2, -\pi/2, 3\pi/2, \ldots \\
\text{Amplitude: } A = 1 \text{ (see definition below)} \\
\text{Period: } p = 2\pi
\end{array}
\]

At $x=0$, the line $y=x$ is tangent to the graph of $\sin(x)$.

Graph $\cos(x)$. ... Done similarly ...

DEFINITION. The amplitude of a function $f$ is half the difference between the max and min values of $f$. Like periods, amplitudes are always positive.

- Find the amplitude and period of $f$, $g$, and $h$.

- **Graph** $y=2\sin(x)$ over one period.

- **Graph** $y=\sin(2x)$ over one period.

Note $\sin(x)=0$ if $x=\ldots, 0, \pi, 2\pi, \ldots$.
Thus $\sin(2x)=0$ if $2x=\ldots, 0, \pi, 2\pi, \ldots$ iff $x=0, \pi/2, \pi, \ldots$.

Amplitude = 1, period = $\pi$.
Increases on $[0, \pi/4]$ and $[3\pi/4, \pi]$.

For $B>0$, $y=\sin(Bx)$ has period $2\pi/B$.
Reason: $\sin(Bx)$ repeats at $Bx=2\pi$. Solving for $x$ gives $x=2\pi/B$.
$B$ is the compression factor.

THEOREM. For $y=\pm A\sin(Bx)$ & $y=\pm A\cos(Bx)$ with $A,B>0$,
- amplitude: $A$
- period: $p=2\pi/B$. Hence also, $B=2\pi/p$.

To graph, mark $\pm A$ on the y-axis, 0 on the x-axis.
Divide $[0,p]$ into four parts.

- **Graph** $y=-3\cos(\pi x)$ over one period.

- List the amplitude, period, x-intercepts and the intervals in the period on which the function increases.

  $y=-3\cos(\pi x)=-A\cos(Bx)$
  $A=3$, $B=\pi$
  amplitude: $A=3$
  period: $p=2\pi/B=2\pi/\pi=2$

  Draw a one-period box of amplitude $A$ and length $p$.
  Divide the period into four parts.

- **Find an equation for the graph. Write it in the form** $y=\pm A\sin(Bx)$ or $y=\pm A\cos(Bx)$ with $A,B>0$.

  \[
  \begin{array}{c}
  \text{The graph has the shape of } y=-A\cos(Bx) \\
  \text{amplitude: } A=2; \quad \text{period: } p=4\text{.}
  \end{array}
  \]

  $B: B = \frac{2\pi}{p} = \frac{2\pi}{4} = \frac{\pi}{2}$
  Equation: $y = -2 \cos(\frac{\pi}{2}x)$. 

1. For the graph below, find the amplitude, period, and an equation of the form $y = A \sin(Bx)$ or $y = A \cos(Bx)$.

Amplitude: 6

Period: $\pi / 3$

Coefficient $B$: $\pi / 3$

Equation: $y = -2 \cos(\pi x / 3)$

2. (a) Graph $-2 \cos(3x)$ over one period. List the period and amplitude.

Amplitude

Period $2\pi / 3$

Graph

(b) Graph $3 \sin(5\pi x)$ over one period. List the period and amplitude.

Amplitude

Period $2\pi / 5$

Graph
1. For the graph below, find the amplitude, period, and an equation of the form $y = \pm A \sin(Bx)$ or $y = \pm A \cos(Bx)$.

Amplitude: $A =$
Must be positive.

Period: $p =$

Coefficient $B$: $B = \frac{2\pi}{p} =$

Equation: ___/3

Example: Graph $y = -5 \sin(x/3)$ over one period.

2. Graph $y = -3 \cos(2x)$ over one period.

Amplitude (must be positive) $A =$

Compression coefficient $B =$

Period $p = \frac{2\pi}{B} =$ ___/1

Draw a box. The length is the period $p$. The height above and below the $x$-axis is the amplitude $A$.
Divide the box into four pieces. Label the divisions on the $x$-axis.
Fill in the graph. ___/2
§5.3 431:80(a)(d). Find the period and amplitude. ½ point per part. 5 answers are integers, one is a fraction. All are positive.

1(1). period= amplitude=

2(1). period= amplitude=

3(1). period= amplitude=

§5.3 429:15-26. One period is an integer, the rest involve π.
Graph over one period. List the amplitude, period, x-intercepts and the intervals (in the period) on which the function increases. graph 1 point, ½ point for other parts.

4(3). \( y = 3 \sin x \)

amplitude=
period=
x-intercepts:
increases on:

5(3). \( y = \sin 3x \)

amplitude=
period=
x-intercepts:
increases on:

6(3). \( y = -2 \cos \left( \frac{x}{4} \right) \)

amplitude=
period=
x-intercepts:
increases on:

7(3). \( y = -2 \cos \left( \frac{\pi x}{4} \right) \)

amplitude=
period=
x-intercepts:
increases on:

Check your graphs at https://www.desmos.com/calculator

§5.3 429:41-48.
Find an equation for the graph of the form \( y = A \sin(Bx) \) or \( y = A \cos(Bx) \) with \( B > 0 \). 1 B involves π, two are fractions.

8(2).

\( y = \quad \)

(3\( \pi \), -2)

9(2).

\( y = \quad \)

(1,0)

10(2).

\( y = \quad \)

(5\( \pi \), 4)
Find the period and amplitude.

A. period= amplitude=

B. period= amplitude=

C. period= amplitude=

Graph over one period. List the amplitude, period, x-intercepts and the intervals (in the period) on which the function increases.

D. \( y = -\sin 2x \)
amplitude= period= x-intercepts: increases on:

E. \( y = 2\cos 2x \)
amplitude= period= x-intercepts: increases on:

Find an equation for the graph of the form \( y = \pm A\sin(Bx) \)
or \( y = \pm A\cos(Bx) \) with \( B>0 \). 2 B's involve \( \pi \), all are fractions.

F. \( y = 3\sin(\pi x/2) \)
amplitude= period= x-intercepts: increases on:

G. \( y = \cos 2\pi x \)
amplitude= period= x-intercepts: increases on:

H. \( y = \) increases on:

I. \( y = \) increases on:

J. \( y = \) increases on:
Find the period and amplitude.

A. period = 4, amplitude = 6

\[ y = 3 \sin (\pi x/2) \]

amplitude = 3

period = 4

x-intercepts: 0, 2, 4

increases on [0, 1], [3, 4]

B. period = 4, amplitude = 2

C. period = 6, amplitude = 3/2

Graph over one period. List the amplitude, period, x-intercepts and the intervals (in the period) on which the function increases. Use the amplitude and period to get the box.

D. \( y = -\sin 2x \)

amplitude = 1

period = \( \pi \)

x-intercepts: 0, \( \pi/2 \), \( \pi \)

increases on: [\( \pi/4, 3\pi/4 \)]

E. \( y = 2 \cos 2x \)

amplitude = 2

period = \( \pi \)

x-intercepts: \( \pi/4 \), \( 3\pi/4 \)

increases on: [\( \pi/2, \pi \)]

Find an equation for the graph of the form \( y = \pm A \sin (Bx) \) or \( y = \pm A \cos (Bx) \) with \( B > 0 \). 2 B’s involve \( \pi \), all are fractions.

H. \( y = \frac{3}{2} \sin (3x/2) \)

Since \( \pi/3 \) is at the one quarter point, the endpoint of the period is at \( 4\pi/3 \). Thus the period \( p = 4\pi/3 \). Since \( 3/2 \) is the high point, the amplitude is \( A = 3/2 \). Now \( B = 2\pi/p = 2\pi/(4\pi/3) = 2\pi(3/4\pi) = 3/2 \).

Since the graph starts at 0 and goes up, it is sin rather than cos, -sin or -cos.

I. \( y = \cos (2\pi x/5) \)

J. \( y = \pi \cos (\pi x/4) \)

Note, the amplitude \( A \) is \( \pi \), not -\( \pi \). Amplitudes and periods are always positive. 4 occurs halfway through the period, hence the period is 8 and \( B = 2\pi/p = 2\pi/8 = \pi/4 \).
RECALL. For $C > 0$, the graph of $f(x+C)$ is the graph of $f(x)$ shifted $C$ units to the left. The phase shift is the amount $C$ subtracted from $x$. $f(x+C) = f(x+(-C))$ has phase shift $-C$. $f(2x-C) = f(2(x-C/2))$ has phase shift $C/2$.

- Graph $y = \sin(x-\pi/2)$.
  Phase shift = $\pi/2$.

To graph $y = A\sin(Bx \pm D)$:
- Factor out $B$, and rewrite in the form $A\sin(B(x-C))$.
- The period is $p = 2\pi/B$. Also find $p/4$. We assume $A, B \geq 0$.
- Draw a box with amplitude $A$, period $p$, shifted by $C$.
- Draw the $\pm \sin$ or $\pm \cos$ shape in the box.

- Graph $y = \sin(2x-\pi)$ over one period.
  $\sin(2x-\pi) = \sin[2(x-\pi/2)]$. Phase shift = $\pi/2, p = \pi$.

- Graph $-2\cos(2\pi x-\pi/2)$ over one period.
  $-2\cos(2\pi x-\pi/2) = -2\cos[2\pi(x-1/4)] \leftrightarrow$ phase shift form
  $\because$ amplitude = 2, period = $2\pi/B = 1$, phase shift = $1/4$.

- Graph $y = \tan(x)$.
  $\tan(x) = \sin(x)/\cos(x)$.
  $\tan(x) = 0$ iff $\sin(x) = 0$ iff $x = ..., 0, \pi, 2\pi, ...$.
  $x$-intercepts: ..., $0, \pi, 2\pi, ...$.
  $\tan(x)$ undefined iff $\cos(x) = 0$ iff $x = ..., -\pi/2, \pi/2, 3\pi/2, ...$.

- Graph $y = -\tan(3x+\pi/2)$.
  $-\tan(3x+\pi/2) = -\tan[3(x-(-\pi/6))]$. Phase shift = $-\pi/6$.
  Period = $\pi/3$. Draw a strip centered at $-\pi/6$ with width $\pi/3$ (half of the width is before the center $-\pi/6$, half is after).
  Then fill in $-\tan(x)$. Add additional copies of this strip.

- Graph $y = \csc(x)$.
  $\csc(x) = 1/\sin(x)$.
  To graph $\csc(x)$, graph $\sin(x)$ and then invert its values.
  $\sin(x) = \frac{1}{2} \Rightarrow \csc(x) = 1/(1/2) = 2$.
  $\sin(x) = 0 \Rightarrow \csc(x) = 1/0 = \text{undefined}$.

- Graph $y = -\csc(3x+\pi)$ over one period.
  $-\csc(3x+\pi) = -\csc(3(x+\pi/3)) = -\csc(3(x-(\pi/3)))$
  Graph $y = -\sin(3x+\pi)$ first. Then invert as with $\csc(x)$ above.
3(7). Graph \( y = -\tan\left(\frac{\pi}{4}x + \frac{\pi}{8}\right) \) over one period. List the period and phase shift. Draw the asymptotes with dotted lines.

\[
= -\tan\left(\frac{\pi}{4}(x + \frac{1}{2})\right)
\]

\[
= -\tan\left(\frac{\pi}{4}(x - (-\frac{1}{2}))\right)
\]

\[
p = \frac{\pi}{B} = \frac{\pi}{\pi/4} = 4, \quad p/2 = 2
\]

Period \( p = p \)

Phase shift \( C = -1/2 \)

2(12). Graph \( y = -2 \sin\left(\frac{x}{4} - \frac{\pi}{2}\right) \) over one period. List the period and phase shift.

Get the box from the amplitude, period and shift.

\[
= -2 \sin\left(\frac{1}{4}(x - 2\pi)\right)
\]

Amplitude \( A = \)

\[
p = \frac{2\pi}{1/4} = 8\pi, \quad p/4 = 2\pi
\]

Period \( p = \)

Phase shift \( C = \)

Except for an incorrect shift, the following would also be acceptable.
2. Graph $-3 \cos\left(\frac{\pi}{3} x + \frac{\pi}{3}\right)$ over one period.
Inside the parentheses, factor out the coefficient of $x$.
Write in the form $A \cos(B(x - C))$

$$= -3 \cos\left(\frac{\pi}{3} x + \frac{\pi}{3}\right)$$

Amplitude $A = \frac{1}{3}$ Amplitudes must be positive.
Compression coefficient $B =$
Period $p = \frac{2\pi}{B} = \frac{6}{\pi}$
Phase shift $C =$

Now draw a box which starts at $C$ has length $p$ and height $A$ above and below the $x$-axis.
Divide the box horizontally into 4 pieces. Then fill in the graph.

3. Graph $y = -\tan\left(\frac{1}{4} x + \frac{\pi}{8}\right)$ over one period.
Again, inside the parentheses, factor out the coefficient of $x$. Write in the form $A \tan(B(x - C))$

Recall the graph of tan($x$):

Score ______/6
§5.3 429:27-40. (1) Graph over one period (start the period at the phase shift). (2) List the x-intercepts. (3) List both coordinates of the highest and lowest points. (4) Give the period and phase shift. 

1(5). \( y = \sin(2x - \frac{\pi}{2}) \) First rewrite in the form \( A \sin(B(x\pm C)). \)

\[ y = \]

---

2(5). \( y = 3 \sin\left(\frac{1}{2}x + \frac{\pi}{6}\right) \) Rewrite: \( y = \)

---

x-intercepts: max point: min point:

---

3(3). \( y = \tan(x - \frac{\pi}{3}) \) Center the period at the phase shift.

---

4(3). \( y = -2 \tan(\pi x) \)

---

x-intercept:

vertical asymptotes:

---

5(3) Put (a) and (b) on the same graph.

First rewrite with the argument in the form: \( B(x\pm C) \)

(a) \( y = \cos(3x + \frac{\pi}{3}) \) \( y = \)

(b) \( y = \sec(3x + \frac{\pi}{3}) \) \( y = \)

---

Check your graphs at https://www.desmos.com/calculator
(1) Graph over one period (period doesn’t have to start at 0).
(2) List the x-intercepts. (3) List both coordinates of the highest and lowest points.

A. \( y = \sin(3x + \frac{\pi}{2}) \) First rewrite in the form \( A\sin(B(x-C)) \).

\[ y = \]

\[ x\text{-intercepts:} \quad \text{period:} \]
\[ \text{max point:} \quad \text{min point:} \quad \text{phase shift:} \]

B. \( y = \cos(x - \frac{\pi}{2}) \)

\[ x\text{-intercepts:} \quad \text{period:} \]
\[ \text{max points:} \quad \text{min point:} \quad \text{phase shift:} \]

(1) Graph over one period. (2) List the x-intercepts.
(3) List the vertical asymptotes.

C. \( y = -\tan(x + \frac{\pi}{4}) \)

\[ x\text{-intercept:} \quad \text{vertical asymptotes:} \]

D. \( y = \frac{1}{2} \tan(\pi x/2) \)

\[ x\text{-intercepts:} \quad \text{vertical asymptotes:} \]

E. Put (a) and (b) on the same graph.
First rewrite with the argument in the form: \( B(x \pm C) \)

(a) \( y = -3 \cos(2\pi x - \frac{\pi}{4}) \) \( y = \)

(b) \( y = -3 \sec(2\pi x - \frac{\pi}{4}) \) \( y = \)

\[ -3 \cos(2\pi x - \frac{\pi}{4}) : x\text{-intercepts:} \]
\[ -3 \sec(2\pi x - \frac{\pi}{4}) : \text{vertical asymptotes:} \]

For answers, see Hw 20 worked examples.
(1) Graph over one period (period doesn’t have to start at 0).  (2) List the x-intercepts.  (3) List both coordinates of the highest and lowest points. Get the box from the amplitude, period and phase shift.

A.  \( y = \sin(3x + \frac{\pi}{2}) \)  First rewrite in the form \( A\sin(B(x-C)) \).

\[
y = \sin(3(x - \frac{\pi}{6}))
\]

\[
\begin{array}{c}
\text{x-intercepts: } -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{2} \\
\text{max point: } (0, 1) \\
\text{min point: } (\frac{\pi}{3}, -1)
\end{array}
\]

\( y = \sin(3(x - \frac{\pi}{6})) \)  (answers depend on choice of interval)

\[
\text{period: } \frac{2\pi}{3}
\]

B.  \( y = \cos(x - \frac{\pi}{2}) \)

\[
\begin{array}{c}
x-intercepts: \pi, 2\pi (answer varies with choice of interval) \ y= \sin(x).
\end{array}
\]

\[
\begin{array}{c}
\text{max points: } (\frac{\pi}{2}, 1), (\frac{5\pi}{2}, 1) \\
\text{min point: } (\frac{3\pi}{2}, -1)
\end{array}
\]

\( \text{period: } 2\pi \)

\( \text{shift: } \frac{\pi}{2} \)
**Math 140    Hw 20    Worked examples continued.**

(1) Graph over one period. (2) List the x-intercepts. (3) List the vertical asymptotes.

C. \( y = -\tan(x + \frac{\pi}{4}) \)

-3\(\pi/4\) \(-\pi/4\) \(\pi/4\)

x-intercept: \(-\pi/4\)
vertical asymptotes: \(x = -3\pi/4, x = \pi/4\)

E. Put (a) and (b) on the same graph.
First rewrite with the argument in the form: \(B(x \pm C)\)

(a) \( y = -3 \cos(2\pi x - \frac{\pi}{4}) \) \( y = -3 \cos(2\pi(x - \frac{1}{8})), p = 1 \)

(b) \( y = -3 \sec(2\pi x - \frac{\pi}{4}) \) \( y = -3 \sec(2\pi(x - \frac{1}{8})) \)

You can start the period at 1/8 or anywhere you like.

-3cos(2\(\pi x - \pi/4\)): x-intercepts: -1/8, 3/8, 7/8, ...

-3sec(2\(\pi x - \pi/4\)): vertical asymptotes: \(x = -1/8, x = 3/8, ...\)

Continued from previous page
In each problem below and in homework:

1. Apply an addition formula. (first point)
2. Simplify. (second point)

- **Cosine Addition Formula:**
  \[ \cos(a + b) - \cos(a - b) = \cos(a) \cos(b) - \sin(a) \sin(b) \]

- **Sine Addition Formula:**
  \[ \sin(a + b) + \sin(a - b) = \sin(a) \cos(b) + \cos(a) \sin(b) \]

- **Example:**
  \[ \cos(\frac{\pi}{3} + \frac{\pi}{6}) = \cos(\frac{\pi}{3}) \cos(\frac{\pi}{6}) - \sin(\frac{\pi}{3}) \sin(\frac{\pi}{6}) \]

**Proof of Cosine Addition Formula:**

\[
\begin{align*}
\cos(s + t) &= \cos(s) \cos(t) - \sin(s) \sin(t) \\
\cos(s - t) &= \cos(s) \cos(t) + \sin(s) \sin(t)
\end{align*}
\]

Proof of **Sine Addition Formula:**

\[
\begin{align*}
\sin(s + t) &= \sin(s) \cos(t) + \cos(s) \sin(t) \\
\sin(s - t) &= \sin(s) \cos(t) - \cos(s) \sin(t)
\end{align*}
\]

In both pictures the obtuse angle is \( s + t \), the short sides are radii of length 1, and the long sides have some common length \( d \). Use the distance formula to calculate \( d \).

In the first,
\[
d = \sqrt{(\cos(s + t) - 1)^2 + (\sin(s + t) - 0)^2}
\]
\[
\therefore d^2 = \cos^2(s + t) - 2\cos(s + t) + 1 + \sin^2(s + t)
\]
\[
\therefore d^2 = 2 - 2\cos(s + t),
\]

since \( \cos^2(s + t) + \sin^2(s + t) = 1 \).

In the second,
\[
d = \sqrt{(\cos t - \cos s)^2 + (\sin t + \sin s)^2}
\]
\[
\therefore d^2 = \cos^2 t - 2\cos t \cos s + \cos^2 s
\]
\[
+ \sin^2 t + 2\sin t \sin s + \sin^2 s
\]
\[
\therefore d^2 = 2 - 2\cos t \cos s + 2\sin t \sin s,
\]

since \( \cos^2 t + \sin^2 t = 1 \), \( \cos^2 s + \sin^2 s = 1 \),

\( \cos(-x) = \cos(s) \), \( \sin(-s) = -\sin(s) \).

