More Limits

Find:

\[ \lim_{x \to \pi} \tan^2 x \]
\[ \lim_{x \to 0} \frac{x^2 + 1}{2 + \cos x} \]
\[ \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \]
\[ \lim_{x \to \infty} \sin \left( \frac{1}{x} \right) \]
\[ \lim_{x \to 0} \frac{x^2 - 4}{x(x+2)} \]
\[ \lim_{x \to 4} \left( \frac{2x}{x+4} + \frac{8}{x+4} \right) \]
\[ \lim_{h \to 0} \frac{1 - \frac{1}{h}}{1 - \frac{1}{h}} \]

Recall: **Limit Laws** If \( \lim_{x \to c} f(x) \) and \( \lim_{x \to c} g(x) \) are defined then:

- \( \lim_{x \to c} d = d \), \( \lim_{x \to c} x = c \), \( \lim_{x \to c} t = t \) independent of \( x \).
- \( \lim_{x \to c} (f(x) \pm g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) \)
- \( \lim_{x \to c} (af(x)) = a \lim_{x \to c} f(x) \)
- \( \lim_{x \to c} (f(x)^r) = \lim_{x \to c} f(x)^r \) for a reduced rational \( r \).
- \( \lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x) \)

Find:

- \( \lim_{x \to c} f(x) = a \) and \( \lim_{x \to c} g(x) = b \) and \( a, b \geq 0 \). Find:
  \[ \lim_{x \to c} (f(x) - xg(x)) \]
  \[ \lim_{x \to c} \sqrt{1 + f(x)g(x)} \]
  \[ \lim_{x \to c} \frac{1 - f(x)}{1 + f(x)} \]

Recall: **Definition** \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \).

Find:

- \( f(x) = x^3 \). Find \( f'(x) \) using the derivative definition.
  \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \]
  \[ = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \]
  \[ = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \]
  \[ = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2 \]
  \[ f'(2) = 3(2^2) = 12 \]. Just replace \( x \) with 2. \( \Box \)

**Limit Definition**

When does a function \( f(x) \) approach a limit \( L \) as the argument \( x \) approaches \( c \)? We want a precise definition, what does it mean to say that \( f(x) \) “approaches \( L \)” as \( x \) approaches \( c \)?

Saying “\( f(x) \) approaches \( L \)” can be made precise by saying that for any measure \( \varepsilon > 0 \) of closeness, eventually \( f(x) \) is \( \varepsilon \)-close to \( L \). I.e.,

- For any \( \varepsilon > 0 \), eventually, \( |f(x) - L| < \varepsilon \).

Now we have to make “eventually” precise, we need to say that when \( x \) is sufficiently close to \( c \), then \( f(x) \) is \( \varepsilon \)-close to \( L \). We formalize this by saying that for some measure \( \delta > 0 \) of closeness, when \( x \) is \( \delta \)-close to \( c \), i.e., \( |x - c| < \delta \) then \( f(x) \) is \( \varepsilon \)-close to \( L \).

Here is the formal definition:

**Limit Definition** For any function \( f \), any \( c, L \):

- \( \lim_{x \to c} f(x) = L \) iff for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that for any \( x \), \( |x - c| < \delta \) \( \Rightarrow |f(x) - L| < \varepsilon \). \( \Box \)