Equate the two terms which equal \( d^2 \):

\[
2 - 2\cos(s + t) = 2 - 2\cos t \cos s + 2\sin t \sin s
\]
\[
\therefore -2\cos(s + t) = -2\cos t \cos s + 2\sin t \sin s
\]
\[
\therefore \cos(s + t) = \cos(s) \cos(t) - \sin(s) \sin(t)
\]

Proof of **Cosine Addition Formula:**

\[
\begin{align*}
\cos(s + t) &= \cos(s) \cos(t) - \sin(s) \sin(t) \\
\cos(s - t) &= \cos(s) \cos(t) + \sin(s) \sin(t)
\end{align*}
\]

The other proofs are similar.
5(5). Write with no products and no π: at most 7 symbols. \( \cos\left(\frac{\pi}{8}\right) \cos(x - \frac{3\pi}{8}) + \sin\left(\frac{\pi}{8}\right) \sin(x - \frac{3\pi}{8}) \)

\[
= \cos\left(\frac{\pi}{8} - (x - \frac{3\pi}{8})\right) = \cos\left(\frac{4\pi}{8} - x\right) \\
= \cos\left(\frac{\pi}{2} - x\right) = \sin x
\]

5(5). Write with no products and no π: at most 7 symbols. \( \sin\left(\frac{5\pi}{8}\right) \cos(x + \frac{3\pi}{8}) + \cos\left(\frac{5\pi}{8}\right) \sin(x + \frac{3\pi}{8}) \)

\[
= \sin\left(\frac{5\pi}{8} + (x + \frac{3\pi}{8})\right) \\
= \sin(x + \frac{8\pi}{8}) = \sin(x + \pi) \\
= -\sin x
\]

6(2). Write in terms of \( \tan x \) and no π: \( \tan(x + \pi/4) \).

\[
\tan(x + \pi/4) = \frac{\tan x + \tan(\pi/4)}{1 - \tan x \tan(\pi/4)} \\
= \frac{\tan x + 1}{1 - \tan x}
\]

6(2). Find the exact answer. \( \tan(x) = 4, \tan(y) = z, \tan(x - y) = \)

\[
\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \\
= \frac{4 - z}{1 + 4z}
\]
\[
\begin{align*}
\sin(s \pm t) &= \sin s \cos t \pm \cos s \sin t \\
\cos(s + t) &= \cos s \cos t - \sin s \sin t \\
\cos(s - t) &= \cos s \cos t + \sin s \sin t \\
\tan(s + t) &= \frac{\tan s + \tan t}{1 - \tan s \tan t} \\
\tan(s - t) &= \frac{\tan s - \tan t}{1 + \tan s \tan t}
\end{align*}
\]

5. Write with no products and no \(\pi\): 4 or 6 symbols. \(\cos\left(\frac{\pi}{8}\right)\) \(\cos(x - \frac{5\pi}{8})\) \(\sin\left(\frac{\pi}{8}\right)\) \(\sin(x - \frac{5\pi}{8})\) = \(\_\)/3

6. Find the exact answer, 6 symbols. \(\tan(x) = 5\), \(\tan(y) = 6\), \(\tan(x + y) = \_\)/1

6. Write in terms of \(\tan(x)\) and no \(\pi\): \(\tan(x - \pi/6) = \_\)/1

Answers involving fractions of fractions must be simplified to a single fraction. Leave \(\sqrt{3}\), in the denominator. 16 symbols, 20 with ( )s.
§7.2 539:13-18.

(a) **Apply an addition formula** (show the step immediately after the addition formula).  
First point.

(b) **Simplify.**  
Second point.

**Examples:**

- \[ \sin(x) \cos(y-x) + \cos(x) \sin(y-x) \]
  \[ = \sin(x + (y-x)) = \sin y \]
  4 symbols

- \[ \cos(\frac{\pi}{4} - \theta) \]
  \[ = \cos \frac{\pi}{4} \cos \theta + \sin \frac{\pi}{4} \sin \theta = \frac{1}{\sqrt{2}} (\cos \theta + \sin \theta) \]

- **Given** \( \tan x = 3, \tan y = 5: \) Find \( \tan(x-y). \)
  \[ = \frac{3 - 5}{1 + 3 \cdot 5} = \frac{-2}{16} = \frac{-1}{8} \]
  4 symbols

**1(2).** \( \sin \frac{\pi}{6} \cos \frac{\pi}{3} + \cos \frac{\pi}{6} \sin \frac{\pi}{3} \)

1 symbol

**2(2).** \( \cos 2u \cos 3u + \sin 2u \sin 3u \)

4 symbols

**3(2).** \( \cos(s-t) \cos t - \sin(s-t) \sin t \)

4 symbols

**4(2).** \( \cos(\frac{3\pi}{2} + \theta) \)

4 symbols

**5(2).** \( \sin(t + \frac{\pi}{6}) - \sin(t - \frac{\pi}{6}) \)

4 symbols

6(2). \( \cos(\theta - \frac{\pi}{4}) + \cos(\theta + \frac{\pi}{4}) \)

6 symbols

7(4). Given \( \tan s = \frac{1}{2}, \tan t = \frac{1}{3}: \)

Find \( \tan(s+t) \)

1 symbol

Find \( \tan(s-t) \)

3 symbols

8(2). \( \frac{\tan(\pi/5) - \tan(\pi/30)}{1 + \tan(\pi/5) \tan(\pi/30)} \)

4 symbols

9(2). \( \frac{2 \tan(\pi/12)}{1 - \tan^2(\pi/12)} \)

4 symbols
(1) Simplify using an addition formula (show the step immediately after the addition formula). First point.

(2) Complete the simplification. Second point.

Examples:

- \[ \sin(x) \cos(y-x) + \cos(x) \sin(y-x) = \sin(x + (y-x)) = \sin y \]

- \[ \cos\left(\frac{\pi}{4} - \theta\right) = \cos \frac{\pi}{4} \cos \theta + \sin \frac{\pi}{4} \sin \theta = \frac{1}{\sqrt{2}} (\cos \theta + \sin \theta) \]

- Given \( \tan x = 3, \tan y = 5 \): Find \( \tan(x - y) \).
  
  \[ \frac{3 - 5}{1 + 3 \cdot 5} = -\frac{1}{8} \]

A. \( \sin 3\theta \cos \theta - \cos 3\theta \sin \theta \)

B. \( \cos 2u \cos 3u - \sin 2u \sin 3u \)

C. \( \sin(A + B) \cos A - \cos(A + B) \sin A \)

D. \( \sin(\theta - \frac{3\pi}{2}) \)

E. \( \sin\left(\frac{\pi}{4} + s\right) - \sin\left(\frac{\pi}{4} - s\right) \)

F. \( \cos\left(\frac{\pi}{3} - \theta\right) - \cos\left(\frac{\pi}{3} + \theta\right) \)

G. Given \( \tan s = 2, \tan t = 3 \): Find \( \tan(s + t) \)
  
  Find \( \tan(s - t) \)

H. \( \frac{\tan(t) + \tan(2t)}{1 - \tan(t) \tan(2t)} \)

I. \( \frac{\tan(x-y) + \tan(y)}{1 - \tan(x-y) \tan(y)} \)

Answers

A. \( \sin 2\theta \)  
B. \( \cos 5u \)  
C. \( \sin B \)  
D. \( \cos \theta \)  
E. \( \sqrt{2} \sin s \)  
F. \( \sqrt{3} \sin \theta \)  
G. \( \tan(s + t) = -1, \tan(s - t) = -1/7 \)  
H. \( \tan 3t \)  
I. \( \tan x \)
(1) Simplify using an addition formula (show the step immediately after the addition formula). First point.

(2) Complete the simplification. Second point.

Examples:

- \(
  \sin(x) \cos(y-x) + \cos(x) \sin(y-x)
  = \sin(x + (y-x)) = \sin y
\)
- \(
  \cos(\frac{\pi}{4} - \theta)
  = \cos \frac{\pi}{4} \cos \theta + \sin \frac{\pi}{4} \sin \theta = \frac{1}{\sqrt{2}} (\cos \theta + \sin \theta)
\)
- Given \( \tan x = 3, \tan y = 5): Find \( \tan(x-y) \).
  \[ \frac{3-5}{1+3 \cdot 5} = -\frac{1}{8} \]

A. \( \sin 3\theta \cos \theta - \cos 3\theta \sin \theta \)

\[ = \sin(3\theta - \theta) \]
\[ = \sin 2\theta \]
Answer: \( \sin 2\theta \)

B. \( \cos 2u \cos 3u - \sin 2u \sin 3u \)

\[ = \cos(2u + 3u) \]
\[ = \cos(5u) \]
Answer: \( \cos 5u \)

C. \( \sin(A+B) \cos A - \cos(A+B) \sin A \)

\[ = \sin((A+B) - A) \]
\[ = \sin(B) \]
Answer: \( \sin B \)

D. \( \sin(\theta - \frac{3\pi}{2}) \)

\[ = \sin \theta \cos \frac{3\pi}{2} - \cos \theta \sin \frac{3\pi}{2} \]
\[ = \sin(0) - \cos(-1) \]
\[ = \cos(\theta) \]
Answer: \( \cos \theta \)

E. \( \sin(\frac{\pi}{4} + s) - \sin(\frac{\pi}{4} - s) \)

\[ = \sin \frac{\pi}{4} \cos s + \cos \frac{\pi}{4} \sin s - (\sin \frac{\pi}{4} \cos s - \cos \frac{\pi}{4} \sin s) \]
\[ = 2 \cos \frac{\pi}{4} \sin s = 2 \frac{1}{\sqrt{2}} \sin s = 2 \frac{\sqrt{2}}{2} \sin s \]
Answer: \( \sqrt{2} \sin s \)

F. \( \cos(\frac{\pi}{3} - \theta) - \cos(\frac{\pi}{3} + \theta) \)

\[ = \cos \frac{\pi}{3} \cos \theta + \sin \frac{\pi}{3} \sin \theta - (\cos \frac{\pi}{3} \cos \theta - \sin \frac{\pi}{3} \sin \theta) \]
\[ = 2 \sin \frac{\pi}{3} \sin \theta = 2 \frac{\sqrt{3}}{2} \sin \theta \]
Answer: \( \sqrt{3} \sin \theta \)

G. Given \( \tan s = 2, \tan t = 3 \)
Find \( \tan(s + t) \)

\[ = \frac{\tan s + \tan t}{1 - \tan s \tan t} = \frac{2 + 3}{1 - 2 \cdot 3} = \frac{5}{-5} = -1 \]
Answer: \( \tan(s + t) = -1 \)

Find \( \tan(s - t) \)

\[ = \frac{\tan s - \tan t}{1 + \tan s \tan t} = \frac{2 - 3}{1 + 2 \cdot 3} = -\frac{1}{7} \]
Answer: \( \tan(s - t) = -1/7 \)

H. \( \frac{\tan(t)+\tan(2t)}{1-\tan(t) \tan(2t)} \)

\[ = \tan(t + 2t) = \tan 3t \]
Answer: \( \tan 3t \)

I. \( \frac{\tan(x-y)+\tan(y)}{1-\tan(x-y) \tan(y)} \)

\[ = \tan((x - y) + y) = \tan x \]
Answer: \( \tan x \)
Proof.
\[
\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi
\]
\[
\Rightarrow \sin 2\theta = \sin \theta \cos \theta + \cos \theta \sin \theta = 2 \sin \theta \cos \theta \checkmark
\]

\[
\cos 2\theta = \cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi
\]
\[
= \cos^2 \theta - \sin^2 \theta = \cos^2 \theta - \sin^2 \theta \checkmark
\]
\[
\tan 2\theta = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{2 \tan \theta}{1 - \tan^2 \theta} \checkmark
\]

**Half-Angle Formulas.**
\[
\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} + \text{ if } \theta/2 \in \text{ quadrant I or II}, -\text{ if not.}
\]
\[
\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} + \text{ if } \theta/2 \in \text{ quadrant I or IV}, -\text{ if not.}
\]
\[
\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}
\]

Proof.
\[
\cos^2 \theta + \sin^2 \theta = 1. \text{ Hence}
\]
\[
\cos^2 \theta = 1 - \sin^2 \theta \quad \text{and}
\]
\[
\sin^2 \theta = 1 - \cos^2 \theta
\]
\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta
\]
\[
= (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2 \sin^2 \theta
\]
\[
\therefore 2 \sin^2 \theta = 1 - \cos 2\theta
\]
\[
\therefore \sin \theta = \pm \sqrt{\frac{1 - \cos 2\theta}{2}}. \text{ Replacing } \theta \text{ by } \theta/2 \text{ gives}
\]
\[
\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \checkmark
\]
\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta
\]
\[
= \cos^2 \theta - (1 - \cos^2 \theta) = 2 \cos^2 \theta - 1
\]
\[
\therefore 2 \cos^2 \theta = 1 + \cos 2\theta
\]
\[
\therefore \cos \theta = \pm \sqrt{\frac{1 + \cos 2\theta}{2}}. \text{ Replacing } \theta \text{ by } \theta/2 \text{ gives}
\]
\[
\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \checkmark
\]
\[
\tan \frac{\theta}{2} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{\sqrt{\frac{1 - \cos \theta}{2}}}{\sqrt{\frac{1 + \cos \theta}{2}}}
\]
\[
= \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \sqrt{\frac{(1 - \cos \theta)(1 + \cos \theta)}{(1 + \cos \theta)(1 + \cos \theta)}}
\]
\[
= \sqrt{\frac{1 - \cos^2 \theta}{(1 + \cos \theta)^2}} = \sqrt{\frac{\sin^2 \theta}{(1 + \cos \theta)^2}} = \frac{\sin \theta}{1 + \cos \theta} \checkmark
\]

- **Evaluate using half-angle formulas.**
  \[
  \sin \frac{\pi}{12} = \pm \sqrt{\frac{1 - \cos 2\pi}{2}} = \pm \sqrt{\frac{1 - \sqrt{3}/2}{2}} = \pm \frac{\sqrt{2 - \sqrt{3}}}{2}
  \]
  \[
  \cos \frac{\pi}{12} = \pm \sqrt{\frac{1 + \cos 2\pi}{2}} = \pm \sqrt{\frac{1 + \sqrt{3}/2}{2}} = \pm \frac{\sqrt{2 + \sqrt{3}}}{2}
  \]
  \[
  \tan \frac{\pi}{12} = \frac{\sin \frac{\pi}{12}}{\cos \frac{\pi}{12}} = \frac{\sqrt{2 - \sqrt{3}}}{\sqrt{2 + \sqrt{3}}} = 2 - \sqrt{3}
  \]
- **For the angle \(t\) in the picture, find \(\cos(2t)\) and \(\sin(t/2)\).**

First find the missing side \(x\), then find \(\sin(t)\) and \(\cos(t)\).
\[
x^2 + 4 = 9, \quad x = \sqrt{9 - 4} = \sqrt{5}
\]
\[
sin t = \sqrt{5}/3, \cos t = 2/3
\]
\[
\cos 2t = \cos^2 t - \sin^2 t = \frac{4}{9} - \frac{5}{9} = -\frac{1}{9}
\]
\[
\sin \frac{t}{2} = \pm \sqrt{\frac{1 - \cos t}{2}} = \sqrt{\frac{1 - 2/3}{2}} = \sqrt{\frac{1/3}{2}} = \frac{1}{\sqrt{6}}
\]
- **Express \(\sin^4 \theta\) in a form which does not use powers of trigonometric functions.**
\[
\sin^4 \theta = (\sin^2 \theta)^2 = \left(\frac{1 - \cos 2\theta}{2}\right)^2 = \frac{1 - \cos 2\theta + \cos 2\theta}{4} = \frac{1 - 2 \cos 2\theta + \cos^2 2\theta}{4}
\]
\[
= \frac{1}{8}(3 - 4 \cos 2\theta + \cos 4\theta)
\]

The calculation for \(\cos^4 \theta\) is similar.
7. Given: $x \tan \theta = 1$, $\pi < \theta < \frac{3}{2} \pi$.

$$\tan \theta = \frac{1}{x}$$

\[\begin{array}{c}
\frac{\theta}{\sqrt{x^2 + 1}} = 1 \\
x \\
\end{array}\]

$x \geq 0$

$$\sin \theta = -\frac{1}{\sqrt{x^2 + 1}}$$

$$\cos \theta = -\frac{x}{\sqrt{x^2 + 1}}$$

(a)(2) Find $\tan\left(\frac{\pi}{2} - \theta\right)$. Hint, draw the triangle, locate the angle $\pi/2 - \theta$.

(b)(5) Find $\sin(\theta/2)$.

$$\frac{\pi}{2} < \frac{\theta}{2} < \frac{3}{4} \pi \quad \text{quad. II}$$

$$= \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$= \frac{\sqrt{\frac{1 - \sin \theta}{2}}}{\sqrt{x^2 + 1}}$$

(c)(5) Find $\cos(2\theta)$.

$$= \cos^2 \theta - \sin^2 \theta$$

$$= \left(-\frac{x}{\sqrt{x^2 + 1}}\right)^2 - \left(\frac{1}{\sqrt{x^2 + 1}}\right)^2$$

$$= \frac{x^2 - 1}{x^2 + 1}$$
7. Given: \( \tan \theta = \frac{3}{2}, \pi < \theta < \frac{3\pi}{2} \).

Draw a triangle with \( \theta \) and the opposite side marked 3 and the adjacent side marked 2 and find the third side. ___/1

Find \( \tan \left( \frac{\pi}{2} - \theta \right) \). In the triangle above, locate the angle \( \pi/2 - \theta \), it is the complementary angle. 3 symbols. ___/1

Find \( \sin(\theta) \). Use \( \pi < \theta < \frac{3\pi}{2} \) to determine the sign \( \pm \). 6 symbols. ___/1

Find \( \cos(\theta) \). 6 symbols. ___/1

Find \( \sin(2\theta) \). 5 symbols. ___/1

Use the double-angle formula
\[
\sin(2\theta) = 2 \sin \theta \cos \theta
\]

Find \( \cos(\theta/2) \).

Use the half-angle formula
\[
\cos(\theta/2) = \pm \sqrt{\frac{1+\cos \theta}{2}}
\]

To determine the sign \( \pm \), find the quadrant for \( \theta/2 \).

To find the quadrant for \( \theta/2 \), divide \( \pi < \theta < \frac{3\pi}{2} \) by 2.

Your answer should not have a fraction of fractions. You may leave \( \sqrt{13} \) in the denominator. 12 symbols.
§7.3 548:1-8, 21, 22, 35-40.

Evaluate the expressions.
4 answers are fractions, one a polynomial. The rest have radicals.

1(4). \( \cos \theta = \frac{2}{5}, \quad \frac{3\pi}{2} < \theta < 2\pi \).
First find \( \sin \theta \). The quadrant determines the sign.

### (a) \( \sin 2\theta \)
8 symbols

### (b) \( \cos 2\theta \)
6 symbols

### (c) \( \sin \frac{\theta}{2} \)
5 symbols

### (d) \( \cos \frac{\theta}{2} \)
6 symbols

2(3). \( \sin \theta = \frac{-1}{10}, \quad 270^\circ < \theta < 360^\circ \).
First find \( \cos \theta \). The quadrant determines the sign.

### (a) \( \sin 2\theta \)
8 symbols

### (b) \( \cos 2\theta \)
5 symbols

### (c) \( \sin \frac{\theta}{2} \)
Double radical

3(3). Evaluate using half-angle formulas.

### (a) \( \sin(\pi/8) \)
Double radical

### (b) \( \cos(\pi/8) \)
Double radical

### (c) \( \tan(\pi/8) \)
4 or 6 symbols


[Diagram of a right triangle with sides 1, 7, and hypotenuse 25]

Note hypotenuse = \( \sqrt{24^2 + 7^2} = 25 \)

### (a) \( \sin 2t \)
7 symbols

### (b) \( \cos(t/2) \)
5 or 6 symbols

5(2). \( x = \sqrt{2} \cos \theta, \quad 0 < \theta < \frac{\pi}{2} \).

### (a) \( \sin 2\theta \)
6 symbols

### (b) \( \cos 2\theta \)
4 symbols
Evaluate the expressions.

A. \( \sin \theta = \frac{3}{4}, \ \frac{\pi}{2} < \theta < \pi. \)
   First find \( \cos \theta. \) The quadrant determines the sign.

(a) \( \sin 2\theta \)

(b) \( \cos 2\theta \)

(c) \( \sin \frac{\theta}{2} \)

(d) \( \cos \frac{\theta}{2} \)

B. \( \cos \theta = \frac{1}{3}, \ 180^\circ < \theta < 270^\circ. \)
   First find \( \sin \theta. \) The quadrant determines the sign.

(a) \( \sin 2\theta \)

(b) \( \cos 2\theta \)

(c) \( \sin \frac{\theta}{2} \)

C. Evaluate using half-angle formulas.

(a) \( \sin(\pi/12) \)

(b) \( \cos(\pi/12) \)

(c) \( \tan(\pi/12) \)

Picture for D and E.

\[ \begin{array}{c}
\beta \\
5 \\
\hline
4 \\
\theta
\end{array} \]

D. (a) \( \sin 2\theta \) (b) \( \cos 2\theta \) (c) \( \tan 2\theta \)

E. (a) \( \sin(\theta/2) \) (b) \( \cos(\theta/2) \) (c) \( \tan(\theta/2) \)

F. \( x = 5 \sin \theta, \ 0 < \theta < \frac{\pi}{2} \)

(a) \( \sin 2\theta \)

(b) \( \cos 2\theta \)

G. \( x - 1 = 2 \sin \theta, \ 0 < \theta < \frac{\pi}{2} \)

(a) \( \sin 2\theta \)

(b) \( \cos 2\theta \)

Answers

A. (a) \( -\frac{3\sqrt{7}}{8} \) (b) \( -\frac{1}{8} \) (c) \( \sqrt{\frac{4+\sqrt{7}}{8}} \) (d) \( \sqrt{\frac{4-\sqrt{7}}{8}} \)

B. (a) \( \frac{4\sqrt{2}}{9} \) (b) \( -\frac{7}{9} \) (c) \( \frac{\sqrt{6}}{3} \)

C. (a) \( \frac{\sqrt{2}-\sqrt{3}}{2} \) (b) \( \frac{\sqrt{2}+\sqrt{3}}{2} \) (c) \( 2 - \sqrt{3} \)

D. (a) \( \frac{24}{25} \) (b) \( \frac{7}{25} \) (c) \( \frac{24}{7} \)

E. (a) \( \frac{1}{\sqrt{10}} \) (b) \( \frac{3}{\sqrt{10}} \) (c) \( \frac{1}{3} \)

F. (a) \( \frac{2x\sqrt{25-x^2}}{25} \) (b) \( \frac{25-2x^2}{25} \)

G. (a) \( \frac{(x-1)\sqrt{3+2x-x^2}}{2} \) (b) \( \frac{1+2x-x^2}{2} \)
Evaluate the expressions.

A. $\sin \theta = \frac{3}{4}, \quad \frac{\pi}{2} < \theta < \pi$.

First find $\cos \theta$. The quadrant determines the sign.

$\frac{\pi}{2} < \theta < \pi \Rightarrow \text{quadrant II} \Rightarrow \cos \theta < 0$

$\cos \theta = -\sqrt{1 - \sin^2 \theta} = -\sqrt{1 - \left(\frac{3}{4}\right)^2} = -\sqrt{\frac{7}{16}}$  

(a) $\sin 2\theta 
\sin 2\theta = 2 \sin \theta \cos \theta = 2\left(\frac{3}{4}\right)\left(-\sqrt{\frac{7}{16}}\right) = -\frac{3\sqrt{7}}{8}$

(b) $\cos 2\theta 
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(-\sqrt{\frac{7}{16}}\right)^2 - \left(\frac{3}{4}\right)^2 
= \frac{7}{16} - \frac{9}{16} = \frac{-2}{16} = -\frac{1}{8}$

$\frac{\pi}{2} < \theta < \pi \Rightarrow \frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2} \Rightarrow \frac{\theta}{2} \in \text{quadrant I} \Rightarrow \sin \frac{\theta}{2} > 0, \cos \frac{\theta}{2} > 0$

(c) $\sin \frac{\theta}{2} 
\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} = \pm \sqrt{\frac{1 - (-\sqrt{\frac{1}{8}})}{2}} = \pm \sqrt{\frac{4 + \sqrt{7}}{8}}$  

(d) $\cos \frac{\theta}{2} 
\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} = \sqrt{\frac{1 + \sqrt{\frac{7}{16}}}{2}} = \sqrt{\frac{4 - \sqrt{7}}{8}}$

Answers: (a) $-\frac{3\sqrt{7}}{8}$ (b) $-\frac{1}{8}$ (c) $\sqrt{\frac{4 + \sqrt{7}}{8}}$ (d) $\sqrt{\frac{4 - \sqrt{7}}{8}}$

B. $\cos \theta = \frac{1}{3}, \quad 180^\circ < \theta < 270^\circ$.

First find $\sin \theta$. The quadrant determines the sign.

$180^\circ < \theta < 270^\circ \Rightarrow \text{quadrant III} \Rightarrow \sin \theta < 0$

$\sin \theta = \pm \sqrt{1 - \cos^2 \theta} = -\sqrt{1 - (-\frac{1}{3})^2}$

$\sin \theta = -\sqrt{1 - 1/9} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$

(a) $\sin 2\theta 
\sin 2\theta = 2 \cos \theta \sin \theta = 2\left(\frac{1}{3}\right)\left(-\frac{2\sqrt{2}}{3}\right) = -\frac{4\sqrt{2}}{9}$

(b) $\cos 2\theta 
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = \left(-\frac{2\sqrt{2}}{3}\right)^2 - \left(-\frac{4\sqrt{2}}{9}\right)^2 
= \frac{8}{9} - \frac{32}{27} = \frac{8}{27}$

$\theta \in \text{quadrant III} \Rightarrow \theta/2 \in \text{quadrant II} \Rightarrow \sin \theta > 0$

$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{1 - (-\frac{1}{3})}{2}} = \sqrt{\frac{4}{3}} = \sqrt{\frac{2}{3}}$

Answers: (a) $\frac{4\sqrt{2}}{9}$ (b) $-\frac{7}{9}$ (c) $\frac{\sqrt{2}}{3}$

C. Evaluate using half-angle formulas.

(a) $\tan(\pi/12)$

$\tan \frac{\pi}{12} = \tan \frac{\pi/6}{2} = \sqrt{\frac{1 - \cos(\pi/6)}{1 + \cos(\pi/6)}} = \sqrt{\frac{1 - \sqrt{3}/2}{1 + \sqrt{3}/2}} = \sqrt{2 - \sqrt{3}}$

(b) $\cos(\pi/12)$

$\cos \frac{\pi}{12} = \sqrt{\frac{1 + \cos(\pi/6)}{2}} = \sqrt{\frac{1 + \sqrt{3}/2}{2}} = \sqrt{\frac{2 + \sqrt{3}}{2}}$

(b) $\tan(\pi/12)$

$\tan \frac{\pi}{12} = \tan \frac{\pi/6}{2} = \frac{\sin(\pi/6)}{1 + \cos(\pi/6)} = \frac{\frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{1 + \sqrt{3}/2} = \frac{1}{1 + \sqrt{3}/2}$

Answers: (a) $\frac{\sqrt{2 - \sqrt{3}}}{2}$ (b) $\frac{\sqrt{2 + \sqrt{3}}}{2}$ (c) $2 - \sqrt{3}$

Picture for 17 and 21.

\[ \begin{align*} 
\text{y} & \quad \beta \\
\text{5} & \quad 4 \\
\text{0} & 
\end{align*} \]

D. (a) $\sin 2\theta$  (b) $\cos 2\theta$  (c) $\tan 2\theta$

$y = \sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3$

$\sin \theta = y/4 = 3/5, \cos \theta = 4/5, \tan \theta = y/4 = 3/4$

(a) $\sin 2\theta = 2 \sin \theta \cos \theta = 2(\frac{3}{5})(\frac{4}{5}) = \frac{24}{25}$

(b) $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = (\frac{4}{5})^2 - (\frac{3}{5})^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

(c) $\tan 2\theta = \frac{\tan \theta}{1 - \tan^2 \theta} = \frac{2(\frac{3}{4})}{1 - (\frac{3}{4})^2} = \frac{3/2}{(16 - 9)/16} = \frac{8(3)}{7} = \frac{24}{7}$

Answers: (a) $\frac{24}{25}$  (b) $\frac{7}{25}$  (c) $\frac{24}{7}$

E. (a) $\sin(\theta/2)$  (b) $\cos(\theta/2)$  (c) $\tan(\theta/2)$

$\sin \theta/2 = \sqrt{\frac{1 - (4/5)^2}{2}} = \sqrt{\frac{1}{10}}$

$\cos \theta/2 = \sqrt{\frac{1 + (4/5)^2}{2}} = \sqrt{\frac{9}{10}} = \frac{3}{\sqrt{10}}$

$\tan \theta/2 = \frac{\sin \theta/2}{\cos \theta/2} = \frac{3/5}{1 + 4/5} = \frac{3/5}{9/5} = \frac{1}{3}$

Answers: (a) $\frac{1}{\sqrt{10}}$  (b) $\frac{3}{\sqrt{10}}$  (c) $\frac{1}{3}$

F. $x = 5 \sin \theta, \quad 0 < \theta < \frac{\pi}{2}$

$\theta \in \text{quadrant I} \Rightarrow \cos \theta > 0$

$\sin \theta = x/5$

$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - (x/5)^2} = \sqrt{\frac{25 - x^2}{5}}$

(a) $\sin 2\theta$

$\sin 2\theta = 2 \sin \theta \cos \theta = 2(\frac{x}{5})(\sqrt{\frac{25 - x^2}{5}}) = \frac{2x\sqrt{25 - x^2}}{25}$

(b) $\cos 2\theta$

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = (\sqrt{\frac{25 - x^2}{5}})^2 - (\frac{x}{5})^2 
= \frac{25 - x^2}{25} - \frac{x^2}{25} = \frac{25 - 2x^2}{25}$

Answers: (a) $\frac{2x\sqrt{25 - x^2}}{25}$  (b) $\frac{25 - 2x^2}{25}$
Math 140  Lecture 23

Product-to-sum rules

Recall

\[
\cos(x-y) = \cos x \cos y + \sin x \sin y \\
\cos(x+y) = \cos x \cos y - \sin x \sin y
\]

Adding these gives

\[
\cos(x-y) + \cos(x+y) = 2 \cos x \cos y
\]

Subtracting gives

\[
\cos(x-y) - \cos(x+y) = 2 \sin x \sin y
\]

Similarly

\[
\sin(x-y) = \sin x \cos y - \cos x \sin y \\
\sin(x+y) = \sin x \cos y + \cos x \sin y
\]

Adding these gives

\[
\sin(x-y) + \sin(x+y) = 2 \sin x \cos y
\]

Dividing by 2 gives

**Product-to-sum formula.**

\[
\frac{\sin x \sin y}{2} = \frac{1}{2}[\cos(x-y) - \cos(x+y)] \\
\frac{\cos x \cos y}{2} = \frac{1}{2}[\cos(x-y) + \cos(x+y)] \\
\frac{\sin x \sin y}{2} = \frac{1}{2}[\sin(x-y) + \sin(x+y)]
\]

Write as a sum or difference of trigonometric functions.

\[
\sin \frac{\pi}{3} \cos \frac{\pi}{6} = \frac{1}{2} [\sin \left( \frac{\pi}{3} - \frac{\pi}{6} \right) + \sin \left( \frac{\pi}{3} + \frac{\pi}{6} \right)]
\]

\[
= \frac{1}{2} (\sin \frac{\pi}{6} + \sin \frac{\pi}{2}) = \frac{\sqrt{3}}{2}
\]

\[
\cos x \cos 2x = \frac{1}{2} [\cos(x - 2x) + \cos(x + 2x)]
\]

\[
= \frac{1}{2} [\cos(-x) + \cos(3x)] = \frac{1}{2} [\cos(x) + \cos(3x)].
\]

Solving trigonometric equations

Recall, sin x = sin(\pi - x), \cos x = \cos(-x).

Note: x and \pi - x are called *supplementary* angles.

Find all solutions for each equation.

\[\sin \theta = \frac{1}{2}\]

The two simplest solutions: \(\theta = \pi/6\), and \(\theta = -\pi/6 = 5\pi/6\).

Since \(\sin(\theta)\) has period \(2\pi\), adding a multiple of \(2\pi\) to either of these also produces a solution.

The general solution consists of the two sets

\[\theta = \pi/6 + 2\pi n \quad \text{or} \quad \theta = 5\pi/6 + 2\pi n \quad \text{for } n \text{ an integer.}\]

\[\cos x = \frac{1}{2}\] Here we use \(x\) for the angle instead of \(\theta\).

The two simplest solutions are \(x = \pi/3\) and \(x = -\pi/3\).

The general solution is

\[x = \pi/3 + 2\pi n \quad \text{or} \quad x = -\pi/3 + 2\pi n \quad \text{for } n \text{ an integer.}\]

**Convention.** Assume \(n\) is an arbitrary, possibly negative, integer. We’ll omit the phrase “for \(n\) an integer”.

\[\tan x = \frac{1}{\sqrt{3}}\]

Since \(\tan \frac{\pi}{6} = \sin \left( \frac{\pi}{6} \right) / \cos \left( \frac{\pi}{6} \right) = \left( \frac{1}{2} \right) / \left( \frac{\sqrt{3}}{2} \right) = \frac{1}{\sqrt{3}}\)

\(\pi/6\) is one solution. Since \(\tan(x)\) has period \(\pi\), adding a multiple of \(\pi\) also gives a solution. The general solution is \(x = \pi/6 + \pi n\).

For \(\sin, \cos\), add \(2\pi n\) to the (usually two) simplest solutions.

For \(\tan\), add \(\pi n\) to the one simplest solution.

\[\sin 2x = 1\]

iff \(2x = \pi/2 + 2\pi n\), \quad Only one simple solution here.

iff \(x = \pi/4 + \pi n\).

Recall, \(\sin(x)\) and \(\cos(x)\) are always between -1 and 1.

\[\cos x = 3\]

iff never. \(\ldots\) no solution.

\[2 \cos^2 x + \cos x = 1\]

\[2 \cos^2 x + \cos x - 1 = 0\]

(\(2 \cos x - 1)(\cos x + 1) = 0\)

\(\cos x = \frac{1}{2}\) or \(\cos x = -1\)

\(x = \pi/3 + 2\pi n\) or \(x = -\pi/3 + 2\pi n\) or \(x = \pi + 2\pi n\).

\[\sin^2 x + \cos x + 1 = 0\]

(1 - \(\cos^2 x\)) + \(\cos x + 1 = 0\)

\(-\cos^2 x + \cos x + 2 = 0\)

\(\cos x - \cos x - 2 = 0\)

(\(\cos x + 1\))\((\cos x - 2\))

\(\cos x = -1\) or \(\cos x = 2\) \(\text{The second is impossible.}\)

\(x = \pi + 2\pi n\).

\[2 \tan^2 x - 3 \tan x \sec x - 2 \sec^2 x = 0\]

\[2 \sin^2 x - 3 \sin x - 2 = 0\]

(\(2 \sin x + 1\))\((\sin x - 2\)) = 0

\(\sin x = -\frac{1}{2}\) or \(\sin x = 2\) \(\text{The second is impossible.}\)

\(x = -\pi/6 + 2\pi n\) or \(x = 5\pi/6 + 2\pi n\).

Recall, A function is 1-1 iff no horizontal line crosses its graph more than once.

\(\sin, \cos \text{ and } \tan \text{ are not 1-1. But they are 1-1 on the} \)

\(\text{heavily marked intervals. These are the largest such} \)

\(\text{intervals containing first quadrant angles.}\)
Write as a sum or difference of trig functions and then simplify. \( \cos(3 + 2y) \sin(3 - 2y) \)

\[
\begin{align*}
\sin((3 - 2y) - (3 + 2y)) & \quad \text{\( \leftarrow \) note, move the sin to the front before applying the product formula} \\
\sin((3 - 2y) + (3 + 2y)) & \\
\sin(-4y) + \sin(6) & \\
\frac{1}{2} \sin(6) - \sin(4y) & \\
\end{align*}
\]

Find all solutions for \( \sin^2 \theta - 3 \cos^2 \theta = 0 \). Hint: divide by \( \cos^2 \theta \).

\[
\begin{align*}
\tan^2 \theta - 3 & = 0 \\
\tan^2 \theta & = 3 \\
\tan \theta & = \pm \sqrt{3} \\
\theta & = -\frac{\pi}{3} + \pi n, \\
\theta & = \frac{\pi}{3} + \pi n \\
\end{align*}
\]

Find all solutions for \( 2 \sin^2 x + \cos x - 1 = 0 \). Hint: write in terms of \( \cos \). Three sets of solutions.

\[
\begin{align*}
2(1 - \cos^2 x) + \cos x - 1 & = 0 \\
2 - 2 \cos^2 x + \cos x - 1 & = 0 \\
-2 \cos^2 x + \cos x + 1 & = 0 \\
2 \cos^2 x - \cos x - 1 & = 0 \\
(2 \cos x + 1)(\cos x - 1) & = 0 \\
\cos x & = -\frac{1}{2}, \; \cos x = 1 \\
x & = \frac{2\pi}{3} + 2\pi n, \\
x & = -\frac{2\pi}{3} + 2\pi n, \; \text{or} \; \frac{4\pi}{3} + 2\pi n, \\
x & = 2\pi n \\
\end{align*}
\]
9. Solve for $\theta$. $\sin^2 \theta + 2 \cos^2 \theta = 0$
Divide by everything by $\cos^2 \theta$.
Rewrite in terms of $\tan \theta$.

Solve for $\theta$. (Write “no solutions” if there are none.) ___/1

10. Solve for $x$. $\sin^2 x + \cos x = 1$. Should be three sets of solutions.
Rewrite in terms of just $\sin$ or just $\cos$, not both.
Since $\sin^2 x + \cos^2 x = 1$, you can solve for $\sin^2$ in terms of $\cos^2$ or $\cos^2$ in terms of $\sin^2$ getting
   $\sin^2 x = 1 - \cos^2 x$ and
   $\cos^2 x = 1 - \sin^2 x$.
Use one of these equations to rewrite the given equation above entirely in terms of $\sin$ or entirely in terms of $\cos$.
___/1

Now solve for $x$. ___/3
§7.3 548:41-46. Write each expression as a sum or difference of trigonometric functions. Don't calculate the answer.

1(1). \( \cos 18^\circ \sin 72^\circ \)

2(1). \( \sin \frac{3\pi}{8} \sin \frac{\pi}{8} \)

3(1). \( \cos 5x \cos 2x \)

§7.5 568:1-12. Find all solutions (write “no solution” if none). Use radian measure. In each answer, write \( n \) to represent an arbitrary integer, e.g., \( -\pi/3 + 2\pi n, \pi/6 + \pi n \).

One problem has no solutions, one has two sets of solutions, one has three sets, and two have one set.

4(2). \( \sin(\theta) + \frac{1}{\sqrt{2}} = 0 \)

5(2). \( \tan(\theta) + \frac{1}{\sqrt{3}} = 0 \)

6(3). \( 2 \sin^2 x - 3 \sin x + 1 = 0 \)

7(2). \( \sin^2 x - \sin x - 6 = 0 \)

8(2). \( \cos \theta + 2 \sec \theta = -3 \)
Write each expression as a sum or difference of trigonometric functions. Don’t calculate the answer.

A. \( \sin 20^\circ \cos 10^\circ \)

B. \( \cos \frac{\pi}{5} \cos \frac{4\pi}{5} \)

C. \( \sin \frac{2\pi}{7} \sin \frac{5\pi}{7} \)

D. \( \sin \frac{7\pi}{12} \cos \frac{\pi}{12} \)

E. \( 3x \sin 4x \)

F. \( 6\theta \cos 5\theta \)

Find all solutions (write “no solution” if none). Use radian measure. In each answer, write \( n \) to represent an arbitrary integer, e.g., \( -\pi/3 + 2\pi n \).

G. \( \sin \theta = \frac{\sqrt{3}}{2} \)

H. \( \sin \theta = -\frac{1}{2} \)

I. \( \cos \theta = -1 \)

J. \( \tan \theta = \sqrt{3} \)

K. \( \tan x = 0 \)

L. \( 2\cos^2 \theta + \cos \theta = 0 \)

M. \( \cos^2 t \sin t - \sin t = 0 \)

N. \( 2\cos^2 x - \sin x - 1 = 0 \)

Answers

A. \( \frac{1}{2} \sin 10^\circ + \frac{1}{4} \)

B. \( \frac{1}{2} \cos \frac{3\pi}{5} - \frac{1}{2} \)

C. \( \frac{1}{2} \cos \frac{3\pi}{7} + \frac{1}{2} \)

D. \( \frac{1}{2} \left[ \sin \frac{\pi}{5} - \sin \frac{2\pi}{3} \right] \)

E. \( \frac{1}{2} \cos x - \frac{1}{2} \cos 7x \)

F. \( \frac{1}{2} \sin \theta + \frac{1}{2} \sin 11\theta \)

G. \( \theta = \frac{\pi}{3} + 2\pi n \) or \( \theta = \frac{2\pi}{3} + 2\pi n \)

H. \( \theta = -\frac{\pi}{6} + 2\pi n \) or \( \theta = -\frac{5\pi}{6} + 2\pi n \)

I. \( \theta = \pi + 2\pi n \)

J. \( \theta = \frac{\pi}{3} + \pi n \)

K. \( x = \pi n \)

L. \( \theta = \pm \frac{\pi}{2} + 2\pi n, \theta = \frac{2\pi}{3} + 2\pi n, \) or \( \theta = -\frac{2\pi}{3} + 2\pi n \)

M. \( t = \pi n \)

N. \( x = \frac{\pi}{6} + 2\pi n, x = \frac{5\pi}{6} + 2\pi n, \) or \( x = \frac{3\pi}{2} + 2\pi n \)
Math 140  Hw 23  Worked examples of selected recommended problems.

Write each expression as a sum or difference of trigonometric functions. Don’t calculate the answer.

A.  \( \sin 20^\circ \cos 10^\circ \)
\[ = \frac{1}{2} [\sin(20^\circ - 10^\circ) + \sin(20^\circ + 10^\circ)] \]
\[ = \frac{1}{2} [\sin(10^\circ) + \sin(30^\circ)] \]
\[ = \frac{1}{2} [\sin(10^\circ) + \frac{1}{2}] \]
\[ \text{Answer: } \frac{1}{2} \sin 10^\circ + \frac{1}{4} \]

B.  \( \cos \frac{\pi}{3} \cos \frac{4\pi}{5} \)
\[ = \frac{1}{2} [\cos(\frac{\pi}{3} - \frac{4\pi}{5}) + \cos(\frac{\pi}{3} + \frac{4\pi}{5})] \]
\[ = \frac{1}{2} [\cos(-\frac{\pi}{5}) + \cos(\pi)] \]
\[ = \frac{1}{2} [\cos(\frac{3\pi}{5}) - 1] \]
\[ \text{Answer: } \frac{1}{2} \cos \frac{3\pi}{5} - \frac{1}{2} \]

C.  \( \sin \frac{2\pi}{7} \sin \frac{5\pi}{7} \)
\[ = \frac{1}{2} [\cos(\frac{2\pi}{7} - \frac{5\pi}{7}) - \cos(\frac{2\pi}{7} + \frac{5\pi}{7})] \]
\[ = \frac{1}{2} [\cos(-\frac{3\pi}{7}) - \cos(\pi)] \]
\[ = \frac{1}{2} [\cos(\frac{\pi}{7}) - 1] \]
\[ \text{Answer: } \frac{1}{2} \cos \frac{\pi}{7} + \frac{1}{2} \]

Find all solutions (write “no solution” if none). Use radian measure. In each answer, write \( n \) to represent an arbitrary integer, e.g., \(-\pi/3+2\pi n\).

G.  \( \sin \theta = \sqrt{3}/2 \)
Two simplest: \( \theta = \frac{\pi}{3}, \frac{2\pi}{3} \)
\[ \text{Answer: } \theta = \frac{\pi}{3} + 2\pi n \text{ or } \theta = \frac{2\pi}{3} + 2\pi n \]

H.  \( \sin \theta = -1/2 \)
Two simplest: \( \theta = \frac{-\pi}{6}, \frac{-5\pi}{6} \)
\[ \text{Answer: } \theta = \frac{-\pi}{6} + 2\pi n, \frac{-5\pi}{6} + 2\pi n \text{ or, equivalently,} \]
\[ \text{Answer: } \theta = \frac{11\pi}{6} + 2\pi n \]

I.  \( \cos \theta = -1 \)
One simplest: \( \theta = \pi \)
\[ \text{Answer: } \theta = \pi + 2\pi n \]

J.  \( \tan \theta = \sqrt{3} \)
Simplest answer: \( \theta = \frac{\pi}{3} \)
\[ \text{Answer: } \theta = \frac{\pi}{3} + \pi n \]

L.  \( 2 \cos^2 \theta + \cos \theta = 0 \)
\[ 2 \cos^2 \theta + \cos \theta = 0 \]
\[ \cos \theta(2 \cos \theta + 1) = 0 \]
\[ \cos \theta = 0 \text{ when } \theta = \frac{\pi}{2}, -\frac{\pi}{2} \]
\[ \cos \theta = -\frac{1}{2} \text{ when } \theta = \frac{2\pi}{3}, -\frac{2\pi}{3} \]
\[ \text{Answer: } \theta = \pm \frac{\pi}{2} + 2\pi n, \theta = \pm \frac{2\pi}{3} + 2\pi n \]
or, equivalently,
\[ \text{Answer: } \theta = \frac{\pi}{2} + \pi n, \theta = \frac{2\pi}{3} + 2\pi n, \theta = \frac{4\pi}{3} + 2\pi n \]

M.  \( \cos^2 t \sin t - \sin t = 0 \)
\[ \sin t(\cos^2 t - 1) = 0 \]
\[ \sin t(-\sin^2 t) = 0 \]
\[ \sin^2 t = 0 \]
\[ \sin t = 0 \]
Simplest answers: \( t = 0, \pi \)
\[ \text{Answer: } t = 2\pi n, \pi + 2\pi n \]

N.  \( 2 \cos^2 x - \sin x - 1 = 0 \)
\[ 2(1 - \sin^2 x) - \sin x - 1 = 0 \]
\[ 2 - 2 \sin^2 x - \sin x - 1 = 0 \]
\[ -2 \sin^2 x - \sin x + 1 = 0 \]
\[ 2 \sin^2 x + \sin x - 1 = 0 \]
\[ (2 \sin x - 1)(\sin x + 1) = 0 \]
\[ \sin x = 1/2, \sin x = -1 \]
\[ \sin x = \frac{1}{2} \text{ when } x = \frac{\pi}{6}, \frac{5\pi}{6} \]
\[ \sin x = -1 \text{ when } x = \frac{-\pi}{2} \]
\[ \text{Answer: } x = \frac{\pi}{6} + 2\pi n, x = \frac{5\pi}{6} + 2\pi n, \text{ or } x = \frac{-\pi}{2} + 2\pi n \]
or, equivalently
\[ \text{Answer: } x = \frac{\pi}{6} + 2\pi n, x = \frac{5\pi}{6} + 2\pi n, \text{ or } x = \frac{3\pi}{2} + 2\pi n \]
Inverse trigonometric functions

RECALL. A function is 1-1 iff no horizontal line crosses its graph more than once.
While not 1-1 in general, sin, cos, and tan are 1-1 on the first quadrant angles in \([0, \pi/2]\) and on the larger heavily marked “1-1” intervals.

\[ \sin^{-1}, \cos^{-1}, \tan^{-1} \] are the inverses of the above restrictions of sin, cos, and tan. Thus
\[ \sin^{-1}(x) = \theta \in [-\pi/2, \pi/2] \text{ such that } \sin(\theta) = x, \]
\[ \cos^{-1}(x) = \theta \in [0, \pi] \text{ such that } \cos(\theta) = x, \]
\[ \tan^{-1}(x) = \theta \in (-\pi/2, \pi/2) \text{ such that } \tan(\theta) = x. \]

\(\sin^{-1}(x)\) and \(\cos^{-1}(x)\) have domain \([-1, 1]\), for \(x \in [-1, 1]\), they are undefined. \(\tan^{-1}(x)\) has domain \((-\infty, \infty)\).

NOTATION. \(\sin^{-1}(x), \cos^{-1}(x)\) and \(\tan^{-1}(x)\) are also written: \(\arcsin(x), \arccos(x), \text{and } \arctan(x)\).

Warning, \(\sin^{-1}(x) \neq 1/\sin(x)\). \(\sin^{-1}(x)\) is the inverse; \((\sin(x))^{-1} = 1/\sin(x)\) is the reciprocal.

<table>
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<tr>
<th>[x]</th>
<th>[\sin^{-1}(x)]</th>
<th>[\cos^{-1}(x)]</th>
<th>[\tan^{-1}(x)]</th>
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<td>1/2</td>
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<td>\cos^{-1}(1/2) = \pi/4</td>
<td>\tan^{-1}(1/2) = \pi/4</td>
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<td>\cos^{-1}(1/\sqrt{2}) = \pi/3</td>
<td>\tan^{-1}(1/\sqrt{2}) = \pi/3</td>
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<td>\cos^{-1}(1) = \pi/2</td>
<td>\tan^{-1}(1) = \pi/2</td>
</tr>
</tbody>
</table>

The unit circle pictures are

The heavily marked half circles are the restricted ranges.

Theorem. For \(x \in [-1, 1]\),
\[ \cos(\sin^{-1}(x)) = \sqrt{1-x^2} \]
\[ \sin(\cos^{-1}(x)) = \sqrt{1-x^2} \]

Proof of 1st. Recall: \(\cos^2 \theta = 1 - \sin^2 \theta. \)
\[ \cos \theta = \pm \sqrt{1 - \sin^2 \theta} \]
\[ \cos(\sin^{-1}(x)) = \sqrt{1-x^2} \]
\[ \sin(\cos^{-1}(x)) = \sqrt{1-x^2} \]

The “+” was chosen over the “−” since \(\sin^{-1}(x) \in [-\pi/2, \pi/2] \Rightarrow \cos(\sin^{-1}(x)) \geq 0. \)

\[ \sin^{-1}(1/2) = \frac{\pi}{6} \]
\[ \cos^{-1}(1/2) = \arcsin(1/2) = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \]
\[ \tan^{-1}(1/2) = \frac{\sin(1/2)}{\cos(1/2)} = \frac{\sqrt{\frac{3}{4}}}{\frac{1}{2}} = \frac{\sqrt{3}}{1/2} = \sqrt{3} \]
\[ \sec^{-1}[-1 - \cos^{-1}(1)] = \sec(\frac{\pi}{2} - 0) = 1/\cos(\frac{\pi}{2}) = 1/0 = \text{undef}. \]

In a triangle with hypotenuse 1 and side \(x, \)
\( \sin^{-1}x = \text{angle opposite } x, \)
\( \cos^{-1}x = \text{angle adjacent to } x. \)

Note that \(\sin^{-1}x + \cos^{-1}x = \pi/2 \quad (= 90^\circ)\)
11. Find the exact value (no credit for a decimal).

(a) $\cos^{-1}(-1/\sqrt{2})$
\[
= \frac{3\pi}{4}
\]

(b) $\sin^{-1}(-1/2)$
\[
= -\frac{\pi}{6}
\]

(c) $\tan^{-1}(-\sqrt{3})$
\[
= -\frac{\pi}{3}
\]

(d) $\sin^{-1}(\sin(-\frac{4\pi}{7}))$
\[
= -\frac{\pi}{2}
\]

(e) $\cos^{-1}(\cos(-\frac{4\pi}{7}))$
\[
= \frac{4\pi}{7}
\]

(f) $\tan(\arcsin(\frac{4}{7}))$
\[
= \frac{\sin^{-1}(\frac{4}{7})}{\cos(\sin^{-1}(\frac{4}{7}))}
\]
\[
= \frac{\frac{4}{7}}{\sqrt{1-\frac{16}{49}}}
\]
\[
= \frac{\frac{4}{7}}{\sqrt{\frac{33}{49}}} = \frac{4}{\sqrt{33}}
\]
11. Find the exact value (no credit for a decimal).

(a) $\cos^{-1}(-1/2)$. The answer is an angle in the second quadrant. 4 symbols

(b) $\sin^{-1}(\sin(-5\pi/9))$

Warning $-5\pi/9$ is not correct. $-5\pi/9$ is not in the restricted region $[-\pi/2, \pi/2]$ for the inverse of $\sin$. Rewrite in terms of an angle in the restricted region for $\sin^{-1}$ before canceling the inverses. 5 symbols.

$\sin^{-1}(\sin(-5\pi/9)) = \sin^{-1}(\sin(\quad)) =$

(c) $\cos^{-1}(\cos(-5\pi/9))$

Warning $-5\pi/9$ is not correct. $-5\pi/9$ is not in the restricted region $[0, \pi]$ for the inverse of $\cos$. Rewrite in terms of an angle in the restricted region for $\cos^{-1}$ before canceling the inverses. 4 symbols.

$\cos^{-1}(\cos(-5\pi/9)) = \cos^{-1}(\cos(\quad)) =$

(f) $\tan(\arcsin(2/3))$.

Write $\tan$ as $\frac{\sin}{\cos}$. Then use the theorem: $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$. 4 or 5 symbols.
§7.4 557:1-8. Evaluate exactly without a calculator: π and $\sqrt{2}$ rather than 3.14 or 1.41. Answer may be “undef.”

1(1). \( \cos^{-1}(-1) \)  
One symbol

2(1). \( \arccos(-\sqrt{2}/2) \)  
4 symbol fraction with \( \pi \)

3(1). \( \arcsin(-1) \)  
4 symbol fraction with \( \pi \)


4(1). \( \sin(\arcsin 2) \)  
Careful

5(1). \( \sin^{-1}(\sin \frac{3\pi}{2}) \)  
4 symbol fraction with \( \pi \)

6(2). \( \tan[\sin^{-1}(\frac{4}{5})] \)  
3 symbol fraction

7(1). \( \sin[\tan^{-1}(1)] \)  
4 symbol fraction with radical

8(2). \( \tan(\arccos \frac{5}{13}) \)  
4 symbol fraction

9(1). \( \cos(\arctan \sqrt{3}) \)  
3 symbol fraction

10(1). \( \sin[\arccos(-\frac{1}{2})] \)  
5 symbol fraction with radical

11(2). \( \sec[\cos^{-1}(1/\sqrt{2}) + \sin^{-1}(-1)] \)  
2 symbol radical
Evaluate exactly without a calculator: \( \pi \) and \( \sqrt{2} \) rather than 3.14 or 1.41. Answer may be “undef.”

A. \( \sin^{-1}(\sqrt{3}/2) \)

B. \( \tan^{-1}\sqrt{3} \)

C. \( \arctan(-1/\sqrt{3}) \)

D. \( \tan^{-1}(1) \)

E. \( \cos^{-1}(2\pi) \)

F. \( \sin[\sin^{-1}(1/4)] \)

G. \( \cos[\cos^{-1}(3/4)] \)

H. \( \arctan[\tan(-\pi/7)] \)

I. \( \arcsin[\sin(\pi/2)] \)

J. \( \arccos[\cos(2\pi)] \)

K. \( \cos(\arcsin \frac{2}{7}) \)

L. \( \sin[\tan^{-1}(-1)] \)

M. \( \cos[\sin^{-1}(\frac{2}{3})] \)

N. \( \sin[\cos^{-1}(\frac{1}{3})] \)

O. \( \tan(\arcsin \frac{20}{21}) \)

P. \( \csc[\sin^{-1}(\frac{1}{2}) - \cos^{-1}(\frac{1}{2})] \)

Answers

A. \( \pi/3 \)    B. \( \pi/3 \)    C. \(-\pi/6 \)    D. \( \pi/4 \)    E. undefined
F. \( 1/4 \)    G. \( 3/4 \)    H. \(-\pi/7 \)    I. \( \pi/2 \)    J. 0
K. \( 3\sqrt{5}/7 \)    L. \(-1/\sqrt{2} \)    M. \( \sqrt{5}/3 \)    N. \( 2\sqrt{2}/3 \)
O. 20/21    P. -2
Evaluate exactly without a calculator: $\pi$ and $\sqrt{2}$ rather than 3.14 or 1.41. Answer may by “undef.”

A. $\sin^{-1}(\sqrt{3}/2)$

$\text{Answer: } \pi/3$

B. $\tan^{-1}(\sqrt{3})$

$\text{Answer: } \pi/3$

C. $\arctan(-1/\sqrt{3})$

$\text{Answer: } -\pi/6$

D. $\tan^{-1}(1)$

$\text{Answer: } \pi/4$

E. $\cos^{-1}(2\pi)$

cos$^{-1}x$ is only defined on $[-1, 1]$.
$2\pi \approx 6.28 > 1.$

$\text{Answer: } \text{undef}$

F. $\sin[\sin^{-1}(1/4)]$

$\text{Answer: } 1/4$

G. $\cos[\cos^{-1}(3/4)]$

$\text{Answer: } 3/4$

H. $\arctan[\tan(-\pi/7)]$

$\text{Answer: } -\pi/7$

I. $\arcsin[\sin(\pi/2)]$

$\text{Answer: } \pi/2$

J. $\arccos[\cos(2\pi)]$

$\arccos[\cos 2\pi] = 2\pi$ since $2\pi$ is not in the restricted range $[0,\pi]$ for $\cos$. But $\cos 2\pi = \cos 0$ and $0$ is in $[0,\pi]$.

$\arccos[\cos 2\pi] = \arccos[\cos 0] = 0.$

$\text{Answer: } 0$

K. $\cos(\arcsin(\frac{2}{7}))$

5 symbol fraction with square root

$= \sqrt{1 - (\frac{2}{7})^2}$

$= \sqrt{1 - \frac{4}{49}} = \frac{3\sqrt{5}}{7}$

L. $\sin[\tan^{-1}(-1)]$

5 symbol fraction with square root

$= \sin(-\pi/4)$

$= -1/\sqrt{2}$

or $-\sqrt{2}/2$

M. $\cos[\sin^{-1}(\frac{2}{3})]$

4 symbol fraction with square root

$= \sqrt{1 - (\frac{2}{3})^2}$

$= \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

N. $\sin[\cos^{-1}(\frac{1}{3})]$

5 symbol fraction with square root

$\sqrt{1 - (\frac{1}{3})^2}$

$= \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$

O. $\tan(\arcsin(\frac{20}{29}))$

5 symbol rational

$\frac{\sin(\sin^{-1}(20/29))}{\cos(\sin^{-1}(20/29))} = \frac{20/29}{\sqrt{1 - 400/29^2}} = \frac{20}{21}$

P. $\csc[\sin^{-1}(\frac{1}{2}) - \cos^{-1}(\frac{1}{2})]$

Integer

$csc(\frac{\pi}{6} - \frac{\pi}{3}) = 1/\sin(-\frac{\pi}{6}) = -2$
Math 140 Lecture 25

Trigonometric word problems

(a) Find the exact answer. (b) Find the decimal answer.

To test if your calculator is in radian or degree mode, calculate \( \sin(1) \).
\( \sin(1^\circ) = .017, \sin(1 \text{ rad}) = .84 \). Give only exact answers on tests.

The bookstore has cheap trig calculators for less than $20.

Point P is level with the base of a 1000 ft high building which is 4000 ft away. Find the angle of elevation from P to the top of the building. (a) the exact radian answer, (b) to nearest degree.

\[ \begin{align*}
\tan(\theta) &= \frac{1000}{4000} = 1/4. \\
(\text{a}) \theta &= \tan^{-1}(1/4) \quad \leftarrow \text{exact answer}. \\
(\text{b}) \theta &= 14^\circ \quad \leftarrow \text{to nearest degree}.
\end{align*} \]

An antenna sits atop a 1000 ft high building. From point P on the ground, the angle of elevation to the top of the building is \( \beta \), the angle of elevation to the top of the antenna is \( \alpha \). Express the height of the antenna in terms of \( \alpha \) and \( \beta \).

\[ \begin{align*}
\tan(\beta) &= \frac{1000}{x} \\
x &= \frac{1000}{\tan(\beta)} \\
\tan(\alpha) &= \frac{x + 1000}{x} \\
x \tan(\alpha) &= x + 1000 \\
h &= x \tan(\alpha) - 1000 \\
h &= \frac{1000}{\tan(\beta)} \tan(\alpha) - 1000
\end{align*} \]

Answer: Height is 1000(\frac{\tan(\alpha)}{\tan(\beta)} - 1) ft. Remember the units.

From a point P level with the base of a mountain, the angle of elevation of the mountain is \( \alpha \). From a point Q 1 mile closer to the mountain’s base, the angle of elevation is \( \beta \). Express the height of the mountain in terms of \( \alpha \) and \( \beta \).

\[ \begin{align*}
\tan(\beta) &= \frac{h}{x} \\
x &= \frac{h}{\tan(\beta)}
\end{align*} \]

\( \tan(\alpha) = \frac{h+1000}{x} \)

\[ \begin{align*}
x \tan(\alpha) &= h + 1000 \\
h &= x \tan(\alpha) - 1000
\end{align*} \]

Answer: Height = \( \frac{1000}{\tan(\beta)} \tan(\alpha) - 1000 \) ft. Remember the units.

From a point P level with the base of a mountain, the angle of elevation of the mountain is \( \alpha \). From a point Q 1 mile closer to the mountain’s base, the angle of elevation is \( \beta \). Express the height of the mountain in terms of \( \alpha \) and \( \beta \).

\[ \begin{align*}
\tan(\beta) &= \frac{h}{x} \\
x &= \frac{h}{\tan(\beta)}
\end{align*} \]

\( \tan(\alpha) = \frac{h + 1000}{x} \)

\[ \begin{align*}
x \tan(\alpha) &= h + 1000 \\
h &= x \tan(\alpha) - 1000
\end{align*} \]

Answer: Height = \( \frac{1000}{\tan(\beta)} \tan(\alpha) - 1000 \) ft. Remember the units.

Find the area of an octagon (stop sign) of radius 1 ft.
Give the exact answer and the decimal answer to 2 places.

\[ \begin{align*}
\text{Area} &= \frac{1}{2} \times 2 \times \frac{1}{2} \times (1 \sin(\frac{\pi}{4})) \times 2 \\
&= \frac{1}{2} \times \frac{\sqrt{2}}{2} \times 2 \\
&= \sqrt{2}
\end{align*} \]

Exact answer = 2\( \sqrt{2} \) sq. ft. Decimal answer = 2.83 sq. ft.

\[ \begin{align*}
\\tan(\beta) &= \frac{\text{height}}{x} \\
\text{height} &= \frac{x}{\tan(\beta)}
\end{align*} \]

\( \tan(\alpha) = \frac{h + 1000}{x} \)

\[ \begin{align*}
x \tan(\alpha) &= h + 1000 \\
h &= x \tan(\alpha) - 1000
\end{align*} \]

Answer: Height = \( \frac{1000}{\tan(\beta)} \tan(\alpha) - 1000 \) ft. Remember the units.

The area of the octagon = 8\times the area of each triangle \( = 8 \times \frac{\sqrt{2}}{4} = 2 \sqrt{2} \).

THEOREM. The area of a triangle with sides \( a \) and \( b \) and included angle \( \theta \) is \( \frac{1}{2} ab \sin \theta \).

PROOF. Case \( \theta \) is acute.

\[ \begin{align*}
\text{Area} &= \frac{1}{2} \times x \times h \\
&= \frac{1}{2} \times \left( \frac{a}{\tan(\beta)} \right) \times \frac{b}{\tan(\alpha)}
\end{align*} \]
2(6). A boat is sighted from two lighthouses which are one mile apart along a straight east-west shoreline. From the first lighthouse, the angle between the boat and due east is 35°. From the second lighthouse, the angle between the boat and due east is 50°. How far is the boat from the shore? Exact answer plus units.

Let \(d\) be the distance between the boat and the nearest point on the shoreline.
Let \(x\) be the distance between the closest lighthouse and the nearest point on the shoreline.

\[
\tan 50^\circ = \frac{d}{x}
\]
\[
\tan 35^\circ = \frac{d}{x+1}
\]
\[
(x + 1) \tan 35^\circ = d
\]
\[
\left(\frac{d}{\tan 50^\circ} + 1\right) \tan 35^\circ = d
\]
\[
d \tan 35^\circ + \tan 50^\circ \tan 35^\circ = d \tan 50^\circ
\]
\[
d \tan 35^\circ - d \tan 50^\circ = -\tan 50^\circ \tan 35^\circ
\]
\[
d = \frac{-\tan 50^\circ \tan 35^\circ}{\tan 35^\circ - \tan 50^\circ}
\]
\[
d = \frac{\tan 50^\circ \tan 35^\circ}{\tan 50^\circ - \tan 35^\circ} \text{ miles.}
\]

6(6). Find the area of a pentagon whose sides each have length 2. Give the exact answer.

5 isosceles triangles.
Inner angle: \(2\pi/5\)
Opposite side: 2
Divide the triangle in half with height \(h\).

\[
\tan \frac{\pi}{5} = \frac{1}{h}
\]
\[
h = 1/\tan \frac{\pi}{5}
\]
Area of triangle: \(\frac{1}{2} \cdot 2/ \tan \frac{\pi}{5}\)
Area of pentagon: \(\frac{5}{\tan \frac{\pi}{5}}\)
6. Find the area of a regular (all sides are equal) 7 sided figure inscribed in a circle of radius 1 inch. Give the exact answer, no decimals. Do not solve for any triangle's height or width. Don't use the area formula \( \frac{1}{2}bh \). Instead use \( \frac{1}{2}ab \sin \theta \) of Lecture 25.

The 7-sided figure divides into 7 triangles each with two sides of length 1. Draw the picture.

What is the an included angle of each triangle? ___/1

What is the area of each triangle? Recall \( A = \frac{1}{2}ab \sin \theta \). ___/1

What is the total area? Remember the units. ___/1

2. A boat is sighted from of two lighthouses which are one mile apart along a straight east-west shoreline. From the first lighthouse, the angle between the boat and due east is 55°. From the second lighthouse, the angle between the boat and due east is 70°. How far is the boat from the shore? Exact answer plus units. This is a variant of the problem in Lecture 25.

Let \( d \) be the distance between the boat and the nearest point on the shoreline.
Let \( x \) be the distance between the closest lighthouse and the nearest point on the shore line.

There are two right triangles in the picture. Apply tangent to the two given angles to get two equations involving \( x \) and \( d \). ___/1

Solve these two equations for \( d \). ___/4
§6.2 484:44. §7.4 558:53-55.

2sinα has 5 symbols, 2sin(α) has 7 symbols.

1(3). From a point level with and 1000 ft away from the base of the Washington Monument, the angle of elevation to the top of the monument is 29°. Find the height of the monument. (a) Draw the picture. (b) Give the exact answer using tan (remember the units). (c) Find the height to the nearest foot. (a) Picture.

(b) 10-12 symbols + units Exact height =

(c) 3 symbols + units Nearest ft =

2(3). In triangle \triangle ABC, the sides are AC = BC = 8 inches and AB=4. Find the angles. (a) Give the exact answer. (b) Find the angle to the nearest degree. Hint. Let P be the midpoint of AB. Then APC is a right triangle.

\[
\angle A = \angle B = \quad =
\]

exact radian answer nearest degree answer

\[
\angle C = \quad =
\]

exact radian answer nearest degree answer

3(2). Find the distance AB if AC = 400m, \angle C = 90°, \angle A = 40°. Find AB. (a) Give the exact answer. (b) Find the 3-digit distance to the closest meter.

\[
\]

(a) 10-12 symbols + units Exact distance =

(b) 3 symbols + units Nearest meter (m) =

4(4). A surveyor stands 30 yards from the base of a building. On top of the building is a vertical radio antenna. Let α be the angle of elevation when the surveyor sights to the top of the building. Let β be the angle of elevation when the surveyor sights to the top of the antenna.

(a) Draw the picture.

(b) Give the length of the antenna in terms of the angles α and β. (a) Picture.

(b) 13-17 symbols + units Antenna height =

5(3). The radius of the circle in the following figure is 1 unit. Express the lengths of OA, AB, and DC in terms of trigonometric functions of α. (No units are given in this case.)

\[
\]

4-6 symbols \quad OA =

4-6 symbols \quad AB =

4-6 symbols \quad DC =
Math 140  Hw  25  Recommended problems, don’t turn this in.

2sinα has 5 symbols, 2sin(α) has 7 symbols.

A. When earth E, Mercury M, and the sun S are lined up so that ∠EMS is a right angle, ∠SEM is 21°. Given that the distance ES from the earth to the sun is 93 million miles, find the distance MS from Mercury to the sun to the nearest million miles.

B. A contractor must fence a plot with the shape of a right triangle. One angle is 40°. The hypotenuse is 60 m. Find the length of fencing required. Round to the nearest meter.

C. Given: θ = 39.4° and x = 43.0 ft.

\[ x \quad x \]
\[ \theta \quad \theta \]

(a) Find the height h to one decimal.

(b) Find, to one decimal, the total area of the triangle (the large isosceles triangle, not the two right-triangle pieces).

D. From a point at ground level, the angle of elevation to the top of a mountain is 38°. 200 m further away from the mountain, the angle is 20°. Find the mountain’s height.

E. The arc in the picture is part of the unit circle.

\[ \theta \]
\[ A \quad P \]

(a) Express in terms of \( \theta \): \( \angle BOA, \angle OAB, \angle BAP, \angle BPA \).

(b) Express in terms of \( \sin \theta \), and \( \cos \theta \): AO, AP, OB, BP.

Answers

A. 33 million miles  
B. 145 m

C. (a) 27.3 ft. (b) 906.9 ft²  
D. 136 m.

E. (a) \( \angle BOA = 90° - \theta \), \( \angle OAB = \theta \), \( \angle BAP = 90° - \theta \), \( \angle BPA = \theta \).  
(b) \( AO = \sin \theta \), \( AP = \cos \theta \), \( OB = \sin^2 \theta \), \( BP = \cos^2 \theta \).
2sin$\alpha$ has 5 symbols, 2sin($\alpha$) has 7 symbols.

A. When earth E, Mercury M, and the sun S are lined up so that $\angle$EMS is a right angle, $\angle$SEM is 21°. Given that the distance ES from the earth to the sun is 93 million miles, find the distance MS from Mercury to the sun to the nearest million miles.

\[ \sin 21^\circ = \frac{x}{93} \]
\[ x = 93 \sin 21^\circ = 93(0.358) = 33 \]

Answer: 33 million miles

B. A contractor must fence a plot with the shape of a right triangle. One angle is 40°. The hypotenuse is 60 m. Find the length of fencing required. Round to the nearest meter.

\[ \sin 40^\circ = \frac{x}{60}, \quad x = 60 \sin 40^\circ = 60(0.643) = 38.567 \]
\[ \cos 40^\circ = \frac{y}{60}, \quad y = 60 \cos 40^\circ = 60(0.766) = 45.963 \]
\[ \text{Fence} = 60 + 38.567 + 45.963 = 144.530 \]

Answer: 145 m

C. Given: $\theta = 39.4^\circ$ and $x = 43.0$ ft.

\[ \sin \theta = \frac{h}{x}, \quad h = \frac{x}{\sin(39.4^\circ)} = 27.293 \]
\[ \cos \theta = \frac{b}{x}, \quad b = x \cos(39.4^\circ) = 33.228 \]

(a) Find the height $h$ to one decimal.
Answer: 27.3 ft.

(b) Find, to one decimal, the total area of the triangle.
Each half has area $\frac{1}{2}hb = \frac{1}{2}(27.293)(33.228) = 453.446$
Answer: 906.9 ft²

D. From a point at ground level, the angle of elevation to the top of a mountain is 38°. 200 m further away from the mountain, the angle is 20°. Find the mountain’s height.

\[ \tan 38^\circ = \frac{h}{x}, \quad x = \frac{h}{\tan 38^\circ} \]
\[ \tan 20^\circ = \frac{h}{200+x} = \frac{h}{200+h/\tan 38^\circ} \]
\[ 200 + \frac{h}{\tan 38^\circ} = \frac{h}{\tan 20^\circ} \]
\[ 200 \frac{h}{\tan 38^\circ} - \frac{h}{\tan 20^\circ} = 200 \]
\[ h = 200/\left[1/\tan 20^\circ - 1/\tan 38\right] \]
\[ h = 200/[1.468] = 136.2 \]

Answer: 136 m.

E. The arc in the picture is part of the unit circle.

(a) Express in terms of $\theta$: $\angle$BOA, $\angle$OAB, $\angle$BAP, $\angle$BPA.

$\angle$BOA = 90° - $\theta$
$\angle$OAB = 90° - $\angle$BOA = 90° - (90° - $\theta$) = $\theta$
$\angle$BAP = 90° - $\angle$OAB = 90° - $\theta$
$\angle$BPA = 90° - $\angle$BAP = 90° - (90° - $\theta$) = $\theta$

(b) Express in terms of sin $\theta$, and cos $\theta$: AO, AP, OB, BP.

AO = height of P = sin $\theta$
AP = x-coordinate of P = cos $\theta$
OB = ?
sin($\angle$OAB) = OB/OA
OB = OA·sin($\angle$OAB) = sin$\theta$·sin$\theta$ = sin$^2$\theta
BP = 1 - OB = 1 - sin$^2$\theta = cos$^2$\theta
Math 140 Lecture 26

Conventions. Assume side \(a\) is opposite angle \(A\), side \(b\) is opposite angle \(B\) and side \(c\) is opposite angle \(C\).

Sine laws. In any triangle, the ratio of one angle’s sine and its opposite side equals the ratio of any other angle’s sine and opposite side.

Although written as one, there are 3 equations. Each involves two sides and two angles.

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \text{ or }
\]

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
\]

Proof. Recall that the area of a triangle is half the product of any two sides times the sine of their included angle. Thus the area of the triangle can be written three ways:

\[
\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C
\]

Multiply by 2:

\[
bc \sin A = ac \sin B = ab \sin C
\]

Divide by \(abc\):

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

Cosine laws. For any two sides of a triangle, the sum of their squares minus twice their product times the cosine of the included angle equals the square of the third side.

\[
a^2 = b^2 + c^2 - 2bc \cos A
\]

\[
b^2 = a^2 + c^2 - 2ac \cos B
\]

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

Proof. We prove the last equality for the case \(C\) acute.

\[
\sin C = \frac{h}{a}, \cos C = \frac{x}{a}, \text{ so } h = a \sin C, x = a \cos C
\]

\[
c^2 = (b-x)^2 + h^2
\]

\[
c^2 = b^2 - 2bx + x^2 + h^2
\]

\[
c^2 = b^2 - 2b(a \cos C) + a^2 \cos^2 C + a^2 \sin^2 C
\]

\[
c^2 = b^2 - 2ab \cos C + a^2 (\cos^2 C + \sin^2 C)
\]

\[
c^2 = a^2 + b^2 - 2ab \cos C
\]

Straight angle sum fact. The sum of a triangle’s 3 angles is a straight angle.

\[
\angle A + \angle B + \angle C = 180^\circ = \pi
\]

If you know two angles, you can solve for the third. Solving for \(C\) gives \(\angle C = 180^\circ - (\angle A + \angle B)\).

Today’s problems involve 4 quantities; each is a side or an angle whose measure is given or wanted. Usually --

- For 2 sides and 2 angles: use the sine law involving the 2 sides (if necessary, get the third angle with the Straight Angle Sum Theorem).
- For 3 sides and 1 angle: use the cosine law involving the angle.

\[
\begin{aligned}
\text{Given } b, \angle A, \angle C: & \text{ find } c. \\
\text{2 side, 2 angle problem. Use the sine law with } b \text{ and } c. \\
& \frac{c}{\sin C} = \frac{b}{\sin B} \text{ where } B = 180^\circ - (A+C).
\end{aligned}
\]

\[
\begin{aligned}
\text{Give an exact answer and a 2-place decimal answer.} \\
\text{\(\angle A = 20^\circ, \angle B = 30^\circ\), } c = 40 \text{ cm. Find } a. \\
\text{2 angle, 2 side problem with sides } a, c. \\
\text{Use the sine law for } c \text{ and } a. \\
& \frac{a}{\sin A} = \frac{c}{\sin C} \text{ where } C = 180^\circ - (\angle A + \angle B) \\
& = 180^\circ - (20^\circ + 30^\circ) = 180^\circ - 50^\circ = 130^\circ.
\end{aligned}
\]

Now solve for \(a\).

\[
\frac{a}{\sin A} = \frac{c}{\sin C}
\]

\[
a = \frac{c \sin A}{\sin C} = \frac{40 \sin 20^\circ}{\sin 130^\circ} \text{ cm} \leftarrow \text{Exact answer}
\]

\[
17.86 \text{ cm} \leftarrow \text{2-place decimal answer}
\]

Additional table-user step:

\[
\sin x = \sin(\pi - x) = \sin(180^\circ - x), \text{ so } \\
\sin C = \sin 130^\circ = \sin(180^\circ - 130^\circ) = \sin 50^\circ.
\]

\[
\text{\(\angle A = 20^\circ, \ b = 50, \ c = 60\). Find } a. \\
\text{3 side, 1 angle problem with angle } \angle A. \\
\text{Use the cosine law for } \angle A. \\
a^2 = b^2 + c^2 - 2bc \cos A
\]

Solve for \(a\).

\[
a = \sqrt{b^2 + c^2 - 2bc \cos A}
\]

\[
= \sqrt{50^2 + 60^2 - 2(50)(60) \cos 20^\circ}
\]

\[
= \sqrt{6100 - 6000 \cos 20^\circ} \leftarrow \text{Exact answer}
\]

\[
= 21.49 \leftarrow \text{2-place decimal answer}
\]

\[
\text{\(\angle A = 20^\circ, \ b = 20, \ c = 30\). Find } \angle A. \\
\text{3 side, 1 angle problem with angle } \angle A. \\
\text{Use the cosine law for } \angle A. \\
a^2 = b^2 + c^2 - 2bc \cos A
\]

Solve for \(\cos A\).

\[
2bc \cos A = b^2 + c^2 - a^2
\]

\[
\cos A = (b^2 + c^2 - a^2)/2bc
\]

\[
A = \cos^{-1}[(b^2 + c^2 - a^2)/2bc] \\
= \cos^{-1}(900/1200) = \cos^{-1}(3/4) = 41.41^\circ
\]
Give exact answers, not decimal answers.

4(9). $a = 3, \ b = 1, \ c = 3$. Find $\angle B$.

3 side, 1 angle problem with angle $\angle B$.
Use the cosine law for $\angle B$.
\[ b^2 = a^2 + c^2 - 2ac \cos B \]
Solve for $\cos B$.
\[ 2ac \cos B = a^2 + c^2 - b^2 \]
\[ \cos B = (a^2 + c^2 - b^2) / 2ac \]
\[ B = \cos^{-1}[(a^2 + c^2 - b^2) / 2ac] \]
\[ = \cos^{-1}[(3^2 + 3^2 - 1^2) / (2 \cdot 3 \cdot 3)] \]
\[ = \cos^{-1}(17/18) \]

3(9). $\angle C = 40^\circ, \ b = 3, \ a = 5$. Find $c$.

3 side, 1 angle problem with angle $\angle C$.
Use the cosine law for $\angle C$.
\[ c^2 = b^2 + a^2 - 2ba \cos C \]
Solve for $c$.
\[ c = \sqrt{b^2 + a^2 - 2ba \cos C} \]
\[ = \sqrt{3^2 + 5^2 - 2(3)(5) \cos 40^\circ} \]
\[ = \sqrt{34 - 30 \cos 40^\circ} \]

5(6). $\angle A = 50^\circ, \ a = 4, \ c = 5$. Find the two values of $\angle C$.
There are two values since the side opposite is < side adjacent.
2 sides, 2 angle problem.
Use the sine law with sides $a, c$.
\[ \frac{\sin C}{c} = \frac{\sin A}{a} \]
\[ \sin C = \frac{c \sin A}{a} = \frac{5 \sin 50^\circ}{4} \]
\[ C = \sin^{-1}\left(\frac{5 \sin 50^\circ}{4}\right) \text{ and} \]
\[ C = 180^\circ - \sin^{-1}\left(\frac{5 \sin 50^\circ}{4}\right) \text{ or} \]
\[ C = \pi - \sin^{-1}\left(\frac{5 \sin 50^\circ}{4}\right) \]

5(10, may omit). A 100 foot tall building is viewed from a point $S$, the angle of inclination from $S$ to point $P$ at the top of the building is $3^\circ$. The angle of declination from $S$ to the point $Q$ at the bottom of the building is $5^\circ$. Find the distance $d$ between $S$ and the bottom $Q$ of the building?
Solve for the one large triangle, not the two smaller right triangles.
Include units. This problem is done only if it can be completed before the hour.

\[ \angle PSQ = 8, \ \angle SPQ = 90^\circ - 3^\circ = 87^\circ, \]
\[ \angle SQP = 90^\circ - 5^\circ = 85^\circ \]
\[ \frac{d}{\sin 87^\circ} = \frac{100}{\sin 8^\circ} \]
\[ d = \frac{100 \sin 87^\circ}{\sin 8^\circ} \text{ feet} \]
Give exact answers, no decimal answers. Use either the sine law (2 sides, 2 angles) or the cosine law (3 sides, 1 angle).

3. \( \angle B = 55^\circ, \ a = 3, \ c = 5 \). Find \( b \). ___/2

5. \( \angle C = 40^\circ, \ c = 4, \ a = 5 \). Find the two values of \( \angle A \). Recall if \( \theta \) is a solution to \( \sin \theta = a \), so is \( 180^\circ - \theta \) when the side opposite is < the side adjacent. ___/3

5. \( \angle C = 60^\circ, \ c = 3, \ a = 2 \). Find the \( b \). Use the quadratic formula \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \). Since triangle sides have positive lengths; discard any negative answers. ___/4
§6.4 §6.5

(a) Give the unsimplified exact answer obtained using the law of sines or cosines or Straight Thm. E.g. 
\[ a = \sqrt{8^2 + 5^2 - 2 \cdot 8 \cdot 5 \cdot \cos(20^\circ)} \] or 
\[ b = \frac{5 \sin(100^\circ)}{\sin(50^\circ)}. \]

(b) Give the decimal answer accurate to two places. “sym” means “symbols” including “.”. 11.23°, 11.23cm have “5 sym+units”

1(2). \( \angle A = 30^\circ, \angle B = 135^\circ, a=4\text{cm}, \) find \( b. \)

Exact \( b = \)

Decimal (4 sym + units) \( b = \)

2(2). \( \angle A = \angle B = 35^\circ, c=16\text{cm}, \) find \( a. \) Hint, 1st find \( \angle C. \)

\[ \angle C = 180 - (35 + 35) = 110^\circ. \]

\[ \frac{a}{\sin A} = \frac{c}{\sin C}, \quad a = \frac{c \sin A}{\sin C}. \]

Exact \( a = \)

Decimal (4 sym + units) \( a = \)

3(4)ab. In figure (a) and (b), find \( x. \)

(a) Exact \( x = \)

Decimal (4 sym+units) \( x = \)

(b) Exact \( x = \)

Decimal (5 sym+units) \( x = \)

4(6). \( a=33, b=7, c=37. \) Find \( \angle A, \angle B, \angle C. \)

An unsimplified exact answer for an angle might be of the form 
\[ A = \cos^{-1}\left(\frac{20^2 + 5^2 - 12^2}{2 \cdot 20 \cdot 5}\right) \] or 
\[ B = 180 - (70 + 85) \]

5(4). Find the perimeter of a regular nine-sided polygon inscribed in a circle of radius 4cm.

1st find \( \theta, \) 2nd find other two two triangle angles. 3rd find \( x. \) 4th find the perimeter.

Exact value of \( x = \)

Decimal answer for perimeter (5 sym + units) =
(a) **Give the unsimplified exact answer obtained using the law of sines or cosines or Straight ∠ Thm.** E.g.
\[ a = \sqrt{8^2 + 5^2 - 2(8)(5)\cos(20^\circ)} \quad \text{or} \quad b = \frac{5\sin(100^\circ)}{\sin(50^\circ)}. \]

(b) **Give the decimal answer accurate to two places.** See text or worked examples for the answers

---

**A** \( \angle A = 60^\circ, \angle B = 45^\circ, a=12\text{cm}, \) find \( b \).

Exact \( b = \)

Decimal \( b = \)

**B** \( \angle B=100^\circ, \angle C=30^\circ, c=10\text{cm}, \) find \( a \).

Exact \( a = \)

Decimal \( a = \)

**C.** In figure (a) and (b), find \( x \).

(a) ** Exact \( x = \)**

Decimal \( x = \)

(b) ** Exact \( x = \)**

Decimal \( x = \)

**D.** In figure (a) and (b), find \( x \).

(a) ** Exact \( x = \)**

Decimal \( x = \)

(b) ** Exact \( x = \)**

Decimal \( x = \)

**E.** \( a=7, b=8, c=13. \) Find \( \angle A, \angle B, \angle C. \)

Exact \( A = \)

Decimal (5 sym + units) \( \angle A = \)

Exact \( C = \)

Decimal (3 sym + units) \( \angle C = \)

Exact \( B = \)

Decimal (5 sym + units) \( \angle B = \)

**F.** \( a=b=2\sqrt{3}, c=2. \) Find \( \angle A, \angle B, \angle C. \)

Exact \( A = \)

Decimal (2 sym + units) \( \angle A = \)

Exact \( C = \)

Decimal (3 sym + units) \( \angle C = \)

Exact \( B = \)

Decimal (2 sym + units) \( \angle B = \)

**G.** Find the perimeter of a regular pentagon inscribed in a circle of radius 1cm.

Exact value of a side \( x = \)

Decimal answer for perimeter =

---

**Answers**

See Hw 26 worked examples.
(a) Give the unsimplified exact answer obtained using the law of sines or cosines or Straight \( \angle \) Thm. E.g.
\[ a = \sqrt{8^2 + 5^2 - 2(8)(5)\cos(20^\circ)} \text{ or } b = \frac{5\sin(100^\circ)}{\sin(50^\circ)}. \]
(b) Give the decimal answer accurate to two places.

A. \( \angle A = 60^\circ, \angle B = 45^\circ, a = 12 \text{ cm}, \text{ find } b. \)

Exact \( b = 12 \times \frac{\sin 45}{\sin 60} = 4\sqrt{6} \text{ cm} \)
Decimal \( b = 9.80 \text{ cm} \)

B. \( \angle B = 100^\circ, \angle C = 30^\circ, c = 10 \text{ cm}, \text{ find } a. \)

Exact \( a = 10 \times \frac{\sin 50}{\sin 30} \text{ cm} \)
Decimal \( a = 15.32 \text{ cm} \)

C. In figure (a) and (b), find \( x. \)

\[ (a) \quad \text{Exact } x = \sqrt{5^2 + 8^2 - 2 \times 5 \times 8 \cos 60^\circ} \text{ cm} \]
Decimal \( x = 7 \text{ cm} \)

\[ (b) \quad \text{Exact } x = \sqrt{5^2 + 8^2 - 2 \times 5 \times 8 \cos 120^\circ} = \sqrt{129} \text{ cm} \]
Decimal \( x = 11.36 \text{ cm} \)

D. In figure (a) and (b), find \( x. \)

\[ (a) \quad \text{Exact } x = \sqrt{7.3^2 + 11.5^2 - 2 \times 7.3 \times 11.5 \cos 40^\circ} \text{ cm} \]
Decimal \( x = 7.54 \text{ cm} \)

\[ (b) \quad \text{Exact } x = \sqrt{7.3^2 + 11.5^2 - 2 \times 7.3 \times 11.5 \cos 140^\circ} \text{ cm} \]
Decimal \( x = 17.72 \text{ cm} \)

E. \( a = 7, b = 8, c = 13. \text{ Find } \angle A, \angle B, \angle C. \)

\[ C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \]

Exact \( A = \cos^{-1}\left(\frac{8^2 + 13^2 - 7^2}{2 \times 8 \times 13}\right) \)
Decimal (5 sym + units) \( \angle A = 27.80^\circ \)

Exact \( C = \cos^{-1}\left(\frac{13^2 + 7^2 - 8^2}{2 \times 13 \times 7}\right) \)
Decimal (3 sym + units) \( \angle C = 120^\circ \)

Exact \( B = \cos^{-1}\left(\frac{12^2 + 7^2 - 8^2}{2 \times 12 \times 7}\right) \)
Decimal (5 sym + units) \( \angle B = 32.20^\circ \)

F. \( a = 2/\sqrt{3}, b = 2. \text{ Find } \angle A, \angle B, \angle C. \)

\[ \text{Exact } A = \cos^{-1}\left(\frac{\left(\frac{2}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{3}}\right)^2 - (\frac{2}{\sqrt{3}})^2}{2 \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}}\right) \]
Decimal (2 sym + units) \( \angle A = 30^\circ \)

Exact \( C = \cos^{-1}\left(\frac{(\frac{2}{\sqrt{3}})^2 + (\frac{2}{\sqrt{3}})^2 - 2^2}{2 \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}}\right) \)
Decimal (3 sym + units) \( \angle C = 120^\circ \)

Exact \( B = \cos^{-1}\left(\frac{\left(\frac{2}{\sqrt{3}}\right)^2 + (\frac{2}{\sqrt{3}})^2 - 2^2}{2 \times \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}}}\right) \)
Decimal (2 sym + units) \( \angle B = 30^\circ \)

G. Find the perimeter of a regular pentagon inscribed in a circle of radius 1 cm.

\[ \theta = \frac{360}{5} = 72^\circ \quad x = \sqrt{1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos 72^\circ} \]

Value of \( x = 1.18 \text{ cm} \)
Decimal answer for perimeter = \( 5x = 5.88 \text{ cm} \)
A triangle is determined uniquely up to congruence (1) by two sides and an included angle, and also, (2) by two angles and an included side.

However, given two sides and a nonincluded angle, there may be 0, 1, or 2 triangles.

- Suppose $\angle A = 30^\circ$, $b = 2$. Then there is no triangle with $a = 0.5$, one triangle with $a = 1$, two triangles with $a = 1.5$, and one triangle with $a = 2.5$.

- Suppose two sides and a nonincluded angle are known. When solving for the third side using a cosine law, you may get an answer of the form $s = \sqrt{t}$. Then there is:
  - No solution if $t < 0$ or both $s \leq \sqrt{t} < 0$.
  - Two solutions if $t > 0$ and both $s \leq \sqrt{t} > 0$.
  - One solution otherwise.

When solving for a second angle using a sine law, you may get an answer of the form $\theta = \sin^{-1} t$, $t \geq 0$. There is:
  - No solution if $t > 1$.
  - Two solutions if $t < 1$ and the larger of the two sides is adjacent to the given angle ($\theta$ is one angle, $\pi - \theta$ the other).
  - One solution otherwise.

- In a triangle, $\sin A = 1/2$. What are the possible angles, in degrees, for $A$? One is $A = 30^\circ$, the other is $180^\circ - 30^\circ = 150^\circ$.

Try doing this and the next problem by drawing accurate pictures.

- Is there a triangle in which $a = 2$, $b = 3$, and $\angle A = 60^\circ$? Such a triangle exists if the third side $c$ exists.

Solving for $c$.

- $a^2 = b^2 + c^2 - 2bc \cos A$
- $9 = 4 + c^2 - 2(3)c \cos 60^\circ$
- $c^2 - 3c + 5 = 0$
- $c = \frac{3 \pm \sqrt{3^2 - 4(1)(5)}}{2(1)} = \frac{3 \pm \sqrt{-11}}{2}$

Since $c$ is undefined, the triangle does not exist.

- $a = 2$, $b = 3$, $\angle A = 30^\circ$. Find $\angle C$ if $\angle B$ is acute.

First find $\angle B$ using the sine law, then find $\angle C$.

$\sin B = \frac{b}{a} \sin A$, $\sin B = \frac{2}{3} \sin 30^\circ$. Thus one answer is $B = \sin^{-1} \left( \frac{2}{3} \sin 30^\circ \right) = \sin^{-1} \left( \frac{1}{2} \right) = 30^\circ$. The other answer is $180^\circ - 30^\circ = 150^\circ$. The acute angle is $30^\circ$. $\angle C = 180^\circ - (\angle A + \angle B) = 180^\circ - (30^\circ + 30^\circ) = 120^\circ$.

Polar coordinates

**Definition.** The polar coordinates $(r, \theta)$ of a point $P$ are its distance $r$ (radius) from the origin and the angle $\theta$ between the positive $x$-axis and the line from $(0,0)$ to $P$. The usual $(x,y)$ are the rectangular coordinates.

**Theorem.** If a point has rectangular coordinates $(x,y)$, then its polar coordinates $(r, \theta)$ are

From the picture we have, for positive $r$

$\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, $\tan \theta = \frac{y}{x}$

$y = r \sin \theta$, $x = r \cos \theta$, $r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1} \frac{y}{x}$ if $\theta \in I, IV$ $\theta = \tan^{-1} \frac{y}{x} + \pi$ if $\theta \in II, III$

**Negative $r$.** $(r, \theta)$ is the point $(r, \theta + \pi)$ on the opposite side of the origin $(0,0)$ as $(r, \theta)$.

- Convert from polar coordinates to rectangular: $(7, \frac{\pi}{6})$.

  $r = 7$, $\theta = \pi/6$, $r \sin \theta = (7 \cos \frac{\pi}{6}, 7 \sin \frac{\pi}{6}) = (\frac{7 \sqrt{3}}{2}, \frac{7}{2})$ $\rightarrow$ answer

- Convert from rectangular coordinates to polar: $(-1, \sqrt{3})$.

  $x = -1$, $y = \sqrt{3}$. Point is in quadrant II. $\cdot$ add $\pi$.

  $(r, \theta) = (\sqrt{x^2 + y^2}, \pi + \tan^{-1} \frac{y}{x}) = (\sqrt{1 + 3}, \pi + \tan^{-1} \frac{\sqrt{3}}{1}) = (2, \pi - \frac{\pi}{3}) = (2, \frac{2\pi}{3})$ $\rightarrow$ answer

- Convert the polar equation to a rectangular equation:

  $r \sin \theta + 2 \cos \theta = 0$.

  $y + 2(x/r) = 0 \rightarrow y + 2 \cdot \frac{x}{\sqrt{x^2 + y^2}} = 0$

  $y\sqrt{x^2 + y^2} + 2x = 0$ $\rightarrow$ answer (hw simplifies more)

- Convert the rectangular equation to a polar equation:

  $2x - y^2 = 0$ $2r \cos \theta - r^2 \sin^2 \theta = 0$ $\rightarrow$ answer
7(a)(2) Find the rectangular coordinates of the point with polar coordinates \((r, \theta) = (-4, 5\pi/4)\).

\[
(x, y) = (r \cos \theta, r \sin \theta) \\
= (-4 \cos(5\pi/4), -4 \sin(5\pi/4)) \\
= (-4 \cos(\pi + \pi/4), -4 \sin(\pi + \pi/4)) \\
= (4 \cos(\pi/4), 4 \sin(\pi/4)) \\
= (4 \cdot \frac{\sqrt{2}}{2}, 4 \cdot \frac{\sqrt{2}}{2}) = (2\sqrt{2}, 2\sqrt{2})
\]

7(b)(3) Find the polar coordinates of the point with rectangular coordinates \((x, y) = (-2, 2)\).

\[
r = \sqrt{x^2 + y^2} \\
= \sqrt{(-2)^2 + 2^2} \\
= 2\sqrt{2} \\
\theta = \tan^{-1}(y/x) + \pi \\
= \tan^{-1}(-1) + \pi \\
= -\pi/4 + \pi \\
(r, \theta) = (2\sqrt{2}, 3\pi/4)
\]

8(a)(3) Convert the polar equation to a rectangular equation: \(\cot \theta \csc \theta = r, \ r \neq 0\).

\[
\frac{\cos \theta - 1}{\sin \theta \sin \theta} = r \\
\cos \theta = r \sin^2 \theta \\
\frac{x}{r} = r \left(\frac{y}{r}\right)^2 \\
x = y^2
\]

8(d)(2) Convert the rectangular equation to a polar equation: \(x = y^2 - y + 1\).

\[
r \cos \theta = r^2 \sin^2 \theta - r \sin \theta + 1
\]
7(a) Find the rectangular coordinates of the point with polar coordinates \((-3, -\pi/4)\).
\[(x, y) = (r \cos \theta, r \sin \theta) = \quad /1\]

7(b) Find the polar coordinates of the point with rectangular coordinates \((x, y) = (-1, -\sqrt{3})\).
What is the quadrant for this point? I?, II?, III?, IV?
Find \(r\) using
\[r = \sqrt{x^2 + y^2} = \quad /1\]

Find \(\theta\).
If \(\theta\) is in quadrant I or IV, \(\theta = \tan^{-1}(y/x)\)
Otherwise use the formula \(\theta = \tan^{-1}(y/x) + \pi\) \quad /1

Now write your answer as an ordered pair:
\[(r, \theta) = \]

8(a) Convert the polar equation to a rectangular equation:
\[\sin \theta = r, \ r \neq 0.\]
First replace \(\sin \theta\) with \(y/r\).

8(b) Convert the rectangular equation to a polar equation:
\[x^2 - y^2 = 1. \ \text{Recall: } x = r \cos \theta, y = r \sin \theta.\]
1(2). \( \sin B = \frac{\sqrt{3}}{2} \),
What are the possible values, in degrees, for \( \angle B \)?
2 and 3 symbols + units

2(1). \( \cos B = -\frac{\sqrt{3}}{2} \),
What is the possible value, in degrees, for \( \angle B \)?
3 symbols + units

3(3). Is there any triangle in which \( a = 2 \), \( b = 3 \), and
\( \angle A = 41^\circ \)? One point for the answer, two for showing your work.

4(4). \( a = 30 \), \( b = 36 \), \( \angle A = 20^\circ \). There are two possible
triangles with these sides and angles. Decimal answer only.
Find the areas of these two triangles. Show your work.
Either find \( c \) or find \( \angle C \). Then use Lecture 25’s area formula.
One answer is almost 40, the other is between 360 and 390.

5(1). \((5, \frac{\pi}{4})\)

6(1). \((-5, \frac{\pi}{4})\)

Convert the rectangular coordinates to polar coordinates. Give exact answers with \( r > 0 \). Both involve \( \pi \).
7(1). \((3, \sqrt{3})\)
9 symbols

8(1). \((0, -2)\)
\(\tan^{-1} \frac{1}{2}\) is undefined, draw the picture instead.
8 symbols

§8.1 587:41-60. Convert the polar equation to a rectangular. Simplify any radicals. E.g. \( x^2 + y^2 = 9 \) instead of \( \sqrt{x^2 + y^2} = 3 \)
9(1). \( 2 \sin \theta - 3 \cos \theta = r \)
11 or 13 symbols

10(1). \( r = 4 \)
8 symbols

Convert the rectangular equation to a polar equation.
11(1). \( x^2 + y^2 = 25 \)
3, 4 or 5 symbols

12(1). \( y = x^2 \)
13 symbols
Math 140  Hw 27  Recommended problems, don't turn this in.

A(a) $\sin B = \frac{1}{\sqrt{2}}$.
What are the possible values, in degrees, for $\angle B$?

(b) $\cos B = \frac{1}{\sqrt{2}}$.
What is the possible value, in degrees, for $\angle B$?

(c) $\sin A = \frac{1}{4}$
What is the possible value, in degrees, for $\angle A$?

(d) $\cos A = -\frac{2}{3}$
What is the possible value, in degrees, for $\angle A$?

B. Find the lengths $a$, $b$, and $c$ in the following picture.

\[ \angle A = 50^\circ, \quad \angle B = 110^\circ, \quad \angle C = 95^\circ. \]

\[ a = \frac{2\sin 70^\circ}{\sin 20^\circ} \text{ cm}, \quad b = \frac{2\sin 50^\circ}{\sin 20^\circ} \text{ cm}, \]

\[ c = \frac{2\sqrt{2}}{5\pi/4} \text{ cm} \]

**Answers**

A. $\{45^\circ, 135^\circ\}$  (b) $45^\circ$  (c) $14.5^\circ, 165.5^\circ$  (d) $131.8^\circ$

B. $a = \frac{2\sin 70^\circ}{\sin 20^\circ} \text{ cm}, \quad b = \frac{2\sin 50^\circ}{\sin 20^\circ} \text{ cm},$

C. $(-3/2, 3\sqrt{3}/2), \quad D. (2\sqrt{3}, -2), \quad E. (2\sqrt{3}, -2)$

F. $(\sqrt{2}, 5\pi/4)$  G. $(x-1)^2 + y^2 = 1$

H. $x^4 + x^2y^2 - y^2 = 0$  I. $(x^2 + y^2)^3 = 9(x^2 - y^2)^2$

J. $r = 2/(3\cos \theta - 4\sin \theta)$  K. $r = \tan^2 \theta \sec \theta$

L. $r^2 = \csc 2\theta$
A(a). $\sin B = 1/\sqrt{2}$.

What are the possible values, in degrees, for $\angle B$?
$B = \sin^{-1}1/\sqrt{2} = 45^0$, or $180^0 - 45^0 = 135^0$

(b). $\cos B = 1/\sqrt{2}$.

What is the possible value, in degrees, for $\angle B$?
$B = \cos^{-1}1/\sqrt{2} = 45^0$

For sin, there may be two values $\theta$ and $\pi - \theta$. For cos, there's only one.

(c). $\sin A = 1/4$

What is the possible value, in degrees, for $\angle A$?
$A = \sin^{-1}1/4 = 14.5^0$, or $180^0 - 14.5^0 = 165.5^0$

(d). $\cos A = -2/3$

What is the possible value, in degrees, for $\angle A$?
$A = \cos^{-1}-2/3 = 180^0 - \cos^{-1}2/3 = 180^0 - 48.19^0 = 131.81^0$
The intermediate step is needed only for table users, calculators can do $\cos^{-1}2/3 = 130.81^0$ directly.

B. Find the lengths $a$, $b$, and $c$ in the following picture.

\[ \angle A = 50^0 , \angle B = 110^0 , \angle C = 95^0 . \]

E. $180^0 - (50^0+110^0) = 20^0$

\[ \frac{a}{\sin B} = \frac{2}{\sin E} , \quad \frac{a}{\sin B} = \frac{2}{\sin 20^0} , \quad a = 2 \sin 110^0 \frac{2}{\sin 20^0} = 2 \sin 70^0 \text{ cm} \]

\[ \frac{b}{\sin A} = \frac{2}{\sin E} , \quad \frac{b}{\sin 50^0} = 2 \sin 20^0 , \quad b = 2 \sin 50^0 \text{ cm} \]

\[ \frac{c}{\sin C} = \frac{2}{\sin 70^0} , \quad \frac{c}{\sin 50^0} = 2 \sin 20^0 , \quad c = 2 \sin 50^0 \sin 70^0 \text{ cm} \]

Convert the polar coordinates to rectangular coordinates.

Give exact answer. Both involve the square root of 2.

C. $(3, \frac{\pi}{4})$

$(x, y) = (r \cos \theta, r \sin \theta) = (3 \cos \frac{\pi}{4}, 3 \sin \frac{\pi}{4})$

\[ = (3(-\frac{1}{2}), 3(\frac{\sqrt{3}}{2})) = (-\frac{3}{2}, \frac{3\sqrt{3}}{2}) \]

D. $(4, 11\pi/6)$

$(x, y) = (r \cos \theta, r \sin \theta) = (4 \cos \frac{11\pi}{6}, 4 \sin \frac{11\pi}{6})$

\[ = (4(\sqrt{3}/2), 4(-1/2)) = (2\sqrt{3}, -2) \]

E. $(4, -\pi/6)$

$(x, y) = (r \cos \theta, r \sin \theta) = (4 \cos \frac{-\pi}{6}, 4 \sin \frac{-\pi}{6})$

\[ = (4(\sqrt{3}/2), 4(-1/2)) = (2\sqrt{3}, -2) \]

Convert the rectangular coordinates to polar coordinates.

Give exact answer with $r > 0$.

F. $(-1, -1)$

$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1}x = \tan^{-1}\frac{-1}{-1} = \tan^{-1}1 = \frac{\pi}{4}$

$\tan^{-1}$ is correct only up to a multiple of $\pi$. Since $(-1, -1)$ is in quadrant III, we must add $\pi$ to $\pi/4$.

Answer: $(\sqrt{2}, 5\pi/4)$

Convert the polar equation to a rectangular equation with $0$ on the right, everything else on the left. E.g.,

\[ x^2 + 2xy + y^2 + 3y = 0. \] Simplify any radicals. E.g. $x^2 + y^2 = 9$ instead of $\sqrt{x^2 + y^2} = 3$

G. $r = 2 \cos \theta$

\[ \sqrt{x^2 + y^2} = 2 \cos \theta \text{. Clear the denominator.} \]

\[ x^2 + y^2 = 2x \text{ or, converting to a circle equation,} \]

Answer: $(x - 1)^2 + y^2 = 1$

H. $r = \tan \theta$

\[ \sqrt{x^2 + y^2} = \frac{y}{x} \text{. Square both sides} \]

\[ x^2 + y^2 = y^2/x^2 \]

\[ x^2 + y^2 = 9 \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2 \]

Answer: $x^2 + y^2 = 9(x^2 - y^2)^2$

Convert the rectangular equation to a polar equation.

J. $3x - 4y = 2$

\[ 3r \cos \theta - 4r \sin \theta = 2 \]

Answer: $r = 2/(3 \cos \theta - 4 \sin \theta)$

K. $y^2 = x^3$

\[ (r \sin \theta)^2 = (r \cos \theta)^3 \]

\[ r^2 \sin^2 \theta = r^3 \cos^3 \theta \]

\[ r = \sin^2 \theta \cos \theta \]

Answer: $r = \tan^2 \theta \sec \theta$
Definition. A parabola consists of all points equidistant between a given focus point and a given directrix line. The axis is the line through the focus and perpendicular to the directrix. The vertex is the intersection of the parabola and the axis.

In a parabola, light rays parallel to the axis are reflected to the focus. Telescope mirrors and satellite antennas have this shape. The vertex lies halfway between the focus and the directrix.

- Find the equation for the parabola with focus $(0, p)$ and directrix $y = -p$.
  
  For any point $(x, y)$, the distance between $(x, y)$ and the directrix $y = -p$ is $y + p$. The distance between $(x, y)$ and the focus $(0, p)$ is $\sqrt{(x - 0)^2 + (y - p)^2} = \sqrt{x^2 + y^2 - 2py + p^2}$.

  $(x, y)$ is on the parabola iff the distances are equal:
  
  iff $y + p = \sqrt{x^2 + y^2 - 2py + p^2}$
  
  iff $(y + p)^2 = x^2 + y^2 - 2py + p^2$
  
  iff $y^2 + 2py + p^2 = x^2 + y^2 - 2py + p^2$
  
  iff $2py = x^2 - 2py$ (if $p > 0$)
  
  iff $x^2 = 4py$ (iff $x^2 = ky$ where $k = 4p$ and $p = k/4$)

  **Vertical Parabola Theorem.** For $k \neq 0$, the graph of $x^2 = ky$ is the vertical parabola with focus $(0, p)$ and directrix $y = -p$ where $p = k/4$. The axis is the $y$-axis; the vertex is $(0, 0)$.

  Exchanging $x$ and $y$ gives —

  **Horizontal Parabola Theorem.** For $k \neq 0$, the graph of $y^2 = kx$ is a horizontal parabola with focus $(p, 0)$ and directrix $x = -p$ where $p = k/4$. The axis is the $x$-axis; the vertex is $(0, 0)$.

  - Find the focus, directrix and graph of $y = -x^2/8$.
    
    $x^2 = -8y$. Parabola is vertical.
    
    $p = k/4 = -8/4 = -2$.
    
    Focus: $(0, p) \rightarrow (0, -2)$.
    
    Directrix: $y = -p \rightarrow y = 2$. Graph ...

    Hint, first mark the two focal-width points on a line through the focus, $||$ to the directrix and equidistant from focus and directrix.

  - Find the focus, directrix and graph of $3y^2 = 4x$.
    
    Write it in horizontal-parabola form: $y^2 = kx$.
    
    $y^2 = \frac{4}{3}x, \quad p = k/4 = \frac{4}{3}/4 = \frac{1}{3}$.
    
    Focus $(p, 0)$: $(\frac{1}{3}, 0)$.
    
    Directrix $x = -p: x = -\frac{1}{3}$. Graph ...

**Theorem.** In any equation, replacing each:

- $x$ by $x-a$ shifts the graph right by $a$ units
- $x$ by $x+a$ shifts the graph left by $a$ units
- $y$ by $y-b$ shifts the graph up $b$ units
- $y$ by $y+b$ shifts the graph down $b$ units

When a parabola is shifted, so are its focus, directrix, vertex, and axis.

To graph a parabola, get the squared variable on the left, the rest on the right. Complete the square if needed.

Write the equation in either the:

**vertical parabola form:** $(x \pm a)^2 = k(y \pm b)$ or

**horizontal parabola form:** $(y \pm b)^2 = k(x \pm a)$

- Find the focus, directrix and graph of $x^2 - 2x + 9 - 8y = 0$.
  
  $x^2 - 2x = 8y - 9, \quad x^2 - 2x + 1 = 8y - 8, \quad$ complete the square
  
  $(x - 1)^2 = 8(y - 1), \quad p = k/4 = 8/4 = 2$. A vertical parabola.
  
  For $x^2 = 8y$: vertex $= (0, 0)$, focus $= (0, 2)$, directrix $y = -2$.
  
  To get $(x - 1)^2 = 8(y - 1)$, shift right 1 unit and up 1 unit.
  
  Vertex: $(0, 0)$ shifted right 1 and up 1 $\rightarrow$ $(1, 1)$.
  
  Focus: $(0, 2)$ shifted right 1, up 1 $\rightarrow$ $(1, 3)$.
  
  Directrix: $y = -2$ shifted right 1, up 1 $\rightarrow$ $y = -1$.

**Math 140 Lecture 28** (Omitted on short semesters)

The graph of $y = ax^2 + bx + c$ is a vertical parabola with a vertical axis of symmetry. Other parabolas have horizontal or slanted axes.
24(15). Find the axes/directrix and the focal point(s). Draw the graph of \( y^2 - 2y + 4x - 3 = 0 \).

\[
\begin{align*}
y^2 - 2y + 4x - 3 &= 0 \\
(y^2 - 2y) &= -4x + 3 \\
(y^2 - 2y + 1) &= -4x + 3 + 1 \\
(y - 1)^2 &= -4x + 4 \\
(y - 1)^2 &= -4(x - 1)
\end{align*}
\]

This is the horizontal parabola \( y^2 = -4x \) shifted right 1 and up 1 to get \((y - 1)^2 = -4(x - 1)\)

\[
k = -4 \quad p = k/4 = -4/4 = -1
\]

Vertex: \((0, 0) \rightarrow (1, 1)\)

Focus: \((p, 0) = (-1, 0) \rightarrow (0, 1)\)

Directrix: \(x = -p, x = 1 \rightarrow x = 2\)

(a)(3) Directrix: \(x = 2\)

(b)(3) Focal point(s): \((0, 1)\)

(c)(9) Graph
Here’s an example, your problem is in the next column.

Example. Find the axes/directrix and the focal point(s).

Draw the graph of \( y^2 - 2y + 4x - 3 = 0 \).

Write in parabolic form \((y - b)^2 = k(x - a)\)

\[
\begin{align*}
y^2 - 2y + 4x - 3 &= 0 \\
(y^2 - 2y) &= -4x + 3 \\
(y^2 - 2y + 1) &= -4x + 3 + 1 \\
(y - 1)^2 &= -4x + 4 \\
(y - 1)^2 &= -4(x - 1)
\end{align*}
\]

This is the horizontal parabola \( y^2 = -4x \)

shifted right 1 and up 1 to get \((y - 1)^2 = -4(x - 1)\)

\( k = -4 \quad p = k/4 = -4/4 = -1 \)

Vertex: \((0, 0) \rightarrow \quad (1, 1) \)

Focus: \((p, 0) = (-1, 0) \rightarrow \quad (0, 1) \)

Directrix: \( x = -p, x = 1 \rightarrow \quad x = 2 \)

(a)(3) Directrix: \( x = 2 \)

(b)(3) Focal point(s): \( (0, 1) \)

(c)(9) Graph

24. Find the axes/directrix and the focal point(s).

Draw the graph of \( x^2 + 2x - 4y - 3 = 0 \).

Write in parabolic form \((x - b)^2 = k(y - a)\)

This is the vertical parabola \_________________________\n
shifted \______\ and \______\ to get

Vertex: \((0, 0) \rightarrow \quad \)

Focus: \((0, p) = (____, ____ ) \rightarrow (____, ____ ) \)

Directrix: \( y = -p, \quad \) \rightarrow \quad

Graph: Mark the focal width points on the graph.
Graph the parabola. On the graph, mark and give the coordinates for focus and vertex. Draw the directrix with a dotted line.

First locate the focus and vertex and draw the directrix. Then mark the two “focal-width points” where the line through the focus which is parallel to the directrix intersects the circle around the focus which touches the directrix.

A. \( x^2 = 4y \)

B. \( y^2 = -8x \)

C. \( y^2 - 6y - 4x + 17 = 0 \)

D. \( x^2 - 8x - y + 18 = 0 \)

Answers
For C and D, see Hw 28 worked examples.
For `A and B, see the next page.
A. \( x^2 = 4y \)
vertex: \((0, 0)\)
focus: \((0, 1)\)
directrix: \(y = -1\)

B. \( y^2 = -8x \)
vertex: \((0, 0)\)
focus: \((-2, 0)\)
directrix: \(x = 2\)
Graph the parabola. On the graph, mark and give the coordinates for focus and vertex. Draw the directrix with a dotted line.

First locate the focus and vertex and draw the directrix. Then mark the two “focal-width points” where the line through the focus which is parallel to the directrix intersects the circle around the focus which touches the directrix.

C. \( y^2 - 6y - 4x + 17 = 0 \)

This implies a horizontal parabola.

Complete the square to put the equation in the form:

\[(y + b)^2 = k(x - a)\]

\[
(y^2 - 6y) = 4x - 17
\]

\[-(6/2)^2 = (-3)^2 = 9. \text{ Add 9 to both sides.}
\]

\[
(y^2 - 6y + 9) = 4x - 8
\]

\[
(y - 3)^2 = 4(x - 2)
\]

Thus \( p = \frac{k}{4} = \frac{4}{4} = 1 \)

For \( y^2 = 4x \): vertex = (0,0), focus = (1,0), directrix \( x = -1 \).

To get \( (y - 3)^2 = 4(x - 2) \), shift right 2 and up 3.

Shifting (0,0) right 2 and up 3 gives the vertex: (2,3).
Shifting (1,0) gives the focus: (3,3).
Shifting \( x = -1 \) gives the directrix: \( x = 1 \).

D. \( x^2 - 8x - y + 18 = 0 \)

This implies a vertical parabola.

Complete the square to put the equation in the form:

\[(x + a)^2 = k(y + b)\]

\[
(x^2 - 8x + 16) = y - 18
\]

\[-8/2)^2 = (-4)^2 = 16. \text{ Add 16 to both sides.}
\]

\[
(x^2 - 8x + 16) = y - 2
\]

\[
(x - 4)^2 = l(y - 2)
\]

Thus \( p = \frac{k}{4} = \frac{1}{4} \)

For \( x^2 = 1y \):

vertex = (0,0), focus = (0,1/4), directrix \( y = -1/4 \).

To get \( (x - 4)^2 = 1(y - 2) \), shift right 4 unit and up 2 unit.

Shifting (0,0) right 4 and up 2 gives the vertex: (4,2).
Shifting (0,1/4) gives the focus: (4,9/4).
Shifting \( y = -1/4 \) gives the directrix: \( y = 7/4 \).

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Math 140  Hw 28  Worked examples of selected recommended problems.
Math 140 Lecture 29

- Find the focus, directrix and graph of \( y^2 + 2y = 4x - 5 \).

\[ y^2 + 2y + 1 = 4x - 4, \quad (y + 1)^2 = 4(x - 1) \]

\[ p = k/4 = 4/4 = 1. \]

This is the parabola \( y^2 = 4x \) shifted: down 1 unit, right 1 unit.

The \( y^2 \) means the parabola is horizontal.

- **Vertex:** \((0,0)\) shifted down 1 and right 1 \( \rightarrow \) \((1,-1)\).
- **Focus:** \((p,0) = (1,0)\) shifted down 1, right 1 \( \rightarrow \) \((2,-1)\).
- **Directrix:** \(x=-p\), i.e., \(x=-1\), down 1, right 1 \( \rightarrow \) \(x=0\).

**Recall.** A circle is the set of all points such that the distance to a center point is some constant \( r \).

**Definition.** An ellipse is the set of all points such that the sum of the distances to two focus points is the distance between the vertices. The vertices are the two points farthest apart. The major axis goes from a vertex at one end of the ellipse through the two foci to the vertex at the opposite end. The minor axis is a perpendicular bisector of the major axis.

A light ray emitted from one focus point is reflected to the opposite focus point. Planetary orbits are ellipses.

Let

\[ a = \text{major radius} = \frac{1}{2} \text{ the major axis length} \]
\[ b = \text{minor radius} = \frac{1}{2} \text{ the minor axis length} \]
\[ c = \text{focal radius} = \frac{1}{2} \text{ the distance between the foci} \]

**Theorem.** \( a^2 = b^2 + c^2 \). \( \therefore \) \( c = \sqrt{a^2 - b^2} \).

The graph of \( x^2 + y^2 = r^2 \) is a circle with center \((0,0)\) and radius \( r \). This can be written as \( \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1 \).

**Theorem.** For \( a \geq b > 0 \), the graphs of \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) and \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) are ellipses. \((0,0)\) is the center. \( a, b, c \) are the major, minor and focal radii where \( c = \sqrt{a^2 - b^2} \).

- \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is a horizontal ellipse.
- Foci: \((-c,0)\) and \((c,0)\).
- Major axis: the line segment \((-a,0)(a,0)\).
- Minor axis: \((0,-b)(0,b)\).
- Here \( x \) has the bigger radius.

- \( \frac{y^2}{a^2} + \frac{x^2}{b^2} = 1 \) is a vertical ellipse.
- Foci: \((0,-c)\) and \((0,c)\).
- Major axis: \((0,-a)(0,a)\).
- Minor axis: \((-b,0)(b,0)\).
- Here \( y \) has the bigger radius.

To graph, complete the squares if necessary. Get 1 on the right. Write the equation in one of the forms above.

- **Find the major and minor axes and foci; draw the graph:** \( 4x^2 + y^2 = 4 \).

\[ x^2 + \frac{y^2}{4} = 1, \quad \frac{x^2}{2^2} + \frac{y^2}{2^2} = 1, \quad \frac{y^2}{2^2} + \frac{x^2}{2^2} = 1 \]

\[ a = 2, b = 1, c = \sqrt{4-1} = \sqrt{3} \]

**Vertical ellipse (\( y \) has the bigger radius).**

- Major axis: \((0,-2)(0,2)\). Write your axes like this on the final.
- Minor axis: \((-1,0)(1,0)\).
- Foci: \((0,-\sqrt{3}),(0,\sqrt{3})\) or \((0,\pm \sqrt{3})\).

Set your compass to a major radius.

Put the point at the end of a minor radius.

Draw an arc.

It intersects the major axis at the two foci.

- **Find the major and minor axes and foci; draw the graph:** \( 16x^2 - 96x + 25y^2 = 256 \).

\[ 16(x^2 - 6x +9) + 25y^2 = 256 + 16 \cdot 9 \]
\[ 16(x-3)^2 + 25y^2 = 400 \]

**Horizontal ellipse (\( x \) has the bigger radius).**

\[ (x-3)^2 + \frac{y^2}{16} = 1 \]
\[ \frac{(x-3)^2}{9} + \frac{y^2}{4} = 1 \]
\[ a = 5, b = 4, c = \sqrt{25-16} = \sqrt{9} = 3 \]

This is \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \) shifted right 3 units.

- Major axis: \((-5,0)(5,0)\) shifted right 3 \( \rightarrow \) \((-2,0)(8,0)\).
- Minor axis: \((-0,-4)(0,4)\) shifted right 3 \( \rightarrow \) \((-3,4)(3,4)\).
- Foci: \((-3,0),(3,0)\) shifted right 3 \( \rightarrow \) \((-2,0),(6,0)\).
Find the axes/directrix and the focal points. Draw the graph of $x^2 + 2x + 4y^2 - 8y = -1$.

$x^2 + 2x + 4y^2 - 8y = -1$

$(x^2 + 2x) + (4y^2 - 8y) = -1$

$(x^2 + 2x + 1) + 4(y^2 - 2y) = -1 + 1$

$(x + 1)^2 + 4(y^2 - 2y + 1) = -1 + 1 + 4$

$(x + 1)^2 + 4(y - 1)^2 = 4$

$$\frac{(x+1)^2}{4} + \frac{(y-1)^2}{1} = 1$$

$$\frac{(x+1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$$

Horizontal ellipse $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$ → shifted left 1, up 1 to $\frac{(x+1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$

$a = 2, b = 1, c = \sqrt{a^2 - b^2} = \sqrt{4 - 1} = \sqrt{3}$

Vertex $(0, 0) \rightarrow (-1, 1)$

Major axis: $(-2,1)(2,1) \rightarrow (-3,1)(1,1)$

Minor axis: $(0,0)(0,2) \rightarrow (-1,0)(-1,2)$

Focal points: $(-\sqrt{3},0), (\sqrt{3},0) \rightarrow (-1 - \sqrt{3},0)(-1 + \sqrt{3},0)$

(a)(3) Axes: $(-3,1)(1,1), (-1,0)(-1,2)$

(b)(3) Focal points: $(-1 - \sqrt{3},1), (-1 + \sqrt{3},1)$

(c)(9) Graph
Here’s an example, your problem in in the next column.

Example. Find the axes and the foci. Draw the graph of

\[ x^2 + 2x + 4y^2 - 8y = -1. \]

\[ (x^2 + 2x) + (4y^2 - 8y) = -1 \]

\[ (x^2 + 2x + 1) + 4(y^2 - 2y) = -1 + 1 \]

\[ (x + 1)^2 + 4(y - 1)^2 = 4 \]

\[ \frac{(x+1)^2}{4} + \frac{(y-1)^2}{1} = 1 \]

\[ \frac{(x-(-1))^2}{2^2} + \frac{(y-1)^2}{1^2} = 1 \quad \text{← must have squares on the bottom.} \]

Horizontal ellipse \[ \frac{x^2}{2^2} + \frac{y^2}{1^2} = 1 \rightarrow \]

shifted left 1, up 1 to \[ \frac{(x+1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1 \]

\[ a = 2, b = 1, c = \sqrt{a^2 - b^2} = \sqrt{4 - 1} = \sqrt{3} \]

Vertex \((0,0)\) → \((-1,1)\)

Major axis: \((-2,0)(2,0)\) → \((-3,1)(1,1)\)

Minor axis: \((0,-1)(0,1)\) → \((-1,0)(-1,2)\)

Focal points:

\[ (\pm \sqrt{3}, 0) \rightarrow (-1 \pm \sqrt{3}, 1) \]

(a)(3) Axes: \((-3,1)(1,1), (-1,0)(-1,2)\)

(b)(3) Focal points: \((-1 \pm \sqrt{3}, 1)\)

(c)(9) Graph

Problem. Find the axes and the foci. Draw the graph of

\[ 4x^2 + 8x + y^2 - 2y = -1. \]

Complete the square. Then write the equation in ellipse form. In this case \( \frac{(y\pm?)^2}{a^2} + \frac{(x\pm?)^2}{b^2} = 1 \) with squares on the bottom.

\[ a = ? \]
\[ b = ? \]
\[ c = ? \]

Major axis:

Minor axis:

Focal points:

Graph:
Graph the ellipse. On the graph, mark and give the coordinates for endpoints of the major and minor axes and the foci.

A. $4x^2 + 9y^2 = 36$

B. $\frac{(x-5)^2}{5^2} + \frac{(y+1)^2}{3^2} = 1$

C. $3x^2 + 4y^2 - 6x + 16y + 7 = 0$

D. $5x^2 + 3y^2 - 40x - 36y + 188 = 0$

Answers
For C and D, see Hw 29 worked examples.
For A and B, see the next page.
Math 140  Hw 29  Recommended problems, answers.

A. \(4x^2 + 9y^2 = 36 \quad \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1\)

Major axis: (-3,0)(3,0).

Minor axis: (0,-2)(0,2).

Foci: \((-\sqrt{5}, 0), (\sqrt{5}, 0)\).

B. \(\frac{(x-5)^2}{5^2} + \frac{(y+1)^2}{3^2} = 1\)

Major axis: (0,-1)(10,1).

Minor axis: (5,-4)(5,2).

Foci: (1, -1), (9, -1).
Graph the ellipse. On the graph, mark and give the coordinates for endpoints of the major and minor axes and the foci.

C. \(3x^2 + 4y^2 - 6x + 16y + 7 = 0\)
\((3x^2 - 6x) + (4y^2 + 16y) = -7\)
\(3(x^2 - 2x) + 4(y^2 + 4y) = -7\)
\(3(x^2 - 2x + 1) + 4(y^2 + 4y + 4) = -7 + 3 + 16\)
\(3(x-1)^2 + 4(y+2)^2 = 12\)
\(\frac{(x-1)^2}{4} + \frac{(y+2)^2}{3} = 1\)

Horizontal ellipse (\(x\) has the bigger denominator).
This is shifted right 1, down 2 units.
Shifting \((-2,0)\), \((2,0)\) right 1 gives major axis: \((-1,-2),(3,-2)\).
Shifting \((0,-\sqrt{3}),(0,\sqrt{3})\) gives minor axis: \((1,-2-\sqrt{3})(1,\sqrt{2}+\sqrt{3})\).
Shifting \((-1,0)\), \((1,0)\) gives foci: \((0,-2),(2,-2)\).

\[a = 2, \ b = \sqrt{3}, \ c = \sqrt{4 - 3} = \sqrt{1} = 1\]

This is \(\frac{x^2}{2^2} + \frac{y^2}{\sqrt{3}} = 1\) shifted right 1, down 2 units.

\[\frac{(x-4)^2}{5} + \frac{(y-6)^2}{3} = 0\]
This is true iff \(x = 4\) and \(y = 6\). Thus it is a degenerate ellipse, it is just the single point \((4,6)\).
DEFINITION. A hyperbola is the set of all points such that the difference of the distances to two focal points is the distance between the vertices. The focal axis through the two foci intersects the hyperbola at its two vertices. The major axis is the segment between the vertices, the minor axis is a perpendicular bisector. The ends of the axes are the midpoints of a box whose diagonals are asymptotes. The box corners and the foci are equidistant from the box’s center. To get the foci, draw an arc around the box center from the box corner to the focal axis.

The path of a stone thrown upward (in a vertical gravitational field) is a parabola. The path of an orbiting comet is an ellipse with the sun at one focus. The path of a comet passing through the solar system is one piece of a hyperbola with the sun at one focal point.

Let
\( a = \) the major radius = \( \frac{1}{2} \) major axis length,
\( b = \) the minor radius = \( \frac{1}{2} \) the minor axis length,
\( c = \) the diagonal radius = \( \frac{1}{2} \) the distance between the foci.

THEOREM. \( c^2 = a^2 + b^2 \), thus \( c = \sqrt{a^2 + b^2} \).

THEOREM. The graphs of
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]
and
\[ \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \]
are hyperbolas (\( a \) must be below the positive square) with
\( a = \) the major radius, \( b = \) the minor radius, and
\( c = \sqrt{a^2 + b^2} \) = the focal radius.

\( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) is a horizontal hyperbola with
foci: \((-c, 0) \) and \((c, 0)\)
major axis: \((\pm a, 0)\)
minor axis: \((0, \pm b)\)

\( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \) is a vertical hyperbola with
foci: \((0, -c) \) and \((0, c)\)
major axis: \((0, \pm a)\)
minor axis: \((-b, 0) \) and \((b, 0)\)

To graph, complete the squares if necessary, then write the equation in one of the above forms.

- Find the axes and foci; draw the graph and asymptotes:
  \( 4y^2 - x^2 = 4 \).
  \[ y^2 - \frac{x^2}{4} = 1, \quad \frac{y^2}{12} - \frac{x^2}{22} = 1 \]
  \( a = 1, b = 2, c = \sqrt{1 + 4} = \sqrt{5} \)
Vertical hyperbola (the \( y^2 \) is positive).
Major axis: \((0, -1)(0,1)\), minor axis: \((-2)(2,0)\),
fooci: \((0, \sqrt{5}), (0, -\sqrt{5})\). On the final, write the axes as above.

- Find the axes and foci; draw the graph and asymptotes:
  \( 16x^2 - 96x - 25y^2 = 256 \).
  \[ 16(x^2 - 6x + 9) - 25y^2 = 256 + 16 \times 9 \]
  \[ 16(x - 3)^2 - 25y^2 = 400 \]
  divide by 400, get 1 on right
  \[ \frac{(x - 3)^2}{25} - \frac{y^2}{16} = 1 \]
  \[ \frac{(x - 3)^2}{5^2} - \frac{y^2}{4^2} = 1 \]
  \( a = 5, b = 4, c = \sqrt{25 + 16} = \sqrt{41} \)
Horizontal hyperbola (the \( x^2 \) is positive).
This is \( \frac{x^2}{25} - \frac{y^2}{16} = 1 \) shifted right 3 units.
Shifting \((-5,0)(5,0)\) right 3 gives the major axis: \((-2)(8)\).
Shifting \((0, -4)(0, 4)\) right 3 gives the minor axis: \((3)(4)\).
Shifting \((-\sqrt{41},0), (\sqrt{41},0)\) gives the foci: \((3 - \sqrt{41}), (3 + \sqrt{41})\).
26(15). Find the axes/directrix and the focal points. Draw the graph of
\[ x^2 + 2x - 4y^2 + 8y = 7. \]

\[
\begin{align*}
(x^2 + 2x) - (4y^2 - 8y) &= 7 \\
(x^2 + 2x + 1) - 4(y^2 - 2y) &= 7 + 1 \\
(x + 1)^2 - 4(y - 1)^2 &= 4 \\
\frac{(x+1)^2}{4} - \frac{(y-1)^2}{1} &= 1 \\
\frac{(x+1)^2}{2^2} - \frac{(y-1)^2}{1^2} &= 1
\end{align*}
\]

Horizontal hyperbola \( \frac{x^2}{2^2} - \frac{y^2}{1^2} = 1 \) \( \rightarrow \) shifted left 1, up 1 to \( \frac{(x+1)^2}{2^2} - \frac{(y-1)^2}{1^2} = 1 \)

\[ a = 2, b = 1, c = \sqrt{a^2 + b^2} = \sqrt{4 + 1} = \sqrt{5} \]

Vertex \((0, 0) \rightarrow (-1, 1)\)

Major axis: \((-2,1)(2,1) \rightarrow (-3,1)(1,1)\)

Minor axis: \((0,0)(0,2) \rightarrow (-1,0)(-1,2)\)

Focal points: \((-\sqrt{5}, 0), (\sqrt{5}, 0) \rightarrow (-1 - \sqrt{5}, 0)(-1 + \sqrt{5}, 0)\)

(a)(3) Axes: \((-3,1)(1,1), (-1,0)(-1,2)\)

(b)(3) Focal points: \((-1 - \sqrt{5}, 1), (-1 + \sqrt{5}, 1)\)

(c)(9) Graph
Here’s an example, your problem is in the next column.

26 Example. Find the axes/directrix and the focal points.

Draw the graph of \( x^2 + 2x - 4y^2 + 8y = 7 \).

\[
\begin{align*}
(2x + 2) - (4y - 8y) &= 7 \\
(x + 1)^2 - 4(y^2 - 2y + 1) &= 7 + 4 \\
(x + 1)^2 - 4(y - 1)^2 &= 4 \\
\frac{(x+1)^2}{4} - \frac{(y-1)^2}{1} &= 1 \\
\frac{(x+1)^2}{2} - \frac{(y-1)^2}{1} &= 1 \\
\end{align*}
\]

Horizontal hyperbola \( \frac{x^2}{2^2} - \frac{y^2}{1^2} = 1 \) shifted left 1, up 1 to \( \frac{(x+1)^2}{2} - \frac{(y-1)^2}{1^2} = 1 \)

\[ a = 2, b = 1, c = \sqrt{a^2 + b^2} = \sqrt{4 + 1} = \sqrt{5} \]

Vertex \((0,0)\) -> \((-1,1)\)

Major axis: \((-2,0)(2,0)\) -> \((-3,1)(1,1)\)

Minor axis: \((0,0)(0,2)\) -> \((-1,0)(-1,2)\)

Focal points:
\((-\sqrt{5},0), (\sqrt{5},0)\) -> \((-1 - \sqrt{5},0)(-1 + \sqrt{5},0)\)

(a) Axes:
\((-3,1)(1,1), (-1,0)(-1,2)\)

(b) Focal points: \((-1 - \sqrt{5},1), (-1 + \sqrt{5},1)\)

(c) Graph:

[Graph of a hyperbola with marked axes and focal points]

26 Problem. Find the axes/directrix and the focal points.

Draw the graph of \( 4y^2 + 8y - x^2 + 2x = 1 \).

Complete the square. Then write the equation in hyperbola form. In this case \( \frac{(y+y_0)^2}{a^2} - \frac{(x-x_0)^2}{b^2} = 1 \)

\[ a = ? \]
\[ b = ? \]
\[ c = ? \]

Major axis:

Minor axis:

Focal points:

Graph:
Graph the hyperbola. On the graph, mark and give the coordinates for endpoints of the major and minor axes and the foci. Draw the asymptotes with a dotted line.

A. \( x^2 - 4y^2 = 4 \)

B. \( \frac{(x-5)^2}{5^2} - \frac{(y+1)^2}{3^2} = 1 \)

C. \( \frac{(y-2)^2}{2^2} - \frac{(x-1)^2}{1^2} = 1 \)

D. \( x^2 - y^2 + 2y - 5 = 0 \)

E. \( y^2 - 25x^2 + 8y - 9 = 0 \)

Answers
For D and E, see Hw 30 worked examples.
For A, B and C, see the next page.
Math 140  Hw 30  Recommended problems, answers.

A. $x^2 - 4y^2 = 4$
   Major axis: (-2,0)(2,0).
   Minor axis: (0,-1)(0,1).
   Foci: (-√5, 0), (√5, 0)

B. $\frac{(x-5)^2}{5^2} - \frac{(y+1)^2}{3^2} = 1$
   Major axis: (0, -1)(10, -1).
   Minor axis: (5,-4)(5,2).
   Foci: (5−√34, -1), (5+√34, -1).

C. $\frac{(y-2)^2}{2^2} - \frac{(x-1)^2}{12} = 1$
   Major axis: (1,0)(1,4).
   Minor axis: (0,1)(2,1).
   Foci: (1, 2−√3), (1, 2+√3).
Graph the hyperbola. On the graph, mark and give the coordinates for endpoints of the major and minor axes and the foci. Draw the asymptotes with a dotted line.

D. \( x^2 - y^2 + 2y - 5 = 0 \)

\[
x^2 - (y^2 - 2y) = 5
\]

\[
x^2 - (y^2 - 2y + 1) = 5 - 1
\]

\[
x^2 - (y - 1)^2 = 4
\]

\[
\frac{x^2}{2^2} - \frac{(y-1)^2}{2^2} = 1
\]

\[a = 2, \ b = 2, \ c = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2} \approx 2.83\]

Horizontal hyperbola (the \( x^2 \) is positive).
This is \( \frac{x^2}{2^2} - \frac{y^2}{2^2} = 1 \) shifted up 1 unit.
Shifting \((-2,0)(2,0)\) up 1 gives major axis: \((-2,1)(2,1)\).
Shifting \((0,-2)(0,2)\) up 1 gives minor axis: \((0,-1)(0,3)\).
Shifting \((-2.83,0), (2.83,0)\) up 1 gives foci: \((-2.83,1), (2.83,1)\).
   or precisely, \((-2\sqrt{2}, 1), (2\sqrt{2}, 1)\).

E. \( y^2 - 25x^2 + 8y - 9 = 0 \)

\[
(y^2 + 8y + 16) - 25x^2 = 9 + 16
\]

\[
(y + 4)^2 - 25x^2 = 25
\]

\[
\frac{(y+4)^2}{5^2} - \frac{x^2}{1^2} = 1
\]

\[a = 5, \ b = 1, \ c = \sqrt{25 + 1} = \sqrt{26} \approx 5.10\]

Vertical hyperbola (the \( y^2 \) is positive).
This is \( \frac{y^2}{5^2} - \frac{x^2}{1^2} = 1 \) shifted down 4 units.
Shifting \((0,-5)(0,5)\) down 4 gives major axis: \((0,-9)(0,1)\).
Shifting \((-1,0)(1,0)\) down 4 gives minor axis: \((-1,-4)(1,-4)\).
Shifting \((0,-5.10), (0,5.10)\) down 4 gives foci: \((0,-9.10), (0,1.10)\).
   or precisely, \((0,-4 - \sqrt{26}), (0,-4 + \sqrt{26})\)
If only one variable is squared, it is a parabola.
Write with the squared variable on the left.
\((x \pm a)^2 = k(y \pm b)\)  \(\text{Vertical parabola like } y = x^2\text{ or } y = -x^2\).
\((y \pm b)^2 = k(x \pm a)\)  \(\text{Horizontal parabola.}\)

Suppose both variables are squared. If necessary, complete and replace \((x \pm c)\) by \(x\) and \((y \pm d)\) by \(y\).
Write with 1 on the right, squares in the denominators.

- **Horizontal ellipse:** \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) with \(a\geq b > 0\). Bigger first.
- **Vertical ellipse:** \(\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1\) with \(a\geq b > 0\).
- **Horizontal hyperbola:** \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\). Positive first.
- **Vertical hyperbola:** \(\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1\).

There is clearly a missing case, \(-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\), but this is impossible since the left side is negative and thus cannot equal 1 which is positive.

Second degree equations such as \(x^2 + xy + y^2 = 1\) which involve a product of \(x\) and \(y\) are also parabolas, ellipses and hyperbolas but their axes may be slanted rather than horizontal or vertical. The simplest example is \(xy = 1\). Solving for \(y\) gives \(y = \frac{1}{x}\). The graph is a hyperbola.

In this case the focal axis of the hyperbola is the major diagonal \(y = x\). Except for this graph you won’t be asked to graph equations with a slanted axis. Such graphs can be quite difficult and require techniques which aren’t covered until upper division linear algebra.

Identify each of the following as a horizontal/vertical parabola/ellipse/hyperbola. Graph.

- \(x^2 + y = 2\). \(x\) squared, \(y\) not: vertical parabola.
  If we didn’t have to find the focus and directrix we could easily graph this by rewriting it as: \(y = 2 - x^2\).
  \(x^2 = -y + 2\)
  \(x^2 = -(y - 2)\)
  This is \(x^2 = -y\) shifted up 2 units.
  \(k = -1\), \(p = k/4 = -1/4\).
  Center: \((0,0)\) \(\rightarrow\) shifts to \((0,2)\).
  Focus: \((0, p) = (0, -1/4)\) \(\rightarrow\) shifts to \((0, -1/4 + 2) = (0, 7/4)\).

- \(y^2 + 4x^2 - 8x = 0\)
  \(y^2 + 4(x^2 - 2x + 1) = 1 \cdot 4\)
  \(y^2 + 4(x - 1)^2 = 4\)
  \(y^2 = 4 - x^2 = 4(x - 1)^2 = 4\)
  \(\frac{y^2}{4^2} + (x - 1)^2 = 1\)
  Both positive, \(y^2\) has the larger denominator, \(\therefore\) a vertical ellipse.
  This is \(\frac{y^2}{4^2} + \frac{x^2}{1^2} = 1\) shifted right 1 unit.
  Center \((0,0)\) shifts to \((1,0)\).
  \(a = \text{major radius} = 2\).
  \(b = \text{minor radius} = 1\).
  \(c = \text{focal radius} = \sqrt{a^2 - b^2} = \sqrt{2^2 - 1^2} = \sqrt{3}\).
  Graph
  To locate the foci, set your compass to the distance between the center and a vertex.
  Then put the compass point at an endpoint of the major axis.
  The arc of the pencil end will intersect the major axis at the two foci.
  \((0,0)\) \(\rightarrow\) vertical ellipse
  \((2,0)\)
  \((1,2)\)
  \((1, -2)\)
  Major axis: \((1,2)(1,-2)\)
  Minor axis: \((0,0)(2,0)\)
  Foci: \((1, -\sqrt{3}), (1, \sqrt{3})\)

- \(x^2 - y^2 = 1\) \(\rightarrow\) \(\frac{x^2}{1^2} - \frac{y^2}{1^2} = 1\).
  \(x^2\) is positive, \(\therefore\) horizontal hyperbola.

- Find the area of a triangle with sides 2, 3, 4.
  Let \(\theta\) be the angle opposite side 4. Then
  \(4^2 = 2^2 + 3^2 - 2(2)(3)\cos \theta\)
  \(\therefore 12 \cos \theta = 4 + 9 - 16 = -3, \theta = \cos^{-1}(-1/4)\)
  \(\text{Area} = \frac{1}{2}(2)(3)\sin(\cos^{-1}(-\frac{1}{4})) = 3 \sqrt{1 - \frac{1}{16}} = \frac{3\sqrt{15}}{4}\